Algebraic Symmetry

Def $\mathcal{A} \subseteq \mathbb{F}^{n \times n}$ is matrix $*$-algebra over field $F \in \{\mathbb{R}, \mathbb{C}\}$ if $\forall X, Y \in \mathcal{A}$

- $\alpha X + \beta Y \in \mathcal{A}$, $\forall \alpha, \beta \in F$
- $X^* \in \mathcal{A}$
- $XY \in \mathcal{A}$

Assumption $\exists$ low dimensional matrix $*$-algebra, $\mathcal{A}_{SOP}$, $\ni A_i$, $\forall i = 0, \ldots, m$

$\dim(\mathcal{A}_{SOP}) \ll n.$

Th^{m} 2 Slater holds for p-d pair (1), (2)

Then $\exists$ optimal p-d pair

$(X, S) \in \mathcal{A}_{SDP} \times \mathcal{A}_{SDP}$

Proof. $\mu > 0$, (X_μ, y_μ, S_μ) unique

opt sol$^{\underline{n}}$ on central path.

Then $S_\mu = A_0 - \sum_i (y_\mu)_i A_i \in \mathcal{A}_{SDP}$

$\Rightarrow X_\mu = \mu S_\mu^{-1} \in \mathcal{A}_{SDP}$

Take limit $\mu \searrow 0$

$\blacksquare$

Representations of Matrix $*$-algebras

– canonical block diagonal (up to ordering of blocks)

Th^m 3: $I \in \mathcal{A}$ matrix $*$-algebra

Then $\exists$ unitary Q, integer s $\ni$

$$Q^* \mathcal{A} Q = \begin{bmatrix} \mathcal{A}_1 & & & \\ & \mathcal{A}_2 & & 0 \\ & 0 & \ddots & \\ & & & \mathcal{A}_s \end{bmatrix}$$

$\mathcal{A}_i$ isomorphic to $\mathbb{C}^{n_i \times n_i}$

$$\mathcal{A}_i = \{ I_{k_i} \otimes A \mid A \in \mathbb{C}^{n_i \times n_i} \}, \; i = 1, \ldots, s$$

k_i integers

$$\dim \mathcal{A} = \sum_{i=1}^{s} n_i^2$$

$$n = \sum_{i=1}^{s} k_i n_i$$

ex 4 circulants

$$\begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ & \ddots & & & \vdots \\ & & & \ddots & c_1 \\ c_1 & & & c_{n-1} & c_0 \end{bmatrix}$$

dim $\mathcal{A}$ is n (over $\mathbb{R}$ or $\mathbb{C}$)

$$Q = (Q_{ij}) = \frac{1}{\sqrt{n}} e^{-2\pi\sqrt{-1}\, ij/n}$$

$$i,j = 0, \ldots, n-1$$

discrete Fourier transform

Q unitary, diagonalizes circulants.

4.2 Regular $*$-representation

Unitary Q in thm 3 generally unknown
Use other "faithful rep."

$B_1, \dots B_d \in \mathbb{R}^{n \times n}$ real, orthog., basis of $\mathcal{A}$ (matrix $*$-algebra over $\mathbb{R}$)

normalize $\quad D_i := \frac{1}{\sqrt{\operatorname{Tr} B_i^T B_i}} B_i, \; i=1,\dots,d$

$$D_i D_j = \sum_k \gamma_{ij}^k D_k \;, \quad \text{multip param } \gamma_{ij}^k$$

$$\underset{d \times d}{(L_k)_{ij}} = \gamma_{ij}^k \;, \quad k=1,\dots,d$$

$\{L_k\}$ basis of a faithful (isomorphic) representation of $\mathcal{A}$, say $\mathcal{A}^{reg}$, Th^m 4 the bijective lin. map

$\varphi : \mathcal{A} \to \mathcal{A}^{reg}$ such that

$\varphi(D_i) = L_i, \; i = 1, \dots, d$

defines a $*$-isomorphism $\mathcal{A} \to \mathcal{A}^{reg}$.

Thus, φ is an algebra isomorphism with

$$\varphi(A^*) = \varphi(A)^*, \quad \forall A \in \mathcal{A}$$

φ homomorphism $\Longrightarrow$ A and $\varphi(A)$ have same eigs

we get

$$\sum_{i=1}^{d} x_i \underbrace{D_i}_{n \times n} \succeq 0 \iff \sum_{i=1}^{d} x_i \underbrace{L_i}_{d \times d} \succeq 0$$

4.3 Symmetry reduction of SDP instances

Rewrite SDP as

$$\min_{X \succeq 0} \{ \operatorname{tr} A_0 X : \operatorname{tr} A_k X = b_k, \; X \in \mathcal{A}_{SDP} \}$$

Set $X = \sum_1^d x_i B_i$ ← basis els

get

$$\min_{\sum_i x_i B_i \succeq 0} \left\{ \sum_i^d x_i \operatorname{tr}(A_0 B_i) : \sum_1^d x_i \operatorname{tr} A_k B_i = b_k \quad k = 1, \ldots m \right\}$$

use $\sum_1^d x_i Q^* B_i Q \succeq 0$

to get block diagonal

Note $A \succeq 0 \Leftrightarrow \begin{bmatrix} \operatorname{Re} A & \operatorname{Im} A^T \\ \operatorname{Im} A & \operatorname{Re} A \end{bmatrix} \succeq 0, \; A = A^*$

4.6 Algebraic Symm. from Permutation Groups

- $\mathcal{S}_n$ symm. group on n elements (all perm. of $\{1, \dots, n\}$)

sub-group $\mathcal{G} \leq \mathcal{S}_n$

multiplicative group $n \times n$ permutation matrices

$$(P_\pi)_{ij} = \begin{cases} 1 & \text{if } \pi(i) = j \\ 0 & \text{otherwise} \end{cases} \qquad \pi \in \mathcal{G}$$

commutant $\{A \in \mathbb{C}^{n \times n} : AP_\pi = P_\pi A, \ \forall \pi \in \mathcal{G}\}$

(centralizer ring)

forms a matrix $*$-algebra over $\mathbb{C}$

commutant has a basis that is a coherent representation/configuration

<u>Def</u> Two orbit (orbital) of an index pair (i,j) is

$$\{(\pi(i), \pi(j)) : \pi \in \mathcal{G}\}$$

The orbitals partition

$$\{1, \dots, n\} \times \{1, \dots, n\}$$

yields 0,1 matrices of coherent configuration

Consider $(A_k)_{ij} = (A_k)_{(\pi(i)\,\pi(j))} \quad \forall \pi \in \mathcal{G}_{SDP}$

perm. gp.

equiv. $P_\pi A_k P_\pi^T = A_k \quad \forall \pi \in \mathcal{G}_{SDP}$

So A_k belong to commutant

Def 4 automorphism group of $A \in \mathbb{C}^{n \times n}$

$$\mathrm{Aut}(A) = \{ \pi \in \mathcal{S}_n : A_{ij} = A_{\pi(i)\pi(j)},\ \forall\, ij \}$$

$$\mathcal{G}_{SDP} = \bigcap_{i=0}^{m} \mathrm{Aut}(A_i)$$

ex