CO367 Nonlinear Optimization
Department of Combinatorics and Optimization
University of Waterloo

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NONLINEAR OPTIMIZATION — Assignment 2

Score composition:

(3.3)	(3.6)	(3.12)	(3.16)	(4.1)	(4.8(a))	(4.8(e))	Add1	Add2	Add3	Add4	Total
5	5	5	6	6	5	5	5	5	5	5	57

(3.3) Some students made the assumption that f is continuously differentiable on (a, b) to argue that g is concave. This assumption need not be true; for example, a convex and increasing piecewise linear function f may not be continuously differentiable on (a, b) but $g : f(x) \mapsto x$ is concave on $(f(a), f(b))$. The fact that g is concave in the general case can be proven using either of the following arguments.

1. One may use the fact that f is convex and increasing, to show that

$$g(\lambda f(x) + (1-\lambda)f(y)) \geq \lambda g(f(x)) + (1-\lambda)g(f(y))$$

   for all $\lambda \in [0, 1]$ and $x, y \in (a, b)$.

2. Alternately, one may argue that the hypograph of g is equal to the epigraph of f reflected in the line $y = x$ and, hence, is a convex set.

(3.6) Be careful to avoid confusing hyperplanes with affine functions. A hyperplane H in $\mathbb{R}^n$ is an affine subspace of dimension $\mathbb{R}^{n-1}$ defined by $H = \{x : \alpha^T x = \beta\}$ for some $\alpha \in \mathbb{R}^n$, $\beta \in \mathbb{R}$. In other words, every hyperplane is exactly the set of roots of some fixed affine function. For the epigraph of function $f : \mathbb{R}^n \to \mathbb{R}$ to be a halfspace, the set $(x, f(x))$ must be a hyperplane in $\mathbb{R}^{n+1}$; therefore, the function f must be affine.

(3.12) Again, be careful with hyperplanes/affine functions. Arguing that because $\operatorname{epi}(f)$ and $\operatorname{hypo}(g)$ are convex and intersect only on their boundaries there must exist a hyperplane that separates $\operatorname{int}(\operatorname{epi}(f))$ and $\operatorname{hypo}(g)$ *does not* prove that there is an affine function h such that $g(x) \leq h(x) \leq f(x)$ for all $x \in \mathbb{R}^n$. One must use the equation for this hyperplane to construct an explicit formula for h.

(3.16) - (4.8e) These questions were generally well-done but please be careful to fully answer the questions. A number of students lost marks for not including optimal values for (4.1), not considering the change of the equality constraint to inequality in (4.8e), etc.

(Add2) To prove that $K = K^{**}$, one must prove that $K \subseteq K^{**}$ and $K^{**} \subseteq K$. That $K \subseteq K^{**}$ follows immediately from the definition of the second dual as $(K^*)^*$. To see that $K^{**} \subseteq K$, assume, on the contrary, that there exists $k \in K^{**}$ such that$k \notin K$. One may reach a contradiction using the following argument.

1. Since K is closed and convex, there exists $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

   $$a^T x \geq b > a^T k$$

   for all $x \in K$ by Hyperplane Separation Theorem.

2. Since K is a cone, we must have $b \leq 0$ and $a^T k < 0$. This implies that $a \notin K^*$ and that there is $\bar{x} \in K$ such that $a^T \bar{x} < 0$.

3. The fact that the ray $\{\lambda \bar{x} : \lambda \geq 0\}$ is in K can then be used to show that there exists $\lambda > 0$ such that

   $$b \leq a^T (\lambda \bar{x}) < b.$$

(Add3) Most students correctly argued that $S := \{A^T v : v \geq 0\}$ is a subset of V^* using the definition of nonnegative polar. To show that $S = V$, one must also prove the opposite inclusion. To do so, apply the hint to show that V is closed. Then it suffices to show that $S^* \subseteq V^{**} = V$ to complete the proof.

Submitted on February 7, 2011.