Enumerative properties of $NC^{(B)}(p,q)$

I.P. Goulden * Alexandru Nica * Ion Oancea

Abstract

We determine the rank generating function, the zeta polynomial and the Möbius function for the poset $NC^{(B)}(p,q)$ of annular non-crossing partitions of type B, where p and q are two positive integers.

1 Introduction

The enumerative properties of the lattice $NC(n)$ of non-crossing partitions of $\{1,\ldots,n\}$ have been studied since the early 1970's, starting with the paper [8] of G. Kreweras. An important feature of this lattice is its connection to the symmetric group $\mathcal{S}_n$. More precisely, one has a natural poset isomorphism

$$NC(n) \simeq [\varepsilon, \alpha_n] := \{\tau \in \mathcal{S}_n \mid \varepsilon \le \tau \le \alpha_n\}, \tag{1.1}$$

where "$\le$" is a natural partial order on $\mathcal{S}_n$, ε is the unit of $\mathcal{S}_n$, and α_n is the long cycle $(1,\ldots,n)$ (see [3], [5]).

In 1997, V. Reiner [10] introduced the lattice $NC^{(B)}(n)$ of non-crossing partitions of type B. Soon after that (see [2], [6], [4]) it was noticed that one has a poset isomorphism analogous to the one from (1.1):

$$NC^{(B)}(n) \simeq [\varepsilon, \gamma_n] := \{\tau \in B_n \mid \varepsilon \le \tau \le \gamma_n\}, \tag{1.2}$$

where "$\le$" is a natural partial order on the hyperoctahedral group B_n, ε is the unit of B_n, and γ_n is the long cycle $(1,\ldots,n,-1,\ldots,-n)$. (Here B_n is viewed as the group of permutations τ of $\{1,\ldots,n\} \cup \{-1,\ldots,-n\}$ which satisfy the condition $\tau(-i) = -\tau(i)$, $1 \le i \le n$.)

The recent paper [9] introduced a family of posets $NC^{(B)}(p,q)$, where p,q are two positive integers. One has a poset isomorphism

$$NC^{(B)}(p,q) \simeq [\varepsilon, \gamma_{p,q}] \subseteq B_{p+q}, \tag{1.3}$$

where the partial order on the hyperoctahedral group B_{p+q} is the same as in (1.2), and where $\gamma_{p,q}$ is now the permutation with two cycles

$$\gamma_{p,q} := (\,1,\ldots,p,-1,\ldots,-p\,)(\,p+1,\ldots,p+q,-(p+1),\ldots,-(p+q)\,) \in B_{p+q}.$$

The elements of $NC^{(B)}(p,q)$ are partitions of the set $\{1,\ldots,p+q\} \cup \{-1,\ldots,-(p+q)\}$, and every partition $\pi \in NC^{(B)}(p,q)$ has the property that if A is a block of π then $-A$ is a

*Supported by a Discovery Grant from NSERC, Canada.

block of π as well. The partial order considered on $NC^{(B)}(p,q)$ is the one given by reverse refinement ($\pi \leq \rho$ if and only if every block of π is contained in a block of ρ). The poset $NC^{(B)}(p,q)$ isn't generally a lattice, but we have a notable exception occurring in the case when $q = 1$. In this case the meet operation coincides with the usual "intersection meet" for partitions – the blocks of the meet $\pi \wedge \rho \in NC^{(B)}(p,1)$ are precisely the non-empty intersections $A \cap B$ where A is a block of π and B is a block of ρ.

A distinctive feature of the partitions in $NC^{(B)}(p,q)$ is that one can draw them as non-crossing diagrams in an *annulus* with $2p$ points marked on its outside circle and $2q$ points marked on its inside circle. (This is unlike the diagrams drawn for partitions in $NC^{(B)}(n)$, which are drawn in a *disc* with $2n$ points marked on its boundary.)

In the present paper we determine the rank generating function, the zeta polynomial and the Möbius function of the poset $NC^{(B)}(p,q)$. Here is how the paper is organized. In the next Section 2 we give a brief review (following [9]) of $NC^{(B)}(p,q)$, and a few of its properties that are needed in the present paper. Then in Section 3 we discuss the special "lattice" case $q = 1$, when the formulas for both the rank generating function and the Möbius function are nicer, and have simpler derivations. It is amusing to note that $NC^{(B)}(n-1,1)$ has the same rank generating function as $NC^{(B)}(n)$. Nevertheless, one has $NC^{(B)}(n-1,1) \not\simeq NC^{(B)}(n)$ for all $n \geq 3$, as one sees by looking at Möbius functions. The Möbius function calculation shown in Section 3 relies on a partial Möbius inversion formula, specific to the framework of a lattice. (This calculation is thus taking advantage of the fact that $NC^{(B)}(n-1,1)$ is a lattice, and does not extend to the case of $NC^{(B)}(p,q)$ for $p,q \geq 2$.)

Section 4 is about the rank generating function of $NC^{(B)}(p,q)$ for general p,q. We observe that we still have nice formulas when we focus on partitions in $NC^{(B)}(p,q)$ which have a given connectivity (the *connectivity* of a partition $\pi \in NC^{(B)}(p,q)$ is the number of pairs of blocks $A, -A$ of π such that $A \neq -A$ and such that A intersects both sets $\{\pm 1, \ldots, \pm p\}$ and $\{\pm(p+1), \ldots, \pm(p+q)\}$). But when we just enumerate the partitions in $NC^{(B)}(p,q)$ by their rank we get 1-parameter sums (which can be summed up to a "closed form" when $q = 1$, but not for general q).

The final Section 5 of the paper is devoted to determining the Möbius function for $NC^{(B)}(p,q)$. The method used here is to count multichains via suitable "systems of parentheses", on the same lines that were used by Edelman [7] to count multichains in $NC(n)$ and then by Reiner [10] to count multichains in $NC^{(B)}(n)$. A benefit of this approach is that it also yields concrete formulas for the zeta polynomial for $NC^{(B)}(p,q)$, and for the number of maximal chains in $NC^{(B)}(p,q)$. The formulas obtained are again not in closed form, but (again) they can be summed up to closed form in the particular case when $q = 1$.

2 Review of $NC^{(B)}(p,q)$

In this section we review, following [9], a few basic facts about the posets $NC^{(B)}(p,q)$. We start by clarifying what kind of objects we have in these posets.

Remark 2.1. *(Some features of $NC^{(B)}(p,q)$.)*
(a) The elements of $NC^{(B)}(p,q)$ are partitions of the set $\{1, \ldots, p+q\} \cup \{-1, \ldots, -(p+q)\}$. Every partition $\pi \in NC^{(B)}(p,q)$ has the property that if A is a block of π then $-A$ is a

block of π as well. If a block A of π has the property that $A = -A$, then we will say that A is *inversion-invariant*, or that A is a *zero-block* of π. Clearly, the blocks of π which are not inversion-invariant come in pairs (A and $-A$, with $A \neq -A$). It is worth keeping in mind that a partition $\pi \in NC^{(B)}(p,q)$ can never have more than one inversion-invariant block (if such a block exists, then it is unique).
(b) The partial order considered on $NC^{(B)}(p,q)$ is the one given by reverse refinement ($\pi \leq \rho$ if and only if every block of π is contained in a block of ρ). We have that $\widehat{0}, \widehat{1} \in NC^{(B)}(p,q)$, where $\widehat{0}$ is the partition of $\{\pm 1, \ldots, \pm(p+q)\}$ into $2(p+q)$ singletons, while $\widehat{1}$ is the partition of $\{\pm 1, \ldots, \pm(p+q)\}$ which has only one block. It is clear that $\widehat{0}$ and $\widehat{1}$ are the minimal and respectively maximal elements of the poset $NC^{(B)}(p,q)$.

The information given above about $NC^{(B)}(p,q)$ indicates what kind of objects we want to look at, but does not provide a consistent definition of what $NC^{(B)}(p,q)$ actually is. In order to really *define* $NC^{(B)}(p,q)$, we will next introduce a set $\mathcal{S}_{nc}^{(B)}(p,q)$ of "annular non-crossing *permutations* of type B", and then we will define $NC^{(B)}(p,q)$ in terms of $\mathcal{S}_{nc}^{(B)}(p,q)$.

Remark 2.2. *(Definition of $\mathcal{S}_{nc}^{(B)}(p,q)$.)*
We will introduce $\mathcal{S}_{nc}^{(B)}(p,q)$ via a natural partial order on the hyperoctahedral group B_{p+q}. Let us denote for convenience $p+q =: n$. Recall that B_n is the group of permutations τ of $\{\pm 1, \ldots, \pm n\}$ that satisfy the condition $\tau(-i) = -\tau(i)$, $\forall\, 1 \leq i \leq n$. We will use the following (non-minimal) set of n^2 generators of B_n:

$$\{(i,j)(-i,-j) \mid 1 \leq i,j \leq n,\ i \neq j\} \cup \{(i,-j)(-i,j) \mid 1 \leq i,j \leq n,\ i \neq j\}$$
$$\cup \{(i,-i) \mid 1 \leq i \leq n\}. \tag{2.1}$$

By using the generators from (2.1) one introduces a *length function* ℓ_B on B_n: for every $\tau \in B_n$ the length $\ell_B(\tau)$ is defined as the smallest possible $k \geq 0$ such that τ can be factored as a product of k generators (with the convention that the product of 0 generators is equal to the unit ε of B_n). It is not hard to see that $\ell_B(\tau)$ can be alternatively described in terms of the cycle structure of τ, by the formula

$$\ell_B(\tau) = n - \frac{1}{2} \cdot \Big(\# \text{ of cycles } A \text{ of } \tau \text{ such that } A \neq -A \Big). \tag{2.2}$$

We next use the length function ℓ_B in order to define a *partial order* on B_n, by postulating that for $\sigma, \tau \in B_n$ we have

$$\sigma \leq \tau \ \overset{\text{def}}{\Longleftrightarrow}\ \ell_B(\tau) = \ell_B(\sigma) + \ell_B(\sigma^{-1}\tau). \tag{2.3}$$

In other words, the inequality $\sigma \leq \tau$ means that one can find minimal factorizations for σ and for $\sigma^{-1}\tau$ into products of generators, such that the concatenation of these two factorizations gives a minimal factorization for τ.

Coming to the point of this remark, we define $\mathcal{S}_{nc}^{(B)}(p,q)$ as the *interval* $[\varepsilon, \gamma_{p,q}]$ in the group B_n, where $\gamma_{p,q} \in B_n$ is the permutation with two cycles

$$\gamma_{p,q} := (\,1, \ldots, p, -1, \ldots, -p\,)(\,p+1, \ldots, p+q, -(p+1), \ldots, -(p+q)\,) \in B_n. \tag{2.4}$$

We thus have

$$\mathcal{S}_{nc}^{(B)}(p,q) := [\varepsilon, \gamma_{p,q}] = \{\tau \in B_n \mid \tau \leq \gamma_{p,q}\} \tag{2.5}$$

(where for the second equality in (2.5) we used the fact that the condition "$\varepsilon \leq \tau$" is automatically satisfied by every $\tau \in B_n$).

We mention here that one can give several other equivalent descriptions for $\mathcal{S}_{nc}^{(B)}(p,q)$. Two such descriptions are discussed in [9] – one of them is in terms of a "genus inequality", and the other is in terms of "annular crossing patterns" (see Theorem 1.1 in [9]). But these alternative descriptions will not be used in the present paper.

Remark 2.3. *(Orbit partitions and the definition of $NC^{(B)}(p,q)$.)*
Let p, q be positive integers, and let us denote $p+q =: n$. For every $\tau \in B_n$ we will use the notation $\Omega(\tau)$ for the partition of $\{\pm 1, \ldots, \pm n\}$ into orbits of τ. (Thus two numbers a, b from $\{\pm 1, \ldots, \pm n\}$ belong to the same block of $\Omega(\tau)$ if and only if there exists $m \in \mathbb{Z}$ such that $\tau^m(a) = b$.) Moreover, we will use the notation $\widetilde{\Omega}(\tau)$ for the partition of $\{\pm 1, \ldots, \pm n\}$ which is obtained from $\Omega(\tau)$ by grouping together all the inversion-invariant blocks of $\Omega(\tau)$ (if such blocks exist) into one block of $\widetilde{\Omega}(\tau)$. That is: if

$$\Omega(\tau) = \{A_1, \ldots, A_k, B_1, -B_1, \ldots, B_l, -B_l\}$$

with $A_i = -A_i$ for $1 \leq i \leq k$, then

$$\widetilde{\Omega}(\tau) = \{A_1 \cup \cdots \cup A_k, B_1, -B_1, \ldots, B_l, -B_l\}.$$

With these notations, the set $NC^{(B)}(p,q)$ of *annular non-crossing partitions of type B* is defined as

$$NC^{(B)}(p,q) := \{\widetilde{\Omega}(\tau) \mid \tau \in \mathcal{S}_{nc}^{(B)}(p,q)\}. \tag{2.6}$$

It is remarkable that the map $\widetilde{\Omega} : \mathcal{S}_{nc}^{(B)}(p,q) \to NC^{(B)}(p,q)$ is a *poset isomorphism*, where $\mathcal{S}_{nc}^{(B)}(p,q)$ is partially ordered as an interval of B_n (and where B_n is partially ordered as in Remark 2.2), while $NC^{(B)}(p,q)$ is partially ordered by reverse refinement. This is the content of Theorem 1.2 in the paper [9].

Remark 2.4. *(Rank and connectivity for a partition in $NC^{(B)}(p,q)$.)*
It is immediate that $\mathcal{S}_{nc}^{(B)}(p,q)$ is a ranked poset, where the rank of a permutation $\tau \in \mathcal{S}_{nc}^{(B)}(p,q)$ is given by the length $\ell_B(\tau)$ discussed in Remark 2.2. As a consequence, we see that $NC^{(B)}(p,q)$ is a ranked poset as well, where the rank of a partition $\pi \in NC^{(B)}(p,q)$ is given by the formula

$$\text{rank}(\pi) = (p+q) - \frac{1}{2} \cdot \Big(\# \text{ of blocks of } \pi \text{ that are not inversion-invariant}\Big). \tag{2.7}$$

Another important statistic for partitions in $NC^{(B)}(p,q)$ is connectivity. For $\pi \in NC^{(B)}(p,q)$ we will call the *connectivity* of π the number

$$c := \frac{1}{2} \left(\begin{array}{c} \# \text{ of blocks } A \text{ of } \pi \text{ such that } A \neq -A \\ \text{and such that } A \text{ intersects both sets} \\ \{\pm 1, \ldots, \pm p\} \text{ and } \{\pm(p+1), \ldots, \pm(p+q)\} \end{array} \right). \tag{2.8}$$

An important fact concerning the concept of connectivity is the following:

$$\left\{ \begin{array}{l} \text{if } \pi \in NC^{(B)}(p,q) \text{ has connectivity } c > 0, \\ \\ \text{then } \pi \text{ has no inversion-invariant blocks} \end{array} \right. \tag{2.9}$$

(see [9], Proposition 3.7). Thus the blocks of a partition π with connectivity $c > 0$ all come in pairs $A, -A$ with $A \neq -A$; there are c pairs of blocks as in (2.8), while each of the remaining pairs is either "exterior" ($A, -A \subseteq \{\pm 1, \ldots, \pm p\}$) or "interior" ($A, -A \subseteq \{\pm(p+1), \ldots, \pm(p+q)\}$). Note moreover that if e and i denote, respectively, the number of exterior and of interior pairs of blocks of π, then the numbers c, e, i satisfy the inequalities:

$$\left\{ \begin{array}{l} 1 \leq c \leq \min\{p,q\}, \text{ and} \\ \\ 0 \leq e \leq p - c, \quad 0 \leq i \leq q - c. \end{array} \right. \tag{2.10}$$

Remark 2.5. It is instructive at this point to give a brief discussion, based on connectivity, about how the adjusted orbit map $\widetilde{\Omega} : \mathcal{S}_{nc}^{(B)}(p,q) \to NC^{(B)}(p,q)$ works. Let π be a partition in $NC^{(B)}(p,q)$, and let c be the connectivity of π. There are two possible cases.
(a) $c > 0$. Then by the fact stated in (2.9) above we have $\pi = \Omega(\tau) = \widetilde{\Omega}(\tau)$, where τ is a (uniquely determined) permutation in $\mathcal{S}_{nc}^{(B)}(p,q)$, and τ has no inversion-invariant orbits.
(b) $c = 0$. Let τ denote the unique permutation in $\mathcal{S}_{nc}^{(B)}(p,q)$ such that $\widetilde{\Omega}(\tau) = \pi$. Then every orbit of τ is contained either in $\{\pm 1, \ldots, \pm p\}$ or in $\{\pm(p+1), \ldots, \pm(p+q)\}$ (see Lemma 3.6 of [9]). Moreover, τ can have at most one self-invariant orbit contained in $\{\pm 1, \ldots, \pm p\}$, and at most one self-invariant orbit contained in $\{\pm(p+1), \ldots, \pm(p+q)\}$ (this is due to the fundamental fact from [10] that partitions in $NC^{(B)}(p)$ or $NC^{(B)}(q)$ can have at most one zero-block). If τ has two self-invariant orbits, then π is obtained from the orbit partition $\Omega(\tau)$ by joining together these two orbits; otherwise (if τ has at most one self-invariant orbit) we just have $\pi = \Omega(\tau)$.

The case (b) of the discussion was the more complicated one to describe, but one should keep in mind that typically this is the simpler case to handle. Indeed, the case (b) can be summarized as follows: if $\pi \in NC^{(B)}(p,q)$ has connectivity equal to 0, then π is obtained by "putting together" a partition $\pi_{ext} \in NC^{(B)}(p)$ and a partition $\pi_{int} \in NC^{(B)}(q)$, with a special rule for what to do when both π_{ext} and π_{int} have zero-blocks.

Remark 2.6. We conclude this section with a comment on "how to draw pictures" of partitions in $NC^{(B)}(p,q)$. In fact, what one does is to draw (equivalently) pictures of permutations in $\mathcal{S}_{nc}^{(B)}(p,q)$. In order to do this, one starts by representing the elements of $\{\pm 1, \ldots, \pm(p+q)\}$ as points on the boundary of an annulus: on the outside circle of the annulus we mark $2p$ points which we label clockwise as $1, \ldots, p, -1, \ldots, -p$ (in this order), and on the inside circle of the annulus we mark $2q$ points which we label counterclockwise as $p+1, \ldots, p+q, -(p+1), \ldots, -(p+q)$ (in this order). In terms of pictures drawn in this annulus, the fact that a permutation $\tau \in B_{p+q}$ belongs to $\mathcal{S}_{nc}^{(B)}(p,q)$ corresponds then to the following prescription: one can draw a closed contour for each of the cycles of τ, such that

(i) each of the contours does not self-intersect, and goes clockwise around the region it encloses;

(ii) the region enclosed by each of the contours is contained in the annulus;

(iii) regions enclosed by different contours are mutually disjoint.

Some concrete examples of such drawings are given in Figure 1 below.

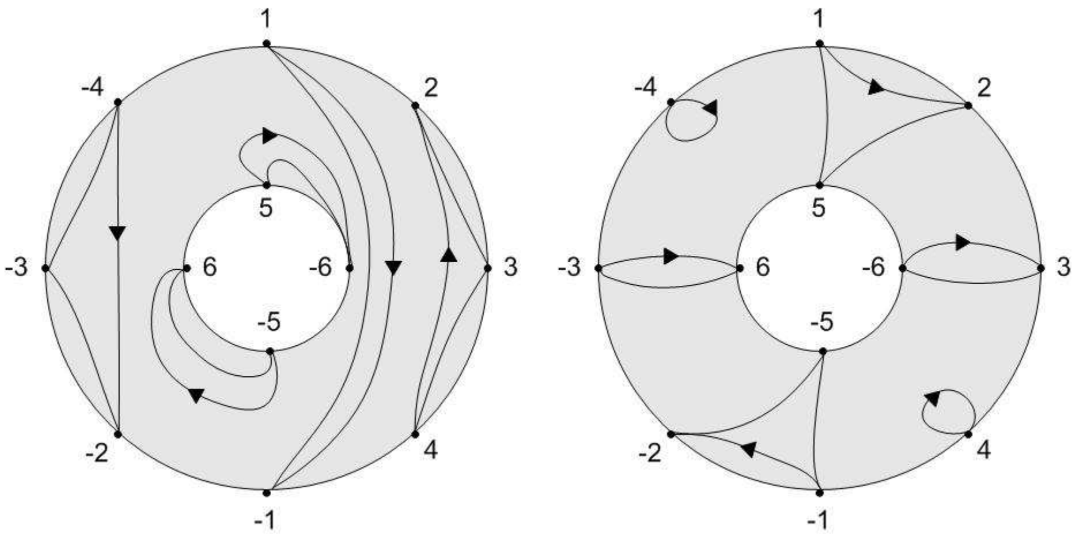

Figure 1. Examples of pictures of permutations in $\mathcal{S}_{nc}^{(B)}(4,2)$.

3 Rank cardinalities and Möbius function for $NC^{(B)}(n-1,1)$

Whereas the posets $NC^{(B)}(p,q)$ aren't lattices in general, it is nevertheless true that $NC^{(B)}(n-1,1)$ is a lattice for every $n \geq 2$; and moreover, the meet operation on $NC^{(B)}(n-1,1)$ coincides with the usual "intersection meet" for partitions – the blocks of the meet $\pi \wedge \rho \in NC^{(B)}(n-1,1)$ are precisely the non-empty intersections $A \cap B$ where A is a block of π and B is a block of ρ. (For a proof of these facts, see Theorem 1.3 of [9].) The present section is devoted to this special "lattice" case, when the formulas for both the rank generating function and the Möbius function are nicer, and can be easily derived from known facts about $NC(n)$ and $NC^{(B)}(n)$.

The rank cardinalities for $NC^{(B)}(n-1,1)$ will be presented in Theorem 3.2. We first record a few known facts that will be used in the proof of this theorem.

Remark 3.1. 1^o We will use the well-known binomial identity

$$\sum_{k=0}^{n-r} \binom{n}{k}\binom{n}{k+r} = \binom{2n}{n-r} \tag{3.1}$$

(holding for any given integers $0 \leq r \leq n$), which is obtained by equating the coefficient of X^{n-r} on the two sides of the polynomial identity $(1+X)^n \cdot (1+X)^n = (1+X)^{2n}$.

2^o We will use the rank generating functions for the posets $NC(n)\Big(= NC^{(A)}(n)\Big)$ and $NC^{(B)}(n)$.

(A) The rank of a partition $\pi \in NC^{(A)}(n)$ is given by the formula

$$\mathrm{rank}(\pi) = n - (\#\text{ of blocks of } \pi)\,.$$

For every $0 \leq k \leq n-1$, we have (see Corollary 4.1 of [8]) that

$$\Big| \Big\{ \pi \in NC^{(A)}(n) \mid \mathrm{rank}(\pi) = k \Big\} \Big| \ = \ \frac{1}{n} \left(\begin{array}{c} n \\ k \end{array} \right) \left(\begin{array}{c} n \\ k+1 \end{array} \right). \tag{3.2}$$

The numbers appearing on the right-hand side of (3.2) are called *Narayana numbers*. The total number of partitions in $NC^{(A)}(n)$ is a Catalan number,

$$\Big| NC^{(A)}(n) \Big| \ = \ \frac{1}{n+1} \binom{2n}{n}. \tag{3.3}$$

(B) The rank of a partition $\pi \in NC^{(B)}(n)$ is given by the formula

$$\mathrm{rank}(\pi) = n - \frac{1}{2} \left(\begin{array}{c} \#\text{ of blocks } A \text{ of } \pi \\ \text{such that } A \neq -A \end{array} \right).$$

For every $0 \leq k \leq n$, we have (see Proposition 6 of [10]) that

$$\Big| \Big\{ \pi \in NC^{(B)}(n) \mid \mathrm{rank}(\pi) = k \Big\} \Big| \ = \ \binom{n}{k}^2. \tag{3.4}$$

The total number of partitions in $NC^{(B)}(n)$ is

$$\Big| NC^{(B)}(n) \Big| \ = \ \binom{2n}{n}. \tag{3.5}$$

3^o We will use a natural "absolute value map" that sends $NC^{(B)}(n)$ to $NC^{(A)}(n)$. We start with the map $\mathrm{Abs} : \{\pm 1, \ldots, \pm n\} \to \{1, \ldots, n\}$ which sends $\pm i$ to i, for every $1 \leq i \leq n$. Note that for every $\pi \in NC^{(B)}(n)$ it makes sense to consider the partition of $\{1, \ldots, n\}$ into blocks of the form $\mathrm{Abs}(B)$, with B a block of π; this partition of $\{1, \ldots, n\}$ will be denoted by "$\mathrm{Abs}(\pi)$". It turns out that $\mathrm{Abs}(\pi) \in NC^{(A)}(n)$ for every $\pi \in NC^{(B)}(n)$, and moreover, that the map

$$NC^{(B)}(n) \ni \pi \mapsto \mathrm{Abs}(\pi) \in NC^{(A)}(n) \tag{3.6}$$

defined in this way is an $(n+1)$-to-1 map (see Section 1.3 of [4]). In the proof of the next theorem we will use the following property (also noticed in Section 1.3 of [4]) of the map Abs from (3.6):

$$\left\{ \begin{array}{l} \text{Given a partition } \pi_o \in NC^{(A)}(n) \text{ and a block } A \text{ of } \pi_o \\ \text{there exists a unique } \pi \in NC^{(B)}(n) \text{ with zero-block } Z \\ \text{such that } \mathrm{Abs}(\pi) = \pi_o \text{ and } \mathrm{Abs}(Z) = A. \end{array} \right. \tag{3.7}$$

Theorem 3.2. *Let $n \geq 2$ be an integer. Then*

$$\Big| NC^{(B)}(n-1,1) \Big| = \binom{2n}{n}, \tag{3.8}$$

and for every $0 \leq k \leq n$ we have that

$$\Big| \Big\{ \pi \in NC^{(B)}(n-1,1) \mid rank(\pi) = k \Big\} \Big| = \binom{n}{k}^2. \tag{3.9}$$

Proof. Equation (3.8) follows from (3.9) by summing over k and by invoking (3.1). Thus it will suffice to verify (3.9). For the whole proof we fix a k for which we will prove that (3.9) holds. We will assume that $k \neq 0$ (the case $k = 0$ is obvious).

From the first inequality (2.10) in Remark 2.4 it is clear that every partition in $NC^{(B)}(n-1,1)$ has connectivity equal to 0 or 1. Let us denote

$$\begin{cases} \mathcal{C} & := \ \{\pi \in NC^{(B)}(n-1,1) \mid \pi \text{ has rank } k \text{ and connectivity } 1\} \\ \mathcal{D} & := \ \{\pi \in NC^{(B)}(n-1,1) \mid \pi \text{ has rank } k \text{ and connectivity } 0\}. \end{cases} \tag{3.10}$$

We note that every partition $\pi \in \mathcal{D}$ must be of the form $\pi = \widetilde{\Omega}(\tau)$, where τ is a permutation in $\mathcal{S}_{nc}^{(B)}(n-1,1)$ which leaves invariant the set $\{n,-n\}$. Clearly, there are only two possibilities for how τ can act on $\{n,-n\}$ – either $\tau(n) = n$ and $\tau(-n) = -n$, or $\tau(n) = -n$ and $\tau(-n) = n$. We will denote by $\mathcal{D}_+$ and respectively by $\mathcal{D}_-$ the set of partitions $\pi \in \mathcal{D}$ for which the first (respectively the second) of these possibilities occurs. We thus have $\mathcal{D} = \mathcal{D}_+ \cup \mathcal{D}_-$, disjoint, and it is clear that

$$\big| \{\pi \in NC^{(B)}(n-1,1) \mid \text{rank}(\pi) = k\} \big| = \big| \mathcal{C} \big| + \big| \mathcal{D}_+ \big| + \big| \mathcal{D}_- \big|. \tag{3.11}$$

We first dispense with the immediate task of counting the partitions in $\mathcal{D}_+$ and in $\mathcal{D}_-$. It is clear that every partition $\pi \in \mathcal{D}_+$ is obtained by taking a partition π_o of rank k from $NC^{(B)}(n-1)$ and by adding to it two singleton blocks $\{n\}$ and $\{-n\}$. This leads to

$$|\ \mathcal{D}_+ \ | = \Big| \ \{\pi_o \in NC^{(B)}(n-1) \mid \text{rank}(\pi_o) = k\} \Big| = \binom{n-1}{k}^2 \quad \text{(by using (3.4))}.$$

On the other hand, every partition $\pi \in \mathcal{D}_-$ is canonically obtained from a partition π_o of rank $k-1$ in $NC^{(B)}(n-1)$: if π_o has no zero-block then we add to it a 2-element block $\{n,-n\}$, while if π_o has a zero-block Z then we replace Z by $Z \cup \{n,-n\}$. We thus get

$$|\ \mathcal{D}_- \ | = \Big| \ \{\pi_o \in NC^{(B)}(n-1) \mid \text{rank}(\pi_o) = k-1\} \Big| = \binom{n-1}{k-1}^2 \quad \text{(by (3.4))}.$$

Let us now count the partitions in the set $\mathcal{C}$ from (3.10). Let π be in $\mathcal{C}$, and let us denote by A the block of π which contains n. We know that $A \neq -A$, and that $A \cap \{\pm 1, \ldots, \pm(n-1)\} \neq \emptyset$. Let π_o be the partition of $\{\pm 1, \ldots, \pm(n-1)\}$ which is obtained from π by taking its blocks A and $-A$ and replacing them with just one block,

$$Z := \Big(A \cup (-A) \Big) \setminus \{n,-n\}.$$

It is immediately seen that $\pi_o \in NC^{(B)}(n-1)$. (In pictorial terms, what one does in order to obtain π_o out of π is to take the union of the regions enclosed by A and by $-A$ with the disc bounded by the inside circle of the annulus; after also removing the labels $\pm n$ from the picture, one remains with a region that corresponds to the block Z of π_o.) From the formulas for rank in $NC^{(B)}(n-1)$ and in $NC^{(B)}(n-1,1)$ it is immediate that the newly created partition $\pi_o \in NC^{(B)}(n-1)$ has rank equal to k.

In the construction $\pi \rightsquigarrow \pi_o$ described in the preceding paragraph, one cannot uniquely recover π from π_o. However, a moment's thought shows that π can be uniquely recovered from the pair $(\pi_o, \tau(n))$, where $\tau \in \mathcal{S}_{nc}^{(B)}(n-1,1)$ is the permutation that corresponds to π. (The number $\tau(n) \in \{\pm 1, \ldots, \pm(n-1)\}$ could simply be described as "the point of A which follows to n", when we move around A in clockwise order.) We thus have a one-to-one map

$$\mathcal{C} \ni \pi \mapsto \Big(\pi_o, \tau(n) \Big) \in \left\{ (\pi_o, m) \;\middle|\; \begin{array}{c} \pi_o \in NC^{(B)}(n-1) \text{ of rank } k \text{ and} \\ \text{with zero-block } Z, \text{ and } m \in Z \end{array} \right\}. \tag{3.12}$$

It is quite easy to see that the map in (3.12) is surjective as well. In pictorial terms: given $\pi_o \in NC^{(B)}(n-1,1)$ with zero-block Z, and given an element $m \in Z$, we always know how to deform the region enclosed by Z so that it becomes a union of three regions – a small disc, and two regions enclosed by blocks of π. (The given element $m \in Z$ determines what side of the region enclosed by Z has to be deformed, and also indicates where on the emerging small disc we should place the labels n and $-n$.)

Let us next observe that by using the "Abs" map and its property reviewed in (3.7) of Remark 3.1.2, we get another bijection

$$(\pi_o, m) \mapsto \Big(\mathrm{Abs}(\pi_o), m \Big) \tag{3.13}$$

which sends the set

$$\left\{ (\pi_o, m) \;\middle|\; \begin{array}{c} \pi_o \in NC^{(B)}(n-1) \text{ of rank } k \text{ and} \\ \text{with zero-block } Z, \text{ and } m \in Z \end{array} \right\}$$

onto the Cartesian product

$$\Big\{ \rho \in NC^{(A)}(n-1) \mid \mathrm{rank}(\rho) = k-1 \Big\} \times \{\pm 1, \ldots, \pm(n-1)\}.$$

By using the bijections (3.12) and (3.13) we thus find that

$$\begin{aligned} |\mathcal{C}| &= \Big| \Big\{ \rho \in NC^{(A)}(n-1) \mid \mathrm{rank}(\rho) = k-1 \Big\} \Big| \cdot 2(n-1) \\ &= \frac{1}{n-1} \binom{n-1}{k-1} \binom{n-1}{k} \cdot 2(n-1) \quad \text{(by (3.2))} \\ &= 2 \binom{n-1}{k-1} \binom{n-1}{k}. \end{aligned}$$

We finally return to (3.11) and substitute on its right-hand side the values found for the cardinalities of $\mathcal{C}, \mathcal{D}_+$ and $\mathcal{D}_-$. We obtain that the number of elements of rank k in $NC^{(B)}(n-1,1)$ is equal to

$$2\binom{n-1}{k-1}\binom{n-1}{k} + \binom{n-1}{k}^2 + \binom{n-1}{k-1}^2 = \left[\binom{n-1}{k} + \binom{n-1}{k-1} \right]^2 = \binom{n}{k}^2,$$

as required. ■

Remark 3.3. We note the somewhat surprising fact that $NC^{(B)}(n-1,1)$ has exactly the same rank generating function as the lattice $NC^{(B)}(n)$. For $n=2$ we have in fact $NC^{(B)}(1,1) = NC^{(B)}(2)$ (equality of sets of partitions of $\{1,2\} \cup \{-1,-2\}$). But already for $n=3$ it is no longer true that $NC^{(B)}(2,1) = NC^{(B)}(3)$; moreover, by comparing the Hasse diagrams of $NC^{(B)}(2,1)$ and of $NC^{(B)}(3)$, one easily sees that $NC^{(B)}(2,1) \not\simeq NC^{(B)}(3)$. (The Hasse diagram for $NC^{(B)}(2,1)$ is drawn in Figure 2 of this paper, and the one for $NC^{(B)}(3)$ appears for instance on page 199 of Reiner's paper [10]. In order to establish that $NC^{(B)}(2,1) \not\simeq NC^{(B)}(3)$ one can for instance count edges in the Hasse diagrams – the Hasse diagram for $NC^{(B)}(2,1)$ has 46 edges, while the one for $NC^{(B)}(3)$ has 44 edges.)

And actually, by comparing the specific formulas which give the Möbius functions for $NC^{(B)}(n)$ and for $NC^{(B)}(n-1,1)$, one sees that in fact $NC^{(B)}(n-1,1) \not\simeq NC^{(B)}(n)$ for all $n \geq 3$; see Remark 3.7 below.

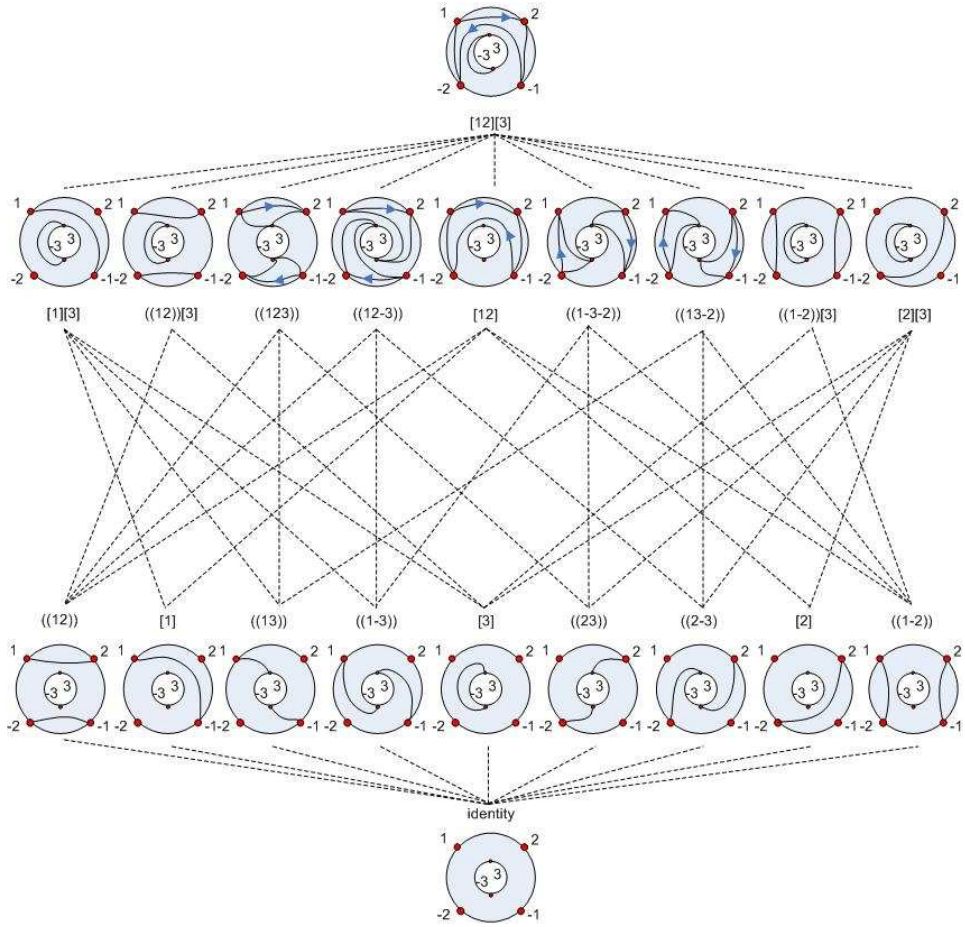

Figure 2. The Hasse diagram for $NC^{(B)}(2,1)$. The bracket notations $((\cdots))$ and $[\cdots]$ refer to the cycles of the corresponding permutations (e.g. $((1,2,-3))$ and $((1,-2))[3]$ are shorthand notations for the permutations $(1,2,-3)(-1,-2,3)$ and $(1,-2)(-1,2)(3,-3)$, respectively).

We now take on the Möbius function of $NC^{(B)}(n-1,1)$. Its calculation will be presented in Theorem 3.6, and will be based on a partial Möbius inversion formula which is described as follows.

Lemma 3.4. *Let P be a finite lattice, let $\widehat{0}$ and $\widehat{1}$ denote the minimal and the maximal element of P, respectively, and let ω be a fixed element of P, where $\omega \neq \widehat{1}$. Then*

$$\sum_{\substack{\pi \in P \\ \pi \wedge \omega = \widehat{0}}} \mu_P(\pi, \widehat{1}\,) = 0. \tag{3.14}$$

For a proof of Lemma 3.4, see Corollary 3.9.3 of [11]. A few other facts needed in the proof of Theorem 3.6 are collected in the next remark.

Remark 3.5. 1^o We will use the explicit formulas known for the Möbius functions of the posets $NC^{(A)}(n)$ and $NC^{(B)}(n)$.

(A) For every $n \geq 1$ we have that

$$\mu_{NC^{(A)}(n)}(\,\widehat{0}, \widehat{1}\,) = (-1)^{n+1} \frac{(2n-2)!}{(n-1)!\; n!}, \tag{3.15}$$

where $\mu_{NC^{(A)}(n)}$ is the Möbius function of $NC^{(A)}(n)$, and $\widehat{0}, \widehat{1}$ are the minimal and respectively the maximal element of $NC^{(A)}(n)$. (See Theorem 6 of [8].)

(B) For every $n \geq 1$ we have that

$$\mu_{NC^{(B)}(n)}(\,\widehat{0}, \widehat{1}\,) = (-1)^{n} \cdot \binom{2n-1}{n}, \tag{3.16}$$

where $\mu_{NC^{(B)}(n)}$ is the Möbius function of $NC^{(B)}(n)$, and $\widehat{0}, \widehat{1}$ now stand for the minimal and respectively the maximal element of $NC^{(B)}(n)$. (See Proposition 7 of [10].)

2^o Let p, q be positive integers. It is an easy exercise (left to the reader) to check that the formula

$$C(\tau) := \tau^{-1}\gamma_{p,q}, \quad \tau \in \mathcal{S}_{nc}^{(B)}(p,q), \tag{3.17}$$

defines a bijection $C : \mathcal{S}_{nc}^{(B)}(p,q) \to \mathcal{S}_{nc}^{(B)}(p,q)$, which is order-reversing – for $\sigma, \tau \in \mathcal{S}_{nc}^{(B)}(p,q)$ one has that $\sigma \leq \tau \Leftrightarrow C(\sigma) \geq C(\tau)$, where the partial order on $\mathcal{S}_{nc}^{(B)}(p,q)$ is as reviewed in Remark 2.2.

Now, by using the canonical isomorphism $\widetilde{\Omega} : \mathcal{S}_{nc}^{(B)}(p,q) \to NC^{(B)}(p,q)$ defined in Remark 2.3, we can transport the map C from (3.17) to an anti-isomorphism $K : NC^{(B)}(p,q) \to NC^{(B)}(p,q)$, defined via the formula

$$K(\,\widetilde{\Omega}(\tau)\,) = \widetilde{\Omega}(\tau^{-1}\gamma_{p,q}), \quad \tau \in \mathcal{S}_{nc}^{(B)}(p,q). \tag{3.18}$$

This anti-isomorphism K is the $NC^{(B)}(p,q)$–analogue for an anti-isomorphism of the lattice $NC^{(A)}(n)$ introduced by Kreweras in [8], and which is commonly called the Kreweras complementation map. Following this trend, we will also refer to the map K from (3.18) by calling it the *Kreweras complementation map* of $NC^{(B)}(p,q)$. Note that, due to the fact that it is an anti-isomorphism, the Kreweras complementation map has the property that

$$\mu(\pi, \rho) = \mu(\,K(\rho), K(\pi)\,), \quad \forall\, \pi, \rho \in NC^{(B)}(p,q) \text{ such that } \pi \leq \rho, \tag{3.19}$$

where μ is the Möbius function of $NC^{(B)}(p,q)$.

Theorem 3.6. *Let $n \geq 2$ be an integer, let $\mu_{NC^{(B)}(n-1,1)}$ be the Möbius function of $NC^{(B)}(n-1,1)$, and let $\widehat{0}$, $\widehat{1}$ be the minimal and respectively maximal elements of $NC^{(B)}(n-1,1)$. Then*

$$\mu_{NC^{(B)}(n-1,1)}(\,\widehat{0},\widehat{1}\,) = (-1)^n \cdot \left(\begin{array}{c} 2n-1 \\ n \end{array} \right) \cdot \frac{5n-4}{4n-2}. \tag{3.20}$$

Proof. Throughout the whole proof we will write for short "μ" instead of "$\mu_{NC^{(B)}(n-1,1)}$".

We will apply Lemma 3.4 to the particular case when $P = NC^{(B)}(n-1,1)$ and

$$\omega := \Big\{ \ \{\pm 1, \ldots, \pm(n-1)\}, \ \{n\}, \ \{-n\} \ \Big\}. \tag{3.21}$$

By taking into account that the meet operation of $NC^{(B)}(n-1,1)$ is just the usual "intersection" meet, one immediately sees that the partitions in the set $\{\, \pi \in NC^{(B)}(n-1,1) \mid \pi \wedge \omega = \widehat{0}\,\}$ can be listed explicitly as $\widehat{0}, \pi_0, \pi_1, \ldots, \pi_{n-1}, \pi_{-1}, \ldots, \pi_{-(n-1)}$, where

$$\pi_0 := \Big\{ \ \{n,-n\}, \ \{1\}, \ \{-1\}, \ldots, \{n-1\}, \ \{-(n-1)\} \ \Big\}$$

and where for every $i \in \{\pm 1, \ldots, \pm(n-1)\}$ we put

$$\pi_i := \Big\{ \ \{i,n\}, \ \{-i,-n\} \ \Big\} \cup \Big\{ \{j\} \ \mid \ j \in \{\pm 1, \ldots, \pm(n-1)\}, \ |j| \neq |i| \Big\}.$$

When applied to this particular situation, Lemma 3.4 thus implies that

$$0 = \mu(\,\widehat{0},\widehat{1}\,) + \mu\big(\pi_0,\widehat{1}\,\big) + \sum_{i=1}^{n-1} \mu\big(\pi_i,\widehat{1}\,\big) + \sum_{i=1}^{n-1} \mu\big(\pi_{-i},\widehat{1}\,\big). \tag{3.22}$$

It is convenient to consider the equivalent restatement of (3.22) which is obtained by taking Kreweras complements and by invoking the formula (3.19) from Remark 3.5.2:

$$0 = \mu(\,\widehat{0},\widehat{1}\,) + \mu\big(\widehat{0},\rho_0\big) + \sum_{i=1}^{n-1} \mu\big(\,\widehat{0},\rho_i\big) + \sum_{i=1}^{n-1} \mu\big(\,\widehat{0},\rho_{-i}\big), \tag{3.23}$$

where we denoted $\rho_i := K(\pi_i)$, for $i \in \{0\} \cup \{\pm 1, \ldots, \pm(n-1)\}$.

Let us now compute explicitly what are the partitions ρ_0 and $\rho_{\pm 1}, \ldots, \rho_{\pm(n-1)}$. We do this by using the corresponding permutations in $\mathcal{S}_{nc}^{(B)}(n-1,1)$, and the formula (3.18) from Remark 3.5.2. For $i \in \{\pm 1, \ldots, \pm(n-1)\}$ we write $\pi = \widetilde{\Omega}(\tau_i)$ with $\tau_i = (i,n)(-i,-n) \in B_n$, and we compute

$$\begin{aligned} \tau_i^{-1}\gamma_{n-1,1} &= \Big((i,n)(-i,-n) \Big) \Big((1, \ldots, n-1, -1, \ldots, -(n-1))(n,-n) \Big) \\ &= \Big((1, \ldots, i-1, n, -i, \ldots, -(n-1) \Big) \Big((-1, \ldots, -(i-1), -n, i, \ldots, n-1 \Big). \end{aligned}$$

Since $\rho_i = K(\,\widetilde{\Omega}(\tau_i)\,) = \widetilde{\Omega}(\,\tau_i^{-1}\gamma_{n-1,1}\,)$, we thus obtain that

$$\rho_i = \Big\{ \{1, \ldots, i-1, n, -i, \ldots, -(n-1)\}, \ \{-1, \ldots, -(i-1), -n, i, \ldots, n-1\} \Big\}. \tag{3.24}$$

For ρ_0 one does a similar calculation, by writing $\pi_0 = \widetilde{\Omega}(\tau_0)$ for $\tau_0 = (n, -n) \in B_n$. The reader should have no difficulty to check that this calculation simply leads to the equality $\rho_0 = \omega$, with ω taken from (3.21).

From the explicit form found in (3.24) for ρ_i with $i \in \{\pm 1, \ldots, \pm(n-1)\}$, one easily infers that the interval $[\widehat{0}, \rho_i]$ of $NC^{(B)}(n-1,1)$ is poset isomorphic with the lattice $NC^{(A)}(n)$. Indeed, the process of constructing a partition $\sigma \in NC^{(B)}(n-1,1)$ such that $\sigma \leq \rho_i$ amounts precisely to breaking in a non-crossing way the block $\{1, \ldots, i-1, n, -i, \ldots, -(n-1)\}$ of ρ_i, where the cyclic order of the n elements of the block is as listed above. (This must be of course matched by the corresponding, uniquely determined, non-crossing breaking of the other block $\{-1, \ldots, -(i-1), -n, i, \ldots, n-1\}$ of ρ_i.) The isomorphism $[\widehat{0}, \rho_i] \simeq NC^{(A)}(n)$ and (3.15) thus give us that

$$\mu(\widehat{0}, \rho_i) = (-1)^{n+1} \frac{(2n-2)!}{(n-1)!\; n!}.$$

In a similar way, one finds that the interval $[\widehat{0}, \rho_0]$ of $NC^{(B)}(n-1,1)$ is isomorphic with $NC^{(B)}(n-1)$, and hence that (by (3.16)) we have

$$\mu\big(\widehat{0}, \rho_0\big) = (-1)^{n-1} \binom{2n-3}{n-1}.$$

Finally, by substituting in (3.23) the concrete values obtained above for the $\mu(\widehat{0}, \rho_i)$, we find that

$$-\mu(\widehat{0}, \widehat{1}\,) = (-1)^{n-1} \binom{2n-3}{n-1} + (2n-2) \cdot (-1)^{n-1} \cdot \frac{(2n-2)!}{(n-1)!\; n!},$$

and the required formula for $\mu(\widehat{0}, \widehat{1}\,)$ follows by straightforward calculation. ∎

Remark 3.7. By comparing the formula (3.20) found in Theorem 3.6 against the corresponding formula (3.16) which holds for $NC^{(B)}(n)$, we see that $\mu_{NC^{(B)}(n)}(\widehat{0}, \widehat{1}\,)$ is different from $\mu_{NC^{(B)}(n-1,1)}(\widehat{0}, \widehat{1}\,)$ for all $n \geq 3$. This implies, of course, that $NC^{(B)}(n-1,1) \not\simeq NC^{(B)}(n)$ for $n \geq 3$.

4 Rank generating function for $NC^{(B)}(p,q)$

In this section, we determine the rank generating function for $NC^{(B)}(p,q)$. Our results follow directly from a bijection, in Proposition 4.2 below, which is similar to Lemma 2.1 of [7] and Proposition 6 of [10]. As a preliminary, we have the following discussion of strings of parentheses.

Remark 4.1. We let $\{(,)\}^*$ be the set of strings of left parentheses "(" and right parentheses ")". With multiplication given by concatenation, this set forms a monoid, with the empty string acting as identity element.

If $s = s_1 \ldots s_n \in \{(,)\}^*$, $n \geq 1$, then the nontrivial left-substrings of s are given by $u_i := s_1 \ldots s_i$, $i = 1, \ldots, n$. If all nontrivial left-substrings of s have (strictly) more left parentheses than right parentheses, then we will say that s is *legal from the left*.

For $s = s_1 \ldots s_n \in \{(,)\}^*$, $n \geq 1$, the *cyclic shifts* of s are the n strings

$$s^{(1)} = s_2 \cdots s_n s_1,\ s^{(2)} = s_3 \cdots s_n s_1 s_2,\ \ldots,\ s^{(n-1)} = s_n s_1 s_2 \cdots s_{n-1},\ s^{(n)} = s.$$

Suppose that s has m more left parentheses than right parentheses, for some $m \geq 1$. Then the well-known Cycle Lemma (see for instance the discussion on page 67 of [12]) says that exactly m of the cyclic shifts of s are legal from the left.

For example, if s is the string $()(()(($ which has 5 left parentheses and 2 right parentheses, then the 3 cyclic shifts of s that are legal from the left are

$$s^{(2)} = (()((((),\quad s^{(5)} = ((()(((),\quad s^{(6)} = (()(()(\,.$$

Symmetrically, if all nontrivial right-substrings of s have more right parentheses than left parentheses, then we say that s is *legal from the right*. For this case, suppose that s has m more right parentheses than left parentheses, for some $m \geq 1$. Then the Cycle Lemma says that exactly m of the cyclic shifts of s are legal from the right.

Proposition 4.2. *Let p, q be positive integers. Suppose that c, e, i are integers satisfying the inequalities stated in* (2.10) *of Remark 2.4, that is:* $1 \leq c \leq \min\{p, q\}$ *and* $0 \leq e \leq p-c$, $0 \leq i \leq q-c$. *Then there exists a bijection between the set*

$$\left\{ (d, L^E, R^E, L^I, R^I) \;\middle|\; \begin{array}{l} 1 \leq d \leq 2c \\ L^E, R^E \subseteq \{1, \ldots, p\},\ |L^E| = e + c,\ |R^E| = e, \\ L^I, R^I \subseteq \{p+1, \ldots, p+q\},\ |L^I| = i,\ |R^I| = i + c \end{array} \right\} \tag{4.1}$$

and the set of partitions in $NC^{(B)}(p, q)$ which have connectivity equal to c, have e exterior pairs of blocks, and have i interior pairs of blocks.

Proof. We will describe explicitly the constructions for two maps $(d, L^E, R^E, L^I, R^I) \mapsto \pi$ and $\pi \mapsto (d, L^E, R^E, L^I, R^I)$, and we will leave it as an exercise to the reader to check that these two maps are inverse to each other (thus giving together a bijection as stated). We recommend that the general descriptions given below for the two maps are read in parallel with Remark 4.3, which illustrates how the maps work on a concrete example.

A. Description of the map $(d, L^E, R^E, L^I, R^I) \mapsto \pi$. Given (d, L^E, R^E, L^I, R^I) as in (4.1), insert left and right parentheses into the string

$$1, \ldots, p, -1, \ldots, -p$$

by placing a left (respectively right) parenthesis before (respectively after) each occurrence of j and $-j$, for each value j in L^E (respectively R^E). In this way we obtain the string u of length $2(p + 2e + c)$, consisting of numbers and parentheses. In u, there are $2c$ more left parentheses than right parentheses, so the Cycle Lemma in Remark 4.1 implies that there are $2c$ cyclic shifts of u beginning with a left parenthesis such that the subsequence consisting of parentheses only is legal from the left. Suppose that these $2c$ cyclic shifts are given by $u^{(i_1)}, \ldots, u^{(i_{2c})}$, ordered so that $i_1 < \cdots < i_{2c}$. Then let $t_1 = u^{(i_d)}$.

Similarly, insert left and right parentheses into the string

$$p+1, \ldots, p+q, -(p+1), \ldots, -(p+q)$$

by placing a left (respectively right) parenthesis before (respectively after) each occurrence of j and $-j$, for each value j in L^I (respectively R^I), to obtain the string v of numbers and parentheses. In v there are $2c$ more right parentheses than left parentheses, so the Cycle Lemma in Remark 4.1 implies that there are $2c$ cyclic shifts of v ending with a right parenthesis such that the subsequence consisting of parentheses only is legal from the right. If these $2c$ cyclic shifts are given by $v^{(j_1)}, \ldots, v^{(j_{2c})}$, ordered so that $j_1 < \cdots < j_{2c}$, then let $t_2 = v^{(j_1)}$.

Now consider the concatenation $t_1 t_2$ of the two strings t_1 and t_2 found above. From the string $t_1 t_2$ we read off a unique partition π in $NC^{(B)}(p, q)$ in the following way: the numbers inside a lowest-level pair of matching parentheses form a block of π; remove these numbers and this pair of parentheses from the string, and iterate until the string is empty. (See part A of Remark 4.3 below, for a concrete example of how this works.)

B. Description of the map $\pi \mapsto (d, L^E, R^E, L^I, R^I)$. Let π be a partition in $NC^{(B)}(p, q)$ which has connectivity equal to c, has e pairs of external blocks and has i pairs of internal blocks. A significant fact we we will use here is that (even though π is drawn on a circular picture) every block of π that is either an external block or an internal block comes with a canonical total order on it, and thus has a *first element* and a *last element*.

Indeed, say for instance that A is an external (i.e. such that $A \subseteq \{\pm 1, \ldots, \pm p\}$) block of π. Let us choose an element $i \in -A$ and, by starting from this i, let us travel around the external circle of the annulus (in the sense that we always use for this circle – that is, clockwisely). When we do this, we encounter the elements of A in a certain order, and this order does not depend on our choice of the starting point $i \in -A$. (The latter fact is an immediate consequence of the fact that the blocks A and $-A$ of π do not cross.) We thus end with a "canonical" total order for the elements of A. Clearly, a similar argument can be made when we deal with an internal block of π. And moreover, this very same argument can be also used to introduce a total order on each of the sets $A \cap \{\pm 1, \ldots, \pm p\}$ and $A \cap \{\pm(p+1), \ldots, \pm(p+q)\}$, in the case when A is a connecting block of π.

So then, starting from the given $\pi \in NC^{(B)}(p, q)$, let us draw some parentheses on the picture representing π, according to the following recipe:

(a) For every block A of π which is either an external block or an internal block, we draw a left parenthesis immediately before the first element of A, and a right parenthesis immediately after the last element of A.

(b) For every connecting block A of π we draw a left parenthesis immediately before the first element of $A \cap \{\pm 1, \ldots, \pm p\}$, and a right parenthesis immediately after the last element of $A \cap \{\pm(p+1), \ldots, \pm(p+q)\}$.

But now, if the parentheses added to the picture of π are read starting from 1 on the outside circle and starting from $p+1$ on the inside circle, then one gets two strings of numbers and parentheses u and v, which are exactly of the same kind as the strings denoted by "u" and "v" in part A of the proof. Furthermore, it is immediate that the strings u and v obtained here correspond to some subsets

$$L^E, R^E \subseteq \{1, \ldots, p\}, \quad L^I, R^I \subseteq \{p+1, \ldots, p+q\}$$

which have the properties required in (4.1).

In order to complete the description of the map $\pi \mapsto (d, L^E, R^E, L^I, R^I)$, we are thus left to explain how we obtain the number $d \in \{1, \ldots, 2c\}$. It is immediate that determining d (in the context where we know already what are the strings u and v) is equivalent to choosing one of the $2c$ cyclic shifts of u which are legal from the left. Expressed directly in terms of the partition π, this in turn amounts to choosing one of the $2c$ connecting blocks of π. (To be precise: if a connecting block A is chosen, then we pick the cyclic shift of u which starts with the left parenthesis placed immediately before the first element of $A \cap \{\pm 1, \ldots, \pm p\}$.) So what we have to do is indicate a procedure for how to canonically select a connecting block of π. The procedure goes by looking at how the connecting blocks intersect the interior circle of the annulus: we start from $p+1$ and move counterclockwisely around the interior circle, and we stop the first time when we meet an element belonging to a connecting block. (See part B of Remark 4.3 below for a concrete example of how this works.) ■

Remark 4.3. Let us illustrate how the two maps described in the proof of the preceding proposition work on a concrete example. Consider the situation when the integers p, q, c, e, i given in Proposition 4.2 are $p = 5$, $q = 3$, $c = 1$, $e = 2$, $i = 1$.

A. Let us determine explicitly the partition $\pi \in NC^{(B)}(5, 3)$ that corresponds (by the first of the two maps described in the proof of Proposition 4.2) to the tuple (d, L^E, R^E, L^I, R^I) where

$$d = 2,\ L^E = \{2, 4, 5\},\ R^E = \{1, 2\},\ L^I = \{7\},\ R^I = \{6, 7\}. \tag{4.2}$$

By inserting parentheses in $1, \ldots, 5, -1, \ldots, -5$ we obtain the following string of length 20, consisting of numbers and parentheses:

$$u = 1\,)\,(\,2\,)\,3\,(\,4\,(\,5\ -1\,)\,(\,-2\,)\ -3\,(\,-4\,(\,-5. \tag{4.3}$$

The two cyclic shifts of u that begin with a left parenthesis and are legal from the left are $u^{(6)}$ and $u^{(16)}$. Since we have $d = 2$, the string t_1 from the description of the above bijection is hence:

$$t_1 = u^{(16)} = (\,-4\,(\,-5\,1\,)\,(\,2\,)\,3\,(\,4\,(\,5\ -1\,)\,(\,-2\,)\ -3.$$

In a similar way, by inserting parentheses in $6, 7, 8, -6, -7, -8$ we get

$$v = 6\,)\,(\,7\,)\,8\ -6\,)\,(\,-7\,)\ -8 \tag{4.4}$$

and then

$$t_2 = v^{(2)} = (\,7\,)\,8\ -6\,)\,(\,-7\,)\ -8\,6\ ).$$

Finally, we concatenate t_1 and t_2, and from the string $t_1 t_2$ we read off the desired partition $\pi \in NC^{(B)}(5, 3)$, which is

$$\pi = \Big\{ \{1, -5\}, \{-1, 5\}, \{2\}, \{-2\}, \{3, -4, 6, -8\}, \{-3, 4, -6, 8\}, \{7\}, \{-7\} \Big\}. \tag{4.5}$$

B. Conversely, let us now start from the partition $\pi \in NC^{(B)}(5, 3)$ that appeared in (4.5) above, and let us determine explicitly the tuple (d, L^E, R^E, L^I, R^I) that corresponds to π by the second map described in the proof of Proposition 4.2.

The annular picture for π and the parentheses that have to be added to it are shown in Figure 3 below. When looking at Figure 3, the reader should keep in mind that the placing of a parenthesis "immediately before" (or "imediately after") a labelled point on one of the circles of the annulus must be always done in agreement with the chosen running direction on that particular circle. Thus for instance the parenthesis sitting next to -5 in Figure 3 is a "left parenthesis placed immediately before -5", since the outside circle is run clockwisely; while next to 6 we have a "right parenthesis placed immediately after 6", as the running direction on the inside circle is counterclockwise.

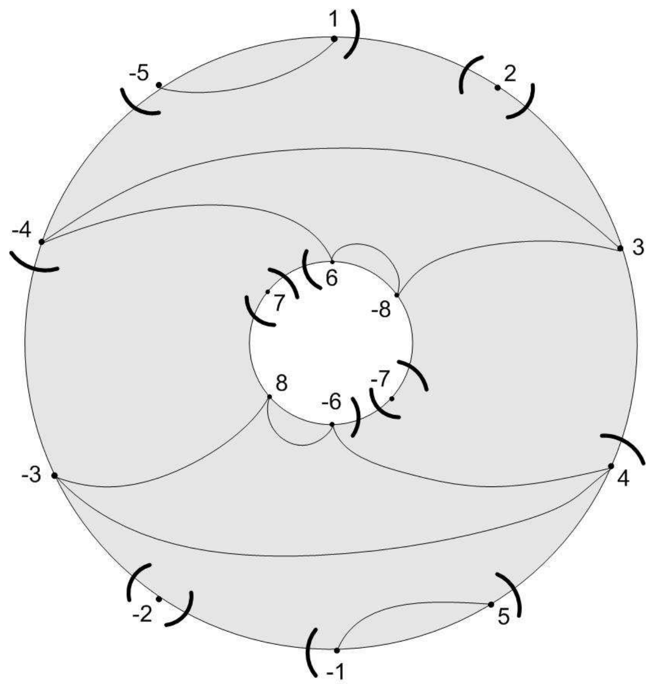

Figure 3. Adding parentheses to the picture of a partition in $NC^{(B)}(p,q)$.

If we read Figure 3 starting with 1 on the outside circle and staring with 6 on the inside circle, we find the strings u and v displayed in (4.3) and respectively (4.4), and from these u and v we clearly get back to the sets L^E, R^E, L^I, R^I indicated in (4.2).

Finally, let us also follow on Figure 3 the procedure for finding the value of d. What we have to do is start from $p+1(=6)$ and move counterclockwisely around the interior circle of the annulus, and stop the first time when we meet an element belonging to a connecting block. But in this example the number 6 belongs itself to the connecting block $A = \{3, -4, 6, -8\}$ of π; so this is the connecting block of π that is chosen. The first element of $A \cap \{\pm 1, \ldots, \pm p\}$ is -4, hence we choose the cyclic shift of u which starts with "(-4", and this corresponds to the value $d = 2$.

Corollary 4.4. *Let p, q, c, e, i be integers such that $1 \leq c \leq \min\{p, q\}$ and such that $0 \leq e \leq p - c$, $0 \leq i \leq q - c$. Then there are exactly*

$$2c \binom{p}{e} \binom{p}{e+c} \binom{q}{i} \binom{q}{i+c} \tag{4.6}$$

partitions in $NC^{(B)}(p,q)$ which have connectivity equal to c, have e exterior pairs of blocks, and have i interior pairs of blocks.

Proof. This follows by taking cardinalities in the bijection from Proposition 4.2. ■

As an immediate consequence of the above corollary, one can enumerate the partitions in $NC^{(B)}(p,q)$ by their connectivity.

Theorem 4.5. *Let p, q be positive integers.*

1^o *For every $1 \leq c \leq \min\{p,q\}$, there are exactly*

$$2c\binom{2p}{p-c}\binom{2q}{q-c} \tag{4.7}$$

partitions in $NC^{(B)}(p,q)$ which have connectivity equal to c.

2^o *There are exactly*

$$\binom{2p}{p}\binom{2q}{q} \tag{4.8}$$

partitions in $NC^{(B)}(p,q)$ which have connectivity equal to 0.

3^o *The total number of partitions in $NC^{(B)}(p,q)$ is*

$$\left|NC^{(B)}(p,q)\right| = \frac{p+q+pq}{p+q}\cdot\binom{2p}{p}\binom{2q}{q}. \tag{4.9}$$

Proof. 1^o From Proposition 4.2, the number of partitions of connectivity c in $NC^{(B)}(p,q)$ equals

$$\begin{aligned} 2c\sum_{e,i\geq 0}\binom{p}{e}\binom{p}{e+c}\binom{q}{i}\binom{q}{i+c} &= 2c\Big(\sum_{e=0}^{p-c}\binom{p}{e}\binom{p}{e+c}\Big)\Big(\sum_{i=0}^{q-c}\binom{q}{i}\binom{q}{i+c}\Big) \\ &= 2c\binom{2p}{p-c}\binom{2q}{q-c} \end{aligned}$$

(where at the second equality sign we used the identity (3.1)).

2^o As observed in Remark 2.5, the partitions with connectivity 0 in $NC^{(B)}(p,q)$ are given by the direct product of $NC^{(B)}(p)$ with $NC^{(B)}(q)$; hence their number is

$$\left|NC^{(B)}(p)\right|\cdot\left|NC^{(B)}(q)\right| = \binom{2p}{p}\cdot\binom{2q}{q} \quad \text{(by using (3.5)).}$$

3^o From the above it follows that

$$\left|NC^{(B)}(p,q)\right| = \binom{2p}{p}\binom{2q}{q} + \sum_{c\geq 1} 2c\binom{2p}{p-c}\binom{2q}{q-c}. \tag{4.10}$$

In the summation over c that has just appeared, we observe that the ratio of two consecutive terms is a rational fraction of c, hence we are dealing with a hypergeometric series. Referring to the standard notations for hypergeometric series one sees, more precisely, that

$$\sum_{c\geq 1} 2c\binom{2p}{p-c}\binom{2q}{q-c} = 2\binom{2p}{p-1}\binom{2q}{q-1}\cdot {}_3F_2\left(\begin{matrix}2,\ -(p-1),\ -(q-1)\\ p+2,\ q+2\end{matrix};1\right). \tag{4.11}$$

(For the precise definition of ${}_3F_2$ see for instance formula (2.1.2) on page 62 of [1].)

Now, it turns out that the special ${}_3F_2$ series on the right-hand side of (4.11) can be summed in closed form; this is by a theorem of Dixon (see formula (2.2.11) on page 72 of [1]), which gives us that

$$ {}_3F_2\left(\begin{matrix} 2, \ -(p-1), \ -(q-1) \\ p+2, \ q+2 \end{matrix} ; 1 \right) = \frac{(p+1)(q+1)}{2(p+q)}. \tag{4.12} $$

By substituting (4.12) into (4.11), and then by plugging the result into (4.10), we obtain the stated formula for the cardinality of $NC^{(B)}(p,q)$. ■

From Corollary 4.4 one can also infer a formula for the rank generating function of $NC^{(B)}(p,q)$.

Theorem 4.6. *Let p, q be positive integers and let $F(x)$ denote the rank generating function for $NC^{(B)}(p,q)$. Then*

$$ F(x) = \sum_{i,j\geq 0} \binom{p}{i}^2 \binom{q}{j}^2 x^{i+j} \ + \sum_{c\geq 1} \sum_{e,i\geq 0} 2c \binom{p}{e}\binom{p}{e+c}\binom{q}{i}\binom{q}{i+c} x^{p+q-e-i-c}. \tag{4.13} $$

Proof. The first summation on the right-hand side of (4.13) gives the contribution to $F(x)$ from partitions $\pi \in NC^{(B)}(p,q)$ which have connectivity equal to 0. Indeed, we saw in Remark 2.5 how such a partition π is obtained by putting together a partition $\pi_{ext} \in NC^{(B)}(p)$ and a partition $\pi_{int} \in NC^{(B)}(q)$; it is moreover immediate that when this is done, the rank of π in $NC^{(B)}(p,q)$ is sum of the ranks of π_{ext} and π_{int} in $NC^{(B)}(p)$ and in $NC^{(B)}(q)$, respectively. Thus when summing over partitions $\pi \in NC^{(B)}(p,q)$ with connectivity equal to 0 we get

$$ \begin{aligned} \sum_{\pi} x^{\mathrm{rank}(\pi)} &= \Big(\sum_{\pi_{ext}\in NC^{(B)}(p)} x^{\mathrm{rank}(\pi_{ext})} \Big) \Big(\sum_{\pi_{int}\in NC^{(B)}(q)} x^{\mathrm{rank}(\pi_{int})} \Big) \\ &= \Big(\sum_{i=0}^{p} \binom{p}{i}^2 x^i \Big) \Big(\sum_{j=0}^{q} \binom{q}{j}^2 x^j \Big) \quad \text{(by (3.4)).} \end{aligned} $$

On the other hand, let us observe that if $\pi \in NC^{(B)}(p,q)$ has connectivity $c \geq 1$, has e pairs of exterior blocks and has i pairs of internal blocks, then from (2.7) it follows that

$$ \mathrm{rank}(\pi) = (p+q) - (c+e+i). $$

Hence in view of Corollary 4.4, the contribution to $F(x)$ of the partitions $\pi \in NC^{(B)}(p,q)$ which have connectivity different from 0 is given precisely by the second summation on the right-hand side of (4.13). ■

Remark 4.7. It can be shown that the second summation on the right-hand side of (4.13) can be reexpressed with only two summation indices instead of three, in the form:

$$ \frac{2pq}{p+q} \sum_{i,j\geq 1} \left(\binom{p}{i}\binom{q}{j-1} + \binom{p}{i-1}\binom{q}{j} \right) \binom{p-1}{i-1}\binom{q-1}{j-1} x^{i+j-1}. \tag{4.14} $$

The proof of this fact is technical, and is omitted.

5 Zeta polynomial and Möbius function for $NC^{(B)}(p,q)$

In this section, we determine the zeta polynomial and Möbius function for $NC^{(B)}(p,q)$. These follow immediately by extending the bijection given in Proposition 4.2 to count multichains in $NC^{(B)}(p,q)$, similar to Theorem 3.2 of [7] and Proposition 7 of [10].

Proposition 5.1. *For $p,q \geq 1$, $m \geq 2$ and $c \geq 1$, the bijection given in Proposition 4.2 extends to a bijection between*

$$\left\{ \begin{array}{c|c} & 1 \leq d \leq 2c \\ & \\ (d; L^E, R_1^E, \quad \ldots, \quad R_{m-1}^E; & L^E, R_1^E, \ldots, R_{m-1}^E \subseteq \{1,\ldots,p\} \\ \qquad\qquad L^I, \quad R_1^I, \ldots, R_{m-1}^I) & |L^E| = |R_1^E| + \cdots + |R_{m-1}^E| + c \\ & \\ & L^I, R_1^I, \ldots, R_{m-1}^I \subseteq \{p+1,\ldots,p+q\} \\ & |L^I| = |R_1^I| + \cdots + |R_{m-1}^I| - c \end{array} \right\} \tag{5.1}$$

and the set of multichains $\pi_1 \leq \cdots \leq \pi_{m-1}$ in $NC^{(B)}(p,q)$, in which π_{m-1} has connectivity c. In this bijection, we have

$$\operatorname{rank}(\pi_i) = p + q - \left(|R_i^E| + \cdots + |R_{m-1}^E| + |R_i^I| + \cdots + |R_{m-1}^I|\right), \quad 1 \leq i \leq m-1. \tag{5.2}$$

Proof. This proof is to a good extent a repetition of the one shown earlier for Proposition 4.2 (which corresponds to the case $m = 2$ of the present proposition). Because of this, we will only give an outline of the argument, with emphasis on the points that are specific to the situation at hand.

Given a tuple $(d; L^E, R_1^E, \ldots, R_{m-1}^E; L^I, R_1^I, \ldots, R_{m-1}^I)$ as in (5.1), insert left and right parentheses into the string

$$1, \ldots, p, -1, \ldots, -p,$$

with $m-1$ types of right parentheses $)^k$ for $k = 1, \ldots, m-1$, as follows: place a left parenthesis before each occurrence of j and $-j$, for each value j in L^E; for $k = 1, \ldots, m-1$, place a right parenthesis of type $)^k$ after each occurrence of j and $-j$, for each value j in R_k^E. (If j occurs in both R_a^E and R_b^E, for $a < b$, then place the corresponding $)^b$ to the right of the $)^a$.) In the resulting string of numbers and parentheses there are $2c$ more left parentheses than right parentheses, so the Cycle Lemma in Remark 4.1 implies that there are $2c$ cyclic shifts of the string beginning with a left parenthesis such that the subsequence consisting of parentheses only is legal from the left. Order these $2c$ cyclic shifts in the canonical way (by the same method as in the proof of Proposition 4.2), and choose the dth them to give the string t_1.

Similarly, insert left and right parentheses into the string

$$p+1, \ldots, p+q, -(p+1), \ldots, -(p+q)$$

by placing a left parenthesis before each occurrence of j and $-j$, for each value j in L^I; for $k = 1, \ldots, m-1$, place a right parenthesis of type $)^k$ after each occurrence of j and $-j$, for each value j in R_k^I. In the resulting string of numbers and parentheses there are $2c$

more right parentheses than left parentheses, so the Cycle Lemma in Remark 4.1 implies that there are $2c$ cyclic shifts of the sequence ending with a right parenthesis such that the subsequence consisting of parentheses only is legal from the right. Let t_2 be the canonical choice (found in the same way as in the proof of Proposition 4.2) from among these $2c$ cyclic shifts.

Now, from the string t_1t_2, we create a partition π_1 in $NC^{(B)}(p,q)$ in the following way: the numbers inside a lowest-level pair of matching parentheses form a block of π_1; remove these numbers and this pair of parentheses from the string, and iterate until the string is empty. Then for $j=2,\dots,m-1$, remove the right parentheses of type $)^1,\dots,)^{j-1}$ from t_1t_2, together with the left parentheses that pair with them, and read the remaining string as above to obtain π_j. This produces the multichain $\pi_1\le\dots\le\pi_{m-1}$ in $NC^{(B)}(p,q)$, and gives a bijection with the required properties. ■

Remark 5.2. As a concrete example for how Proposition 5.1 works, suppose we have $p=6, q=3, m=3, c=2$, with $d=1$, $L^E=\{1,2,3,5,6\}$, $R_1^E=\{1,3\}$, $R_2^E=\{3\}$, and $L^I=\{8,9\}$, $R_1^I=\{7,8,9\}$, $R_2^I=\{7\}$. By inserting parentheses in $1,\dots,6,-1,\dots,-6$ we obtain the string

$$(1)^1(2(3)^1)^2 4(5(6(-1)^1(-2(-3)^1)^2-4(-5(-6,$$

which has 4 cyclic shifts that we might consider. Since we are given that $d=1$, the cyclic shift that we select is the one that begins with "(5", thus getting

$$t_1=(5(6(-1)^1(-2(-3)^1)^2-4(-5(-6(1)^1(2(3)^1)^2 4.$$

Similarly, we obtain

$$t_2=(8)^1(9)^1-7)^1)^2(-8)^1(-9)^1 7)^1)^2,$$

and from the string t_1t_2, we obtain the partitions

$$\pi_1=\Big\{\{4,-6,-7\},\{-4,6,7\}\Big\}\cup\Big\{\{i\}\ \Big|\ 1\le|i|\le 9,\ |i|\ne 4,6,7\Big\}$$

$$\pi_2=\Big\{\{1,4,-5,-6,-7,8,9\},\{-1,-4,5,6,7,-8,-9\},\{2,3\},\{-2,-3\}\Big\}.$$

Note that $\pi_1\le\pi_2$, and that π_2 has connectivity $c=2$, as claimed.

As an immediate enumerative consequence of Proposition 5.1, we obtain the zeta polynomial for $NC^{(B)}(p,q)$.

Theorem 5.3. *Let p,q be positive integers.*

1^o *The zeta polynomial of $NC^{(B)}(p,q)$ is given by the formula:*

$$Z_{NC^{(B)}(p,q)}(m)=\binom{mp}{p}\binom{mq}{q}+\sum_{c=1}^{p}2c\binom{mp}{p-c}\binom{mq}{q+c}. \tag{5.3}$$

2^o *The number of maximal chains in $NC^{(B)}(p,q)$ is equal to*

$$\binom{p+q}{p}p^pq^q+\sum_{c\ge1}2c\binom{p+q}{p-c}p^{p-c}q^{q+c}. \tag{5.4}$$

Proof. 1^o The zeta polynomial Z_P of a partially ordered set P is defined via the condition that for every $m \geq 2$, the value $Z_P(m)$ is equal to the number of multichains $\pi_1 \leq \cdots \leq \pi_{m-1}$ in P (see Section 3.11 of [11]). From Proposition 5.1, the number of such multichains in which π_{m-1} has connectivity $c \geq 1$ is given by

$$2c \sum_{\substack{a_j, b_j \geq 0, \\ j=0,\ldots,m-1}} \binom{p}{a_1 + \ldots + a_{m-1} + c} \binom{q}{b_1 + \ldots + b_{m-1} - c} \prod_{j=1}^{m-1} \binom{p}{a_j} \binom{q}{b_j}. \tag{5.5}$$

The number from (5.5) can be written in a simpler form if one invokes a well-known multinomial formula (which incidentally is a generalization of the binomial identity (3.1) used in the preceding sections), stating that for any integers $A_1, \ldots, A_k, A_{k+1} \geq 0$ and $0 \leq b \leq A_{k+1}$ we have

$$\sum_{a_1,\ldots,a_k \geq 0} \binom{A_1}{a_1} \cdots \binom{A_k}{a_k} \binom{A_{k+1}}{a_1 + \cdots + a_k + b} = \binom{A_1 + \cdots + A_{k+1}}{A_{k+1} - b}. \tag{5.6}$$

By applying (5.6) to (5.5), we find that the quantity in (5.5) is equal to just

$$2c \binom{mp}{p-c} \binom{mq}{q+c}. \tag{5.7}$$

On the other hand, we also have a simple formula for multichains $\pi_1 \leq \cdots \leq \pi_{m-1}$ where π_{m-1} has connectivity equal to 0. Indeed, these are simply multichains in the direct product of $NC^{(B)}(p)$ with $NC^{(B)}(q)$, and thus the number of these is given by

$$Z_{NC^{(B)}(p)}(m) \cdot Z_{NC^{(B)}(q)}(m) = \binom{mp}{p} \binom{mq}{q}, \tag{5.8}$$

from Proposition 7 of [10].

The expression for $Z_{NC^{(B)}(p,q)}(m)$ now follows by summing over $c \geq 0$, and by taking (5.7) and (5.8) into account.

2^o For any partially ordered set P, the number of maximal chains is given by $d!$ times the coefficient of m^d in $Z_P(m)$, where the zeta polynomial $Z_P(m)$ has degree d (see Proposition 3.11.1(a) of [11]). By part 1^o of the theorem, here we have $d = p + q$, and the result follows from the expression for $Z_{NC^{(B)}(p,q)}(m)$ that was obtained above. ■

Corollary 5.4. *For $p, q \geq 1$, $NC^{(B)}(p,q)$ has Möbius function*

$$\mu_{NC^{(B)}(p,q)}(\widehat{0}, \widehat{1}) = (-1)^{p+q} \left(\binom{2p-1}{p} \binom{2q-1}{q} + \sum_{c=1}^{p} 2c \binom{2p-c-1}{p-1} \binom{2q+c-1}{q-1} \right).$$

Proof. This follows immediately from Theorem 5.3, using the fact that for any partially ordered set P one has $\mu_P(\widehat{0}, \widehat{1}) = Z_P(-1)$ (see Proposition 3.11.1(c) of [11]). ■

Remark 5.5. It is straightforward to specialize Corollary 5.4 to the case $p = n-1$, $q = 1$, either by directly evaluating the summation or by setting $p = 1$, $q = n-1$ and using the symmetry between p and q. Using either of these means, we obtain the expression given in

Theorem 3.6. Note further that we can specialize Theorem 5.3 itself in the same way, to obtain that for every $n \geq 2$, the zeta polynomial of $NC^{(B)}(n-1,1)$ is given by the formula

$$Z_{NC^{(B)}(n-1,1)}(m) = \Big(2 + \frac{mn}{(m-1)(n-1)}\Big) \cdot \binom{m(n-1)}{n}. \tag{5.9}$$

References

[1] G.E. Andrews, R. Askey, R. Roy. *Special Functions*, Encyclopedia of Mathematics and its Applications, vol. 71, Cambridge University Press, 1999.

[2] D. Bessis. The dual braid monoid, *Annales Scientifiques de l'Ecole Normale Superieure* no. 36(5) (2003), 647–683.

[3] P. Biane. Some properties of crossings and partitions, *Discrete Mathematics* 175 (1997), 41–53.

[4] P. Biane, F. Goodman, A. Nica. Non-crossing cumulants of type B, *Transactions of the American Mathematical Society* 355 (2003), 2263–2303.

[5] T. Brady. A partial order on the symmetric group and new $K(\pi, 1)$'s for the braid groups, *Advances in Mathematics* 161 (2001), 20–40.

[6] T. Brady, C. Watt. $K(\pi, 1)$'s for Artin groups of finite type, Proceedings of the Conference on Geometric and Combinatorial Group Theory, *Geometria Dedicata* 94 (2002), 225–250.

[7] P.H. Edelman. Chain enumeration and non-crossing partitions, *Discrete Mathematics* 31 (1980), 171–180

[8] G. Kreweras. Sur les partitions non-croisées d'un cycle, *Discrete Mathematics* 1 (1972), 333–350.

[9] A. Nica, I. Oancea. Lattices of non-crossing partitions of type B and D, preprint 2007, available on www.arXiv.org under arXiv:0705.3076 [math.CO].

[10] V. Reiner. Non-crossing partitions for classical reflection groups, *Discrete Mathematics* 177 (1997), 195–222.

[11] R.P. Stanley. *Enumerative Combinatorics, volume 1,* Cambridge University Press, 1997.

[12] R.P. Stanley. *Enumerative Combinatorics, volume 2,* Cambridge University Press, 1999.

I.P. Goulden: University of Waterloo.
Address: Department of Combinatorics and Optimization, University of Waterloo,

Waterloo, Ontario N2L 3G1, Canada.
Email: ipgoulden@math.uwaterloo.ca

Alexandru Nica, Ion Oancea: University of Waterloo.
Address: Department of Pure Mathematics, University of Waterloo,
Waterloo, Ontario N2L 3G1, Canada.
Email: anica@math.uwaterloo.ca, ioancea@math.uwaterloo.ca