Survey 3 types of special structure in SDP

1. a common "chordal" sparsity pattern of the data matrices
   A_i, $i=1,\dots,m$, C

2. Low rank for all data matrices

3. Data matrices are invariant under the action of a permutation group; or where they belong to a low dim. matrix algebra.

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many applications for "symmetry reduction"

<u>Notation</u> $\mathbb{R}^{p\times q}$ ($\mathbb{C}^{p\times q}$) matrix spaces

$\mathcal{S}^{k\times k}$ symmetric matrices

I_n identity J_n all ones

e_i unit vectors e all ones

$A \in \mathbb{C}^{n\times n}$

$$A = \operatorname{Re}(A) + \sqrt{-1}\, \operatorname{Im}(A)$$

$$A^* = \ldots\ldots^T \ldots\ldots\ldots\ldots^T$$ (complex conjugate transp.)

$A = A^*$ <u>Hermitian</u>

$Q \in \mathbb{C}^{n\times n}$, $Q^*Q = I$ <u>unitary</u>

real unitary is <u>orthogonal</u>

$\operatorname{vec}(\cdot)$, $\operatorname{diag}(\cdot)$

<u>Kronecker product</u> $A \otimes B$

Structured instances

(1) $$\min\{A_0\cdot X : A_k\cdot X = b_k,\ k=1,\dots,m,\ X\succeq 0\}$$

$$\mathcal{A}(X) = (A_k\cdot X) \in \mathbb{R}^m \ (\mathbb{C}^m)$$

$$A_k\cdot X = \text{trace } A_k^* X \underbrace{= \text{trace } A_k X}_{\text{in real case}}$$

most applications (real) $A_k \in S^{n\times n}$.

dual

(2) $$\max\{b^T y : \sum_{i=1}^m y_i A_i + S = A_0,\ S\succeq 0\}$$

$$\mathcal{A}^*(y) = \sum_{i=1}^m y_i A_i$$

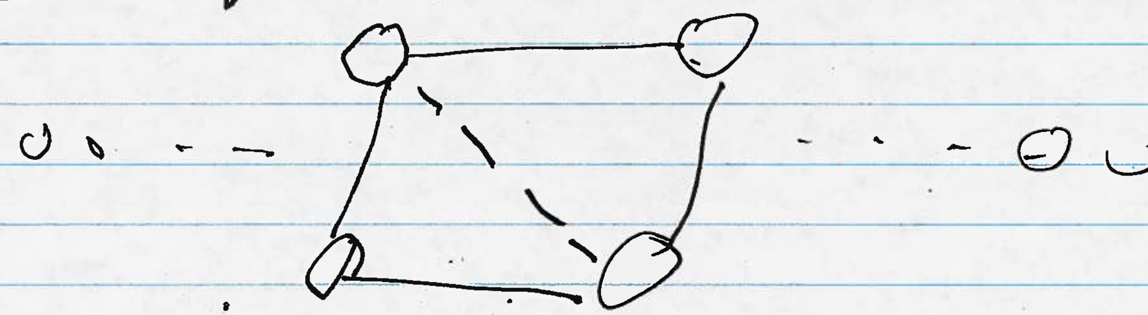
Chordal Structure

$$A_i, \; i=0,1,\dots,m$$

have a common sparsity pattern of the adjacency matrix of some chordal graph $\mathcal{G}=(V,E)$

i.e. $\mathcal{G}$ does not contain a cycle of length 4 or more

$\mathcal{G}$



A adjacency matrix $A_{ij}=1$ if edge $ij \in E$.

$$A_{ij}=0 \Rightarrow (A_k)_{ij}=0, \; \forall k=0,1,\dots,m$$

Low Rank

A_i , $i=1,\dots,m$ are low rank

e.g. for max-cut $A_i = E_{ii} = e_i e_i^T$
is rank 1, $i=1,\dots,m$ (here $m=n$)
(A_0 can have high rank)

Algebraic Symmetry

A_i, $i=0,1,\dots,m$ belong to a
matrix $*$-algebra of low dimension
i.e. over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, it
is a subspace of $\mathbb{F}^{n\times n}$ that is
also closed under matrix multiplication
and taking (conjugate) transposes.
(mostly when data matrices are
invariant under a permutation group)

2 Data Matrices of Low Rank (c.f. facial reduction)

(*) eg $A_i = a_i a_i^T$, $a_i \in \mathbb{R}^n$

Recall simplicity of forming Schur complement M for Max-Cut Problem. In general,

$$M = (m_{ij})$$

$$m_{ij} = \operatorname{trace} A_i Z_1 A_j Z_2, \quad i,j = 1,\dots,m$$

depend on choice of search direction

e.g. (*) gives for

$$\begin{aligned} m_{ij} &= \operatorname{trace} A_i Z_1 A_j Z_2 \\ &= \operatorname{trace} (a_i^T Z_1 a_j)(a_i^T Z_2 a_j) \\ &= (a_i^T S^{-1} a_j) \quad \text{for } Z_1 = Z_2 = S^{-1} \\ &= (S^{-1})^2_{ij} \quad \text{for max-cut} \end{aligned}$$

Can exploit sparsity of $S = -\sum y_i A_i + A_0$