The Traveling Tournament Problem: Complexity Aspects

by Joseph Cheriyan, C&O Dept., U.Waterloo
(based on joint work with
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Salomon Bendayan (2022). The Traveling Tournament Problem. UWSpace.
http://hdl.handle.net/10012/18553

Traveling Tournament Problem

(Easton, Nemhauser, Trick 2001)

Given n teams (n is even) and inter-team metric distances (i.e., $n \times n$ matrix D of inter-team distances satisfying the triangle inequality) find a tournament (i.e., a double round-robin schedule) such that

- each pair of teams i, j plays two games, one at home-venue of i and one at home-venue of j,
- the total distance travelled is minimized (assume "round-trip travel" for each team, that starts & ends at home-venue).

Notes:

Number of rounds $=$ number of *days* $= 2(n-1)$

OBJ: $\min \sum_{\text{TEAM } i} \left(\text{distance traveled by } i \text{ over } 2(n-1) \text{ days}\right)$

TTP: Example with $n = 6$ teams

Day ↓ / Team →	1	2	3	4	5	6
1	@5	@4	@6	2	1	3
2	6	5	4	@3	@2	@1
3	2	@1	@5	@6	3	4
4	@3	@6	1	5	@4	2
5	4	3	@2	@1	6	@5
6	5	4	6	@2	@1	@3
7	@6	@5	@4	3	2	1
8	@2	1	5	6	@3	@4
9	3	6	@1	@5	4	@2
10	@4	@3	2	1	@6	5

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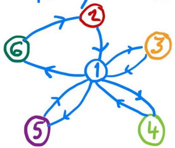
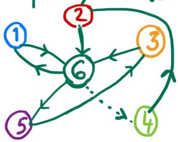
Trips by Team 1:



Trips by Team 6:



TTP: Questions

- Is there a poly-time algorithm for optimally solving TTP?

- Is there a PTAS (i.e., poly-time approximation scheme) for solving TTP?

TTP: Results

- (Bhattacharyya'09, ORL'16)
  (unrestricted) TTP is NP-hard.
  ($\implies \nexists$ poly-time algorithm, assuming $\mathbf{P} \neq \mathbf{NP}$)

- (BCC) (unrestricted) TTP is APX-hard.
  ($\implies \nexists$ PTAS, assuming $\mathbf{P} \neq \mathbf{NP}$)

TTP: Results & Open Question

- (Bhattacharyya'09, ORL'16)
  (unrestricted) TTP is NP-hard.
  ($\implies \nexists$ poly-time algorithm, assuming $\mathbf{P} \neq \mathbf{NP}$)
- (BCC) (unrestricted) TTP is APX-hard.
  ($\implies \nexists$ PTAS, assuming $\mathbf{P} \neq \mathbf{NP}$)

- (Thielen, Westphal TCS'11) 3-restricted TTP is NP-hard.
- (**Open**) 3-restricted TTP is APX-hard?

Bhattacharyya/BCC construction: uTTP is NP-hard

- Let $c > 5$ be a constant.
- G is a complete graph on n vertices, with edge costs in $\{1, 2\}$, v is a designated vertex of G.
- $G^{(c)}$ is the "one sum" of c copies of G with common vertex v, it has $c(n-1)+1$ vertices.
- G' is the metric completion of $G^{(c)}$.

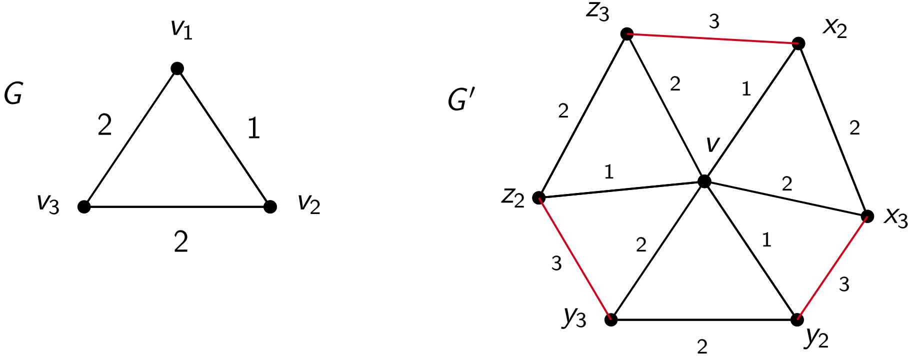
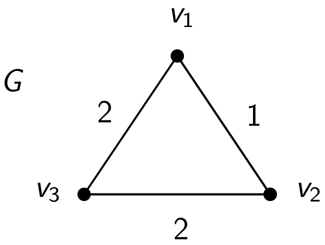


Figure: Small example: G' from G with $c = 3$.

Bhattacharyya/BCC construction: uTTP is NP-hard

- G is a complete graph on n vertices, with edge costs in $\{1, 2\}$, v is a designated vertex of G.
- G' is the metric completion of $G^{(c)}$, where $G^{(c)}$ is the "one sum" of c ($c > 5$) copies of G with common vertex v.
- **Claim**: an optimal TSP tour on G has cost $K \iff$ an optimal TSP tour on G' has cost cK.



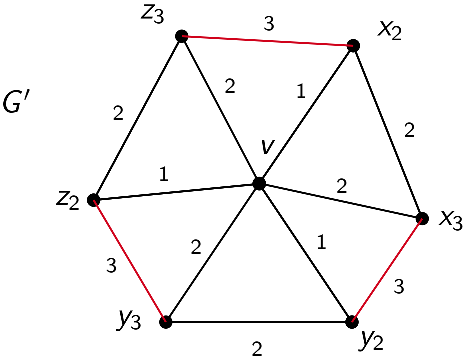


Figure: Small example: G' from G with $c = 3$.

Bhattacharyya/BCC construction: uTTP is NP-hard

- G is a complete graph on n vertices, with edge costs in $\{1, 2\}$, v is a designated vertex of G.
  $K :=$ cost of optimal TSP tour of G.
- G' is the metric completion of $G^{(c)}$, where $G^{(c)}$ is the "one sum" of c ($c > 5$) copies of G with common vertex v.
  $cK :=$ cost of optimal TSP tour of G',
  $m := |V(G')| = c(n-1)+1$.
- Construct a new graph H by adding vertex u to G', and connecting u to all vertices of G' with edges of cost $w_u = (cK-1)/2$.
- Construct the TTP instance on H by placing 2 teams at v (the central vertex of G'), one team at every other vertex of G', and $(m+1)(10m-1)$ teams at u to get $n' = 10m(m+1)$ teams in total.

Lemma

H has a feasible TTP solution of cost $\leq 20m(m+1)cK$.

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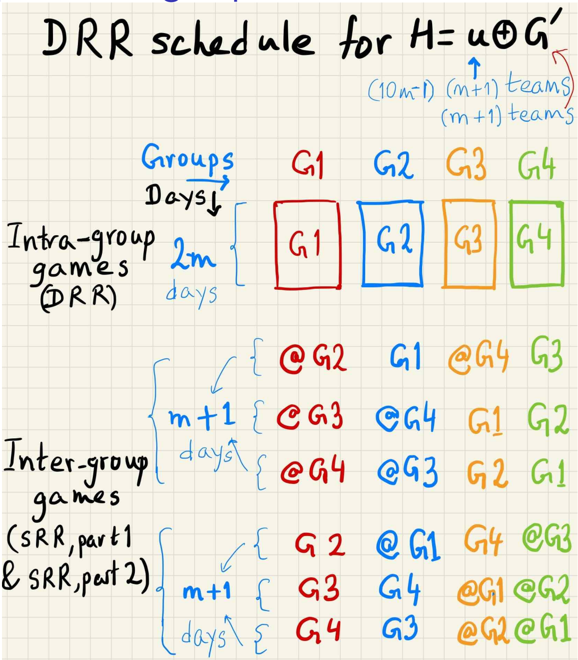
Lemma

H has a feasible TTP solution of cost $\leq 20m(m+1)cK$.

Proof.

- Partition the teams into $10m$ groups, each of size $m+1$.
- Group 1 := the $m+1$ teams placed on G'.
- DRR schedule consists of batch of **intra-group** games (with $2m$ games) followed by batch of **inter-group** games.
- **Claim**: Cost of DRR schedule $\leq 8m(m+1) + \Big((m+1)2w_u + (10m-1)\big((m+1)2w_u + mcK\big)\Big)$.

TTP: Example with 4 groups



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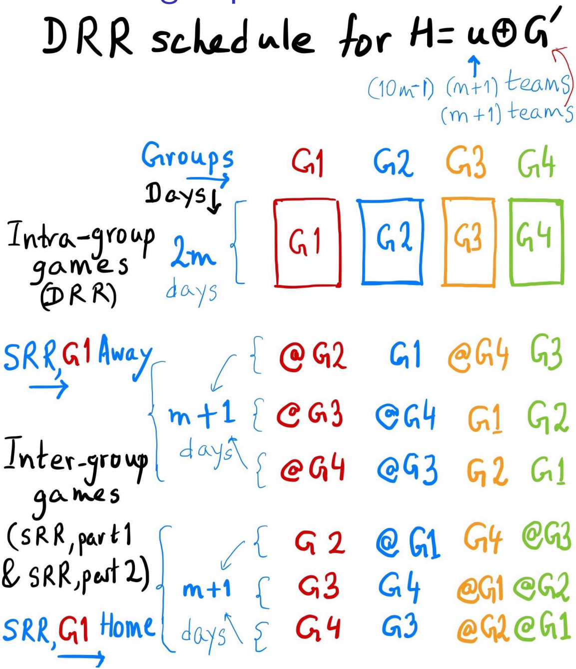

Bhattacharyya/BCC construction: uTTP is NP-hard

Lemma: H has a feasible TTP solution of cost $\leq 20m(m+1)cK$.
Proof of lemma:

- Partition the teams into $10m$ groups, each of size $m+1$.
- Group 1 $:=$ the $m+1$ teams placed on G'.
- DRR schedule consists of batch of **intra-group** games (with $2m$ games) followed by batch of **inter-group** games.
  - **Intra-group** games: All groups, except Group 1, are located at u and incur zero cost. Each team in Group 1 plays m Away-games and incurs cost $\leq (2c_{max})m = 8m$ ($c_{max} = \max_{e \in G'}(c_e) = 4$).
    Group 1 has $(m+1)$ teams, hence, total cost $\leq (2c_{max})m(m+1) = 8m(m+1)$.

Bhattacharyya/BCC construction: uTTP is NP-hard: Proof of lemma

- Partition the teams into $10m$ groups, each of size $m+1$. Group 1 := the $m+1$ teams placed on G'.
- DRR schedule consists of batch of **intra-group** games (with $2m$ games) followed by batch of **inter-group** games.
  - **Intra-group** games:
  - **Inter-group** games, Part 1: Construct an "outer" Single Round-Robin (SRR) schedule on the $10m$ groups. Each "outer" game between Groups i, j maps to $(m+1)$ games where each team of Group i plays against all $(m+1)$ teams of Group j (i.e., edge-coloring of $K_{m+1,m+1}$ by $(m+1)$ colors), and
    Group i plays @Home $\iff$ each team of Group i plays @Home.
    Cost analysis: Assume: Group 1 plays none of its games @Home (each of the $(m+1)$ teams of Group 1 visits vertex u, plays all its games at u, then returns to G').
    Total cost: $\leq (m+1)2w_u$.
  - **Inter-group** games, Part 2:

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Lemma: H has a feasible TTP solution of cost $\leq 20m(m+1)cK$.
Proof of lemma:

- Partition the teams into $10m$ groups, each of size $m+1$.
- Group 1 := the $m+1$ teams placed on G'.
- DRR schedule consists of batch of **intra-group** games (with $2m$ games) followed by batch of **inter-group** games.
  - **Intra-group** games:
  - **Inter-group** games, Part 1:
  - **Inter-group** games, Part 2: Similar to Part 1, but flip Home-Away designations for the "outer" Single Round-Robin (SRR) schedule on the $10m$ groups.
    Cost analysis: Assume: Group 1 plays Home (each of the teams at vertex u visits the vertices of G' according to an optimal TSP tour of G', then returns to u).
    Cost incurred by each group at vertex u:
    $(m+1)2w_u + (m+1)cK - cK$.
    Total cost: $\leq (10m-1)(2w_u(m+1) + mcK)$.

Bhattacharyya/BCC construction: uTTP is NP-hard

Lemma

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Proof of lemma:

- Partition the teams into $10m$ groups, each of size $m+1$.
- Group 1 := the $m+1$ teams placed on G'.
- DRR schedule consists of batch of **intra-group** games (with $2m$ games) followed by batch of **inter-group** games.
  - **Intra-group** games:
    Cost $\leq 8m(m+1)$.
  - **Inter-group** games, Part 1:
    Cost $\leq (m+1)2w_u$.
  - **Inter-group** games, Part 2:
    Cost $\leq (10m-1)(2w_u(m+1)+mcK)$.
- **Claim**: Cost of DRR schedule
  $\leq 8m(m+1)+\Big((m+1)2w_u+(10m-1)\big((m+1)2w_u+mcK\big)\Big)$.

Bhattacharyya/BCC construction: uTTP is NP-hard

Recall: Input graph G has edge-costs of 1 or 2.
$K :=$ cost of an optimal TSP tour of G.

Lemma

H has a feasible TTP solution of cost $\leq 20m(m+1)cK$.

Lemma

Suppose the cost of an optimal TSP tour of G is $\geq K+1$.
Then, any feasible TTP solution of H has cost $> 20m(m+1)cK$.

L-reductions . . . for APX-hardness

L-reduction from optimization problem Π_1 to optimization problem Π_2:

Intent (informal):

α-approximation algorithm for Π_2	$\Longrightarrow$	$f(\alpha)$-approximation algorithm for Π_1, for some linear function $f()$

Suppose there is an L-reduction from Π_1 to Π_2, and suppose Π_1 has **no** β-approximation algorithm (for some $\beta \geq 1$), then
Π_2 has **no** $f^{-1}(\beta)$-approximation algorithm.

L-reductions ... for APX-hardness

L-reduction from optimization problem Π_1 to optimization problem Π_2:

Definition

Given two optimization problems Π_1 and Π_2, we say we have an L-reduction from Π_1 to Π_2 if for some parameters $a, b > 0$:

1. For each instance I_1 of Π_1 we can compute in polynomial time an instance I_2 of Π_2.
2. $\mathrm{OPT}(I_2) \leq a \cdot \mathrm{OPT}(I_1)$.
3. Given a solution of value V_2 to I_2, we can compute in polynomial time a solution of value V_1 to I_1 such that

$$|\mathrm{OPT}(I_1) - V_1| \leq b \cdot |\mathrm{OPT}(I_2) - V_2|$$

L-reductions . . . for APX-hardness

Fact:

- Suppose there is an L-reduction, with parameters a, b, from minimization problem Π_1 to minimization problem Π_2, and
- suppose there is an α-approximation algorithm for Π_2,
- then there is an $(1 + ab(\alpha - 1))$-approximation algorithm for Π_1.

- Instance I_1 of Π_1 $\implies$ instance I_2 of Π_2.
- Compute solution S_2 to I_2 of value $V_2 \leq \alpha\,\mathrm{OPT}(I_2)$.
- "Map" S_2 to a solution S_1 of I_1 of value V_1 such that

$$\begin{aligned} V_1 &\leq \mathrm{OPT}(I_1) + b(V_2 - \mathrm{OPT}(I_2)) \\ &\leq \mathrm{OPT}(I_1) + b\big(\alpha\,\mathrm{OPT}(I_2) - \mathrm{OPT}(I_2)\big) \\ &= \mathrm{OPT}(I_1)(1 + ab(\alpha - 1)) \end{aligned}$$

TTP is APX-hard: L-reduction from $m(m+1)\otimes$TSP(1,2)

- L-reduction from Π_1 to Π_2, where
- Π_1 is the "$m(m+1)\otimes$TSP" on input graph G which has edge-costs of 1 or 2.
  OBJ $= m(m+1)$ times cost of an optimal TSP tour of G, where $m = c(n-1)+1$ and $n = |V(G)|$.
- Π_2 is TTP on the graph H.

$\Big($Recall: G' is the metric-completion of $G^{(c)}$, where $G^{(c)}$ is the "one-sum" of $c > 5$ copies of G with common vertex v. H is obtained from G' by adding a new vertex u, adding the edges ux, for all vertices x of G', and taking the cost of each new edge to be $w_u = (cK-1)/2.\Big)$

TTP is APX-hard: L-reduction from $m(m+1)\otimes$TSP(1,2)

- L-reduction from Π_1 to Π_2, where
- Π_1 is the "$m(m+1)\otimes$TSP" on G $\big($edge-costs $\in \{1,2\}\big)$.
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Recall the three conditions for an L-reduction with parameters $a, b > 0$:

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2. $\mathrm{OPT}(I_2) \leq a \cdot \mathrm{OPT}(I_1)$.
3. Given a solution of value V_2 to I_2, we can compute in polynomial time a solution of value V_1 to I_1 such that

$$|\mathrm{OPT}(I_1) - V_1| \leq b \cdot |\mathrm{OPT}(I_2) - V_2|$$

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- L-reduction from Π_1 to Π_2, where
- Π_1 is the "$m(m+1)\otimes$TSP" on G $\big($edge-costs $\in \{1,2\}\big)$.
  OBJ $= m(m+1)$ times cost of an optimal TSP tour of G.
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Recall the third conditions for an L-reduction with parameters $a, b > 0$:
Given a solution of value V_2 to I_2, we can compute in polynomial time a solution of value V_1 to I_1 such that

$$|\mathrm{OPT}(I_1) - V_1| \leq b \cdot |\mathrm{OPT}(I_2) - V_2|$$

Lemma

Given a TTP schedule S of H with additive slack Δ (i.e., $cost(S) - \mathrm{OPT}_{TTP}(H) \leq \Delta$), one can construct a TSP tour T of G with additive slack $\leq 10\Delta$ (i.e., $cost(T) - \mathrm{OPT}_{TSP}(G) \leq 10\Delta$).

TTP is APX-hard: L-reduction from $m(m+1)\otimes$TSP(1,2)

- L-reduction from Π_1 to Π_2, where
- Π_1 is the "$m(m+1)\otimes$TSP" on G $\big(\text{edge-costs } \in \{1,2\}\big)$.
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Lemma

Given a TTP schedule S of H with additive slack Δ (i.e., $cost(S) - \mathrm{OPT}_{TTP}(H) \leq \Delta$),
one can construct a TSP tour T of G with additive slack $\leq 10\Delta$ (i.e., $cost(T) - \mathrm{OPT}_{TSP}(G) \leq 10\Delta$).

Claim

Given an **optimal** *TTP schedule S of H,*
one can construct a TSP tour T of G that is **optimal**.

Claim

Given a **non-optimal** *TTP schedule S of H,*
one can construct a TSP tour T of G such that
$cost(T) - \mathrm{OPT}_{TSP}(G) \leq 10\,(cost(S) - \mathrm{OPT}_{TTP}(H))$.

TTP: Results & Open Question

- (BCC) (unrestricted) TTP is APX-hard.
  ($\implies \nexists$ PTAS, assuming $\mathbf{P} \neq \mathbf{NP}$)

- (**Open**) 3-restricted TTP is APX-hard?
  (Thielen, Westphal TCS'11) proved 3-restricted TTP is NP-hard.
- Prove Hardness-of-Approximation results for other variants of TTP.
- Design improved approximation algorithms for variants of TTP.
- Concorde-Plus to optimally solve TTP instances with 100 teams?