Combinatorial Optimization on k-Connected Graphs

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Abstract

The topic of *network design* occupies a central place in Combinatorial Optimization. There are a number of key open questions pertaining to k-node connected graphs on topics such as minimum-cost k-connected spanning subgraphs, connectivity augmentation, and orientations. These questions/conjectures are analogous to similar statements for edge connectivity. Most of the statements for edge connectivity have been resolved, but the questions for node connectivity are wide open.

Network design is a sub-area of Combinatorial Optimization that focuses on problems of the following type: Find a minimum-cost sub-network H of a given network G such that H satisfies some pre-specified connectivity requirements. A *network* means an undirected graph or a directed graph together with non-negative costs for the edges. Fundamental examples include the *minimum spanning tree* (MST) problem and the *traveling salesman problem* (TSP). The connectivity requirements may specify edge connectivity or node connectivity. We use n to denote the number of nodes, and k to denote a nonnegative integer. For example, in the *node-connectivity survivable network design problem* (NC-SNDP), we are given a graph G, as well as a connectivity requirement $r_{i,j}$ between every pair of nodes i and j. The goal is to find a minimum-cost subgraph H of G that satisfies these requirements, that is, H should have $r_{i,j}$ openly node-disjoint paths between every pair of nodes i and j. The well-known *minimum-cost k-node connected spanning subgraph problem* (k-NCSS) is a special case of NC-SNDP where $r_{i,j} = k$ for every pair of nodes. (A notational remark: all of our abbreviated problem names, such as MST, k-NCSS, etc., refer to problems of computing minimum-cost subgraphs with additional properties.) Most network design problems are NP-hard and APX-hard. This remains true even for small connectivity requirements; for example, the Steiner tree problem (a classical special case of NC-SNDP) is APX-hard even with edge costs of 1 and 2.

Over the past two decades, there has been significant research on approximation algorithms for network design. Below, we mention a few results, old and new, and we list some open questions.

Consider a complete graph K_n together with edge-costs that satisfy the triangle inequality; we call this a *metric graph.* Together with A.Vetta, we proved the following, see [2]: In a metric graph and for each $k = 1, 2, \ldots, n-1$, the minimum cost of a k-node connected spanning subgraph is within a constant factor of the minimum cost of a simple k-edge connected spanning subgraph. The constant factor in [2] is 22. By using different methods, it may be possible to prove the stated result with a constant factor ≤ 2.

Together with B.Laekhanukit, we are studying another type of network design problem on directed graphs. This type of problem was first studied by [7], see also [12]. Those authors focused on tractable versions of the problem, that could be solved in polynomial time; moreover, they proved min-max results for some of these problems. Our focus is on NP-hard versions of the problem, and our primary goal is to design and analyze approximation algorithms and to prove hardness-of-approximation results.

In the *minimum-cost* (S,T)*connected digraph problem*, we are given a directed graph $G = (V, E_0 \cup E)$, two sets of nodes $S, T \subseteq V$, and positive costs on the edges in E; we may assume that each edge in E_0 has zero cost. We call E the set of *augmenting edges*, and we call $G_0 = (V, E_0)$ the *initial digraph.* In this article, we assume that every augmenting edge has its tail in S and its head in T. A directed graph on V is said to (S,T)*connected* if it has an s, t directed path for each node $s \in S$ and each node $t \in T$. The goal is to find a subset of edges $F \subseteq E$ of minimum cost such that the subgraph $(V, E_0 \cup F)$ is (S,T)*connected.*

Moreover, we focus on a generalization called the *minimum-cost* k-(S,T)*connected digraph problem.* The input to this problem is as above, and in addition we are given a positive integer k. A directed graph is called k-(S,T)connected if it has k edge-disjoint s, t directed paths between every node $s \in S$ and every node $t \in T$. Our goal is to find a subset F of the augmenting edges of minimum cost such that the subgraph $(V, E_0 \cup F)$ is k-(S,T)connected.

Although the above problem pertains to edge-connectivity, it can be seen that k-NCSS is a special case of this problem. Here is a sketch of the reduction, which is standard: we split each node v of the k-NCSS directed graph into a pair of nodes v_{in}, v_{out}, replace each edge vw of the k-NCSS directed graph by an augmenting edge $v_{out}w_{in}$, and finally, we add the edges $v_{in}v_{out}$ to E_0 (the initial digraph). We let S

consist of all the nodes of the form v_{out}, and we let T consist of all the nodes of the form v_{in}. It can be seen that the k-NCSS directed graph has k openly node-disjoint paths between every pair of nodes v, w iff the constructed directed graph has k edge-disjoint paths between every node $v_{out} \in S$ and every node $w_{in} \in T$.

We design approximation algorithms, running in polynomial time, that achieve an approximation guarantee of $O((\log k)(\log n))$ for the general problem, see [1].

A long-standing open question in the area is whether or not there exist polynomial-time approximation algorithms for k-NCSS that achieve an approximation guarantee of $O(1)$. If the answer is no, then it may be possible to prove this using methods from the hardness-of-approximation.

Another open question pertains to edge connectivity. Currently, several approximation algorithms are known for the *minimum-cost k-edge connected spanning subgraph problem* (k-ECSS) that achieve an approximation guarantee of 2, see [9, 8]. Does there exist a polynomial-time approximation algorithm for 2-ECSS that achieves a significantly better approximation guarantee, say 1.5 rather than 2?

Next, we mention a different type of problem from network design; in these problems, all of the edges of the complete (directed or undirected) graph are available as augmenting edges, and each augmenting edge has unit cost.

In the *connectivity augmentation problem* we are given an (undirected or directed) graph G, and a positive integer k; the goal is to find the minimum number of edges of the complete graph to be added to G in order to obtain a k-connected graph. There are four basic versions of this problem, for undirected/directed graphs, and for edge/node connectivity, see [5]. Three of these four basic problems have been solved over the years, see [13, 4, 7], but the last one, for undirected node-connectivity, is still open. Recently, a key special case has been solved, see [11].

Finally, we mention Frank's conjecture on node-connectivity orientation, see [6].

Let $G = (V, E)$ be an undirected graph with $|V| > k$. Then G has a k-node connected orientation if and only if for all $S \subseteq V$ with $|S| < k$, $G - S$ is $(2k - 2|S|)$-edge connected.

This has been proved for the special case of Eulerian graphs and $k = 2$, see [3], and for a short proof see [10].

References

[1] J. Cheriyan and B. Laekhanukit, "Approximation algorithms for mincost (S,T) connected digraphs," manuscript, April 2010.

[2] J. Cheriyan and A. Vetta, "Approximation algorithms for network design with metric costs," *SIAM J. Disc. Math.*, 21(3): 612–636, 2007.

[3] A.R. Berg and T. Jordán, "Two-connected orientations of Eulerian graphs," *Journal of Graph Theory*, 52(3):230–242, 2006.

[4] A. Frank, "Augmenting graphs to meet edge-connectivity requirements," *SIAM J. Disc. Math.*, 5:25–53, 1992.

[5] A. Frank, "Connectivity augmentation problems in network design," in *Mathematical Programming: State of the Art 1994*, (Eds. J. R. Birge and K. G. Murty), pp. 34–63, The University of Michigan, Ann Arbor, MI, 1994.

[6] A. Frank, "Connectivity and network flows," in *Handbook of Combinatorics* vol. 1, pp. 111–177, Elsevier, Amsterdam, 1995.

[7] A. Frank and T. Jordán, "Minimal edge-coverings of pairs of sets," *J. Combinatorial Theory, Series B*, 65:73–110, 1995.

[8] K. Jain, "A factor 2 approximation algorithm for the generalized Steiner network problem", *Combinatorica*, 21(1):39–60, 2001.

[9] S. Khuller, "Approximation algorithms for finding highly connected subgraphs," in *Approximation algorithms for NP-hard problems*, Ed. D.S.Hochbaum, PWS publishing co., Boston, 1996.

[10] Z. Király and Z. Szigeti, "Simultaneous well-balanced orientations of graphs," *J. Comb. Theory, Ser. B* 96(5):684–692, 2006.

[11] L.A. Végh, "Augmenting undirected node-connectivity by one," In *Proc. ACM STOC*, 2010. *EGRES Technical Report* TR–2009–10. (http://www.cs.elte.hu/egres/tr/egres-09-10.pdf)

[12] L.A. Végh and A.A. Benczúr, "Primal-dual approach for directed vertex connectivity augmentation and generalizations," *ACM Transactions on Algorithms*, 4(2), 2008.

[13] T. Watanabe and A. Nakamura, "Edge-connectivity augmentation problems," *J. Comput. Syst. Sci.*, 35(1):96–144, 1987.