Graph Minors Structure Theorem

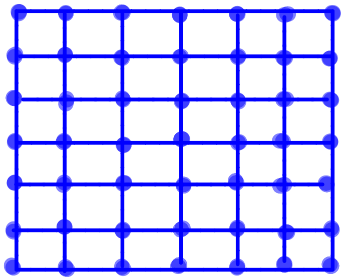
Overview

Question: For a given graph H, what is the structure of a graph with no H-minor?

Note: We usually consider $H = K_t$.

G has tree-width $\leq k$ iff G can be obtained from a set of graphs each with at most $k+1$ vertices via clique-sums.

Grid Theorem: If tree-width(G) $\geq f(k)$, then G has a $k \times k$-grid minor.



Equivalently: Let H be a planar graph. If G has no H-minor, then treewidth(G) $\leq f(H)$.

Tangles.

If $\mathcal{T}$ is a tangle of order k in G then
- each separation of order $< k$ has a unique $\mathcal{T}$-small side,
- no three $\mathcal{T}$-small sides cover G, and
- no $\mathcal{T}$-small side is spanning.

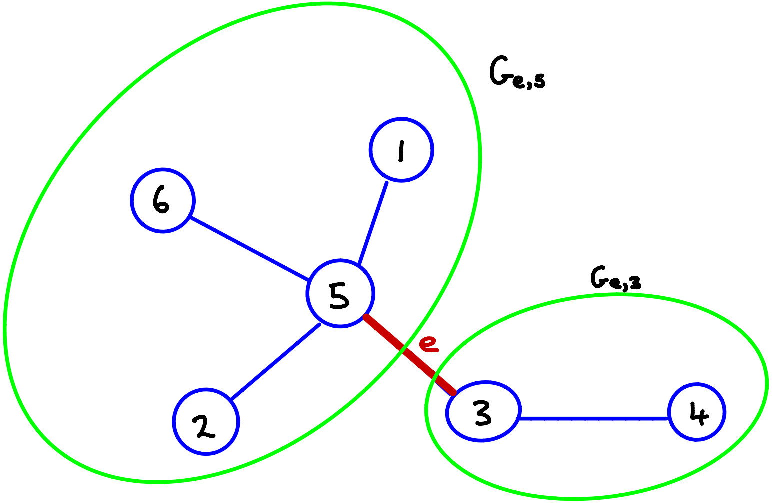
Duality Theorem: If G has tree-width $\geq 10k$, then G has a tangle of order k.

Lemma: $\kappa_{\mathcal{T}}(X)$ is the rank-function of a matroid on $V(G)$.

Tree of tangles.

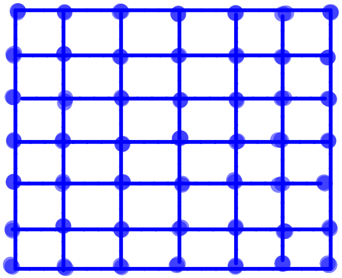
Let $\mathcal{T}_1, \dots, \mathcal{T}_n$ denote the tangles of order k.

Theorem. There is a tree-decomposition $(T, \mathcal{B})$ with $V(T) = \{1, \dots, n\}$ such that, for each edge $e = ij$ of T, $(G_{e,i}, G_{e,j}) \in \mathcal{T}_j$ and $(G_{e,j}, G_{e,i}) \in \mathcal{T}_i$.

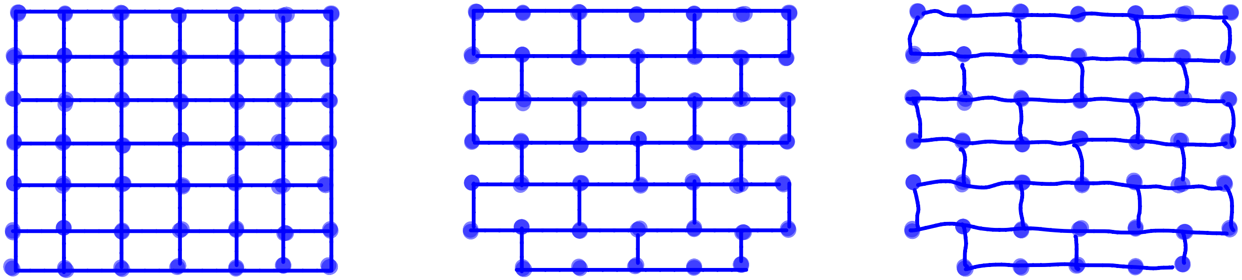


Grids from tangles.

Grid Theorem (refined.) If $\mathrm{ord}(\mathcal{T}) \geq f(k)$, then G has a $k \times k$-grid-minor that is controlled by $\mathcal{T}$.



Gridworks



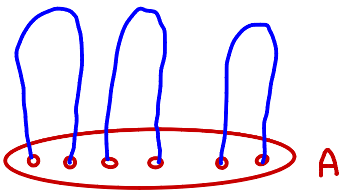
minor

subgraph

Key Tools.

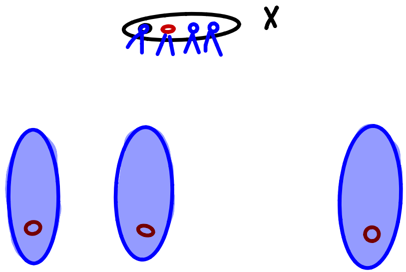
Gallai's A-Paths Theorem. If A is a set of vertices in a graph G, then either

(1) there exist k disjoint A-paths,



or

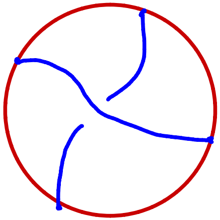
(2) there exists $X \subseteq V(G)$ with $|X| < 2k$ such that $G \setminus X$ has no A-paths.



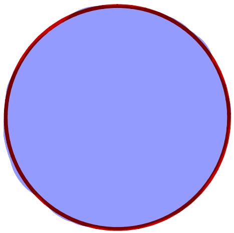
Key Tools.

The Two-Path Theorem: If C is a cycle in a graph $G=(V,E)$ then either

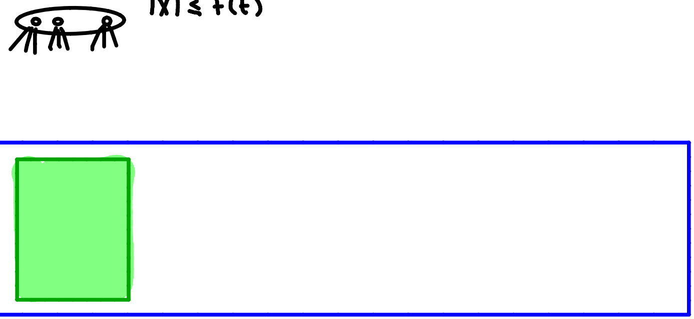
(1) there exist a disjoint pair of paths that cross on C, or



(2) G has a flat embedding in the disc with C in the boundary.



Local structure relative to a gridwork.



$|X| \leq f(t)$

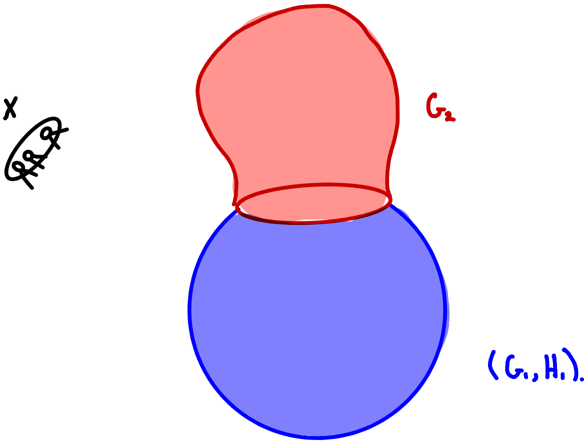
"flat" $(n_1, k-2d)$-gridwork

$d = d(t)$

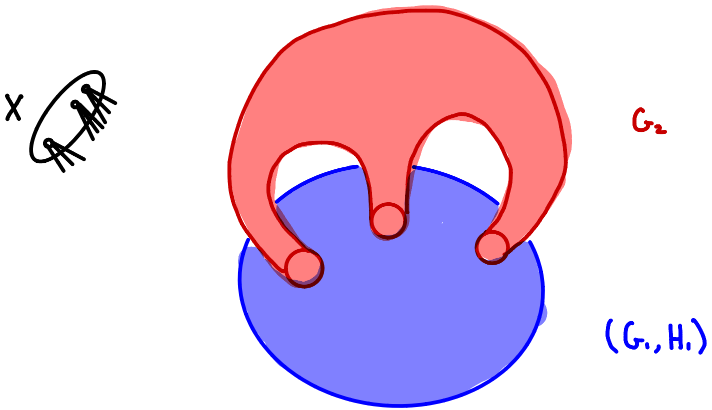
$(n \times k)$-gridwork

$n \gg n_1$

Weak Structure Theorem. Let G be a graph with no K_t-minor and let $\mathcal{T}$ be a tangle of order $\geq \Theta(t,r)$, then there is a $(G,\mathcal{T})$-structure $(X, D, G_1, G_2, H_1, \mathcal{T}_1)$ where $D = \Sigma(0,0;1)$ is a disc, $|X| \leq \xi(t)$, and $\mathcal{T}_1$ is r-representative.



Corollary. Let G be a graph with no K_t-minor and let $\mathcal{T}$ be a tangle of order $\geq \Theta(t,r)$, then there is a $(G,\mathcal{T})$-structure $(X, \Sigma, G_1, G_2, H_1, \mathcal{T}_1)$ where $\Sigma = \Sigma(0,0;\ell)$ is a sphere with $\ell \leq \xi(t)$ holes, $|X| = m \leq \xi(t)$, $\mathcal{T}_1$ has representativity $r \geq r(t,m)$ and each vertex in X has $n(t,m)$ neighbours in G_1 whose pairwise distance is at least $d(t,m)$.



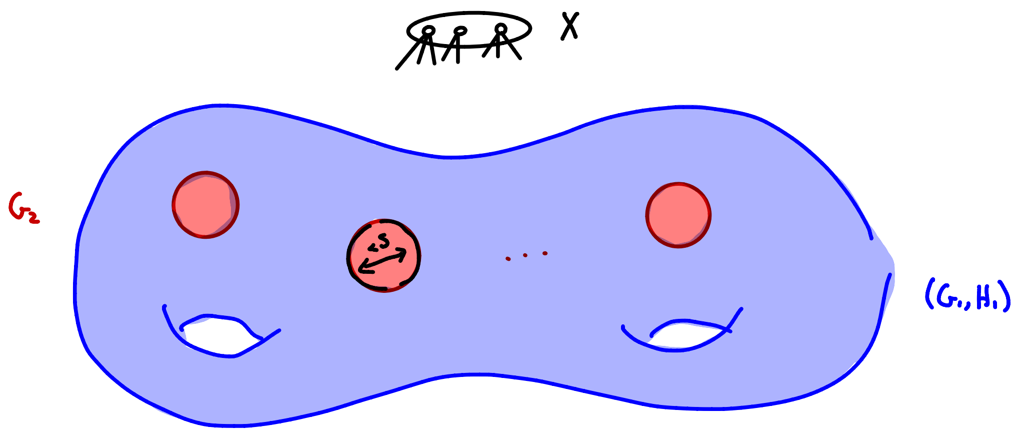
Graph Minors Structure Theorem. Let G be a graph with no K_t-minor and let $\mathcal{T}$ be a tangle with order $\Theta(t,r)$. Then there is a localized $(G,\mathcal{T})$-structure $(X,\Sigma,G_1,G_2,H_1,\mathcal{T}_1)$ such that:

(i) $\Sigma = \Sigma(h,c;\ell)$ where $2h+c \le 2t^2$ and $\ell \le \ell(t)$,

(ii) $\mathcal{T}_1$ is r-representative,

(iii) $|X| \le m(t)$, and

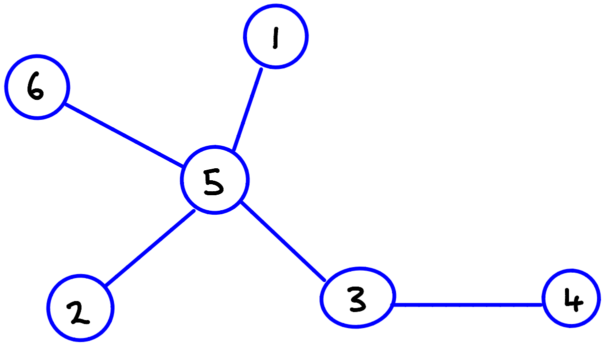
(iv) each hole has tension at most $s(t)$.



Tree of tangles.

Let $\mathcal{T}_1, \dots, \mathcal{T}_n$ denote the tangles of order k.

Theorem. There is a tree-decomposition $(T, \mathcal{B})$ with $V(T) = \{1, \dots, n\}$ such that, for each edge $e = ij$ of T, $(G_{e,i}, G_{e,j}) \in \mathcal{T}_j$ and $(G_{e,j}, G_{e,i}) \in \mathcal{T}_i$.

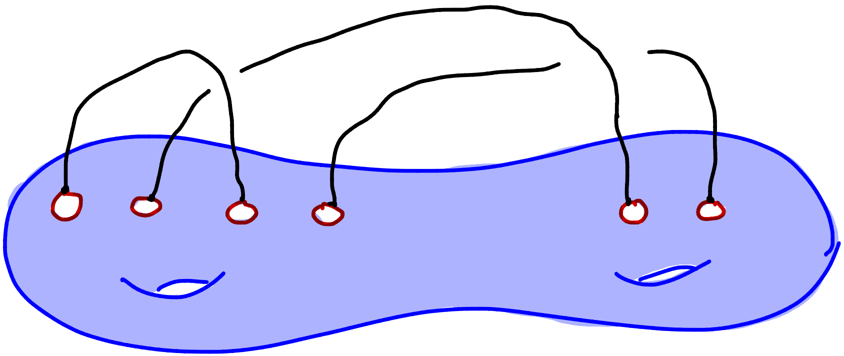


Finding a K_t-minor (High genus.)

Theorem A. If $G \hookrightarrow \Sigma(h,c)$ where $2h+c \in \{2t^2, 2t^2-1\}$ and G has representativity at least $r(t)$, then G has a K_t-minor.

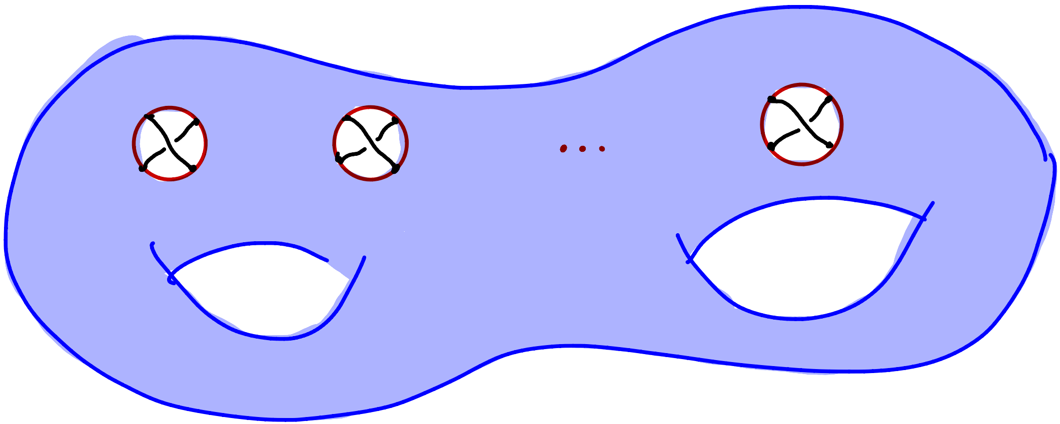
Finding a K_t-minor (Many far apart long jumps)

Theorem B. Let (G_1, G_2) be a separation in a graph G, let $G_1 \hookrightarrow \Sigma = \Sigma(h,c;\ell)$ where $2h+c \leq 2t^2$, let $\mathcal{T}_1$ be an r-representative tangle in G_1 where $r \geq r(t)$, and let $P_1, \ldots, P_m$ be disjoint paths in G_2 with ends in $2m$ distinct holes where $m \geq m(t)$. Then G has a K_t-minor.



Finding a K_t-minor (Many far apart crossings)

Theorem B. Let (G_1, G_2) be a separation in a graph G, let $G_1 \hookrightarrow \Sigma = \Sigma(h, c; l)$ with holes $h_1, \dots, h_l$ where $2h + c \leq 2t^2$ and $l \geq l(t)$, let $\mathcal{T}_1$ be an r-representative tangle in G_1 where $r \geq r(t)$, and let $P_1, \dots, P_l, Q_1, \dots, Q_l$ be disjoint paths where P_i and Q_i cross on the hole h_i for $i \in \{1, \dots, l\}$. Then G has a K_t-minor.



Proof of the Graph Minors Structure Theorem.

Let G be a graph with no K_t-minor, let $\mathcal{T}$ be a tangle in G, and let $(X, \Sigma, G_1, G_2, H_1, \mathcal{T}_1)$ be a $(G,\mathcal{T})$-structure where $\Sigma = \Sigma(h,c;\ell)$ such that:

(1) $2h+c \leq 2t^2$,

(2) $|X| = m \leq m(t)$,

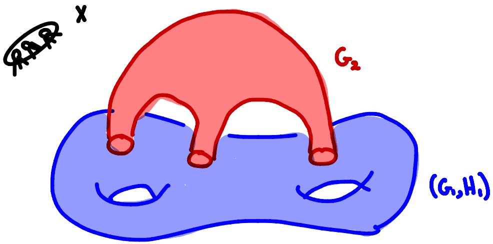
(3) $\ell \leq \ell(2h+c, m, t)$,

(4) $\mathcal{T}_1$ has representativity $r \geq r(2h+c, \ell, m, t)$,

(5) each vertex $v \in X$ has $n(2h+c, \ell, m, t)$ neighbours in G_1 whose pairwise distance in $(G_1, H_1, \mathcal{T}_1)$ is at least $d(2h+c, \ell, m, t)$, and

(6) For each hole h of Σ either there is a path in G_2 from h to another hole, or there are two disjoint paths in G_2 that cross on h and whose ends form a $\mathcal{T}_1$-independent set.

The functions are decreasing in $2h+c$, m, and ℓ and increasing in t.



Maximality Conditions

Among all such (G, J)-structures we choose $(X, \Sigma, G_1, G_2, H_1, J_1)$ such that:

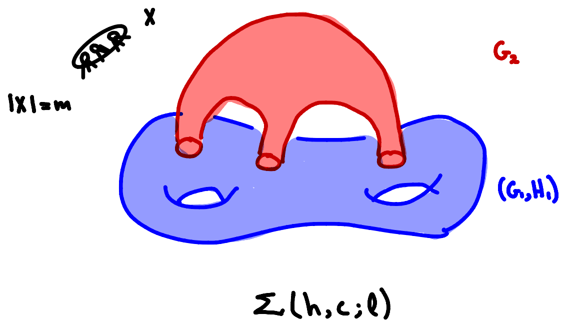
(a) $2h+c$ is maximum,

(b) m is maximum subject to (a), and

(c) ℓ is maximum subject to (b).

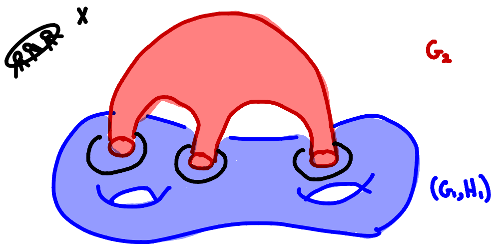
Note that:

- $2h+c < 2t^2-1$, (by Theorem A)
- $m < m(t)$, (by Theorem B)
- $\ell < \ell(2h+c, m, t)$. (by the maximality conditions)



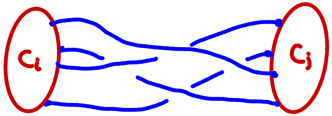
Disembedding.

Let $h_1, \dots, h_\ell$ be the holes in Σ, and for each $i \in \{1, \dots, \ell\}$, let C_i be a cycle at distance $\xi(2h+c, \ell, m, t)$ around hole h_i. Now let $(X, \Sigma, G_1', G_2', H_1', J_1')$ be obtained from $(X, \Sigma, G_1, G_2, H_1, J_1)$ by disembedding (G_1, H_1) around each hole h_i out to C_i, so that C_i is in the boundary of h_i.

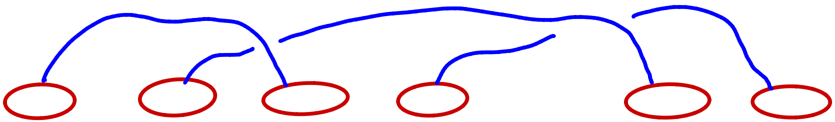


Theorem. Let $C_1, \dots, C_\ell$ be cycles in a graph G. Then either

(1) There exist distinct $i,j \in \{1,\dots,\ell\}$ such that $\kappa_G(V(C_i), V(C_j)) \geq p$.

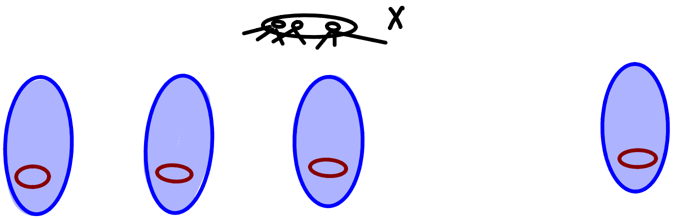


(2) There exist disjoint paths $P_1,\dots,P_q$ with ends in $2q$ distinct cycles among $C_1,\dots,C_\ell$,



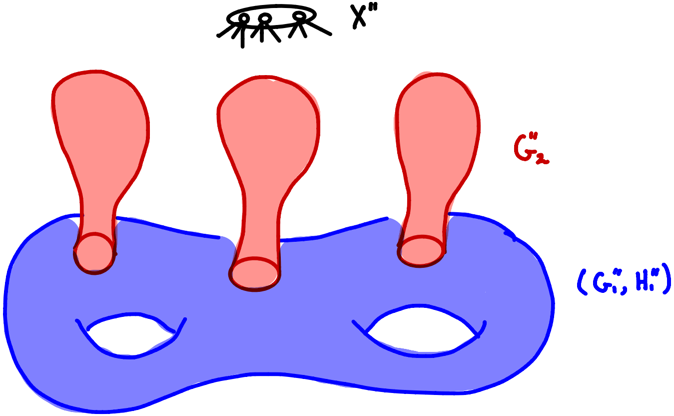
or

(3) There exists $X \subseteq V(G)$ with $|X| \leq 4pq^2 + 2q$ such that there is no path in $G \setminus X$ that connects two of $C_1,\dots,C_\ell$.

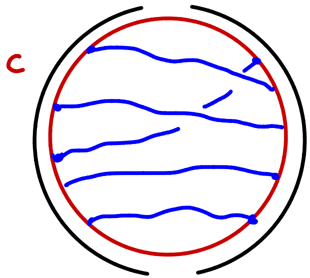


A localized (G,J)-structure.

By the theorem, there is a localized (G,J)-structure $(X'', \Sigma, G_1'', G_2'', H_1'', J_1'')$ such that $|X''| \leq \tilde{m}(2h+c, t)$ and J_1'' has representativity $\geq r''(2h+g, m, \ell, t)$. We may assume that $c_i' = G_1'' \cap h_i$ is a cycle.

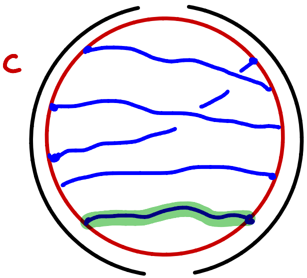


Crooked connectors



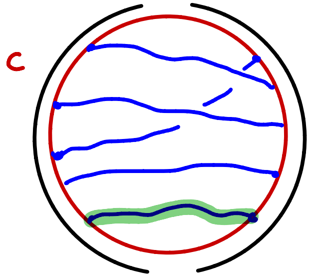
A connector for C

Crooked connectors



A crooked connector is one with no peripheral path.

Crooked connectors

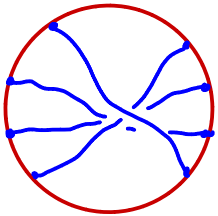


A connector for C

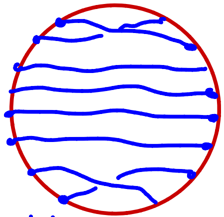
a peripheral path

A crooked connector is one with no peripheral path.

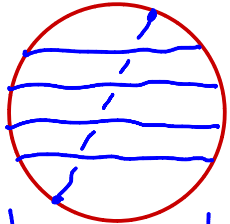
Examples.



crosscap connector

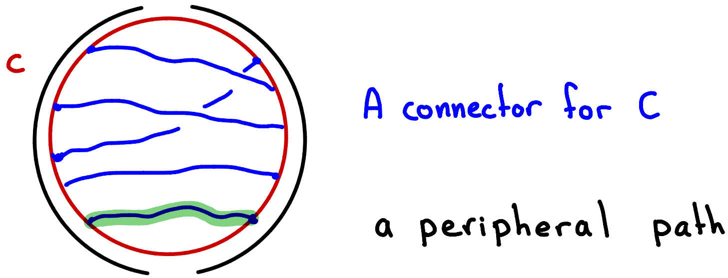


doublecross connector



leap connector

Crooked connectors



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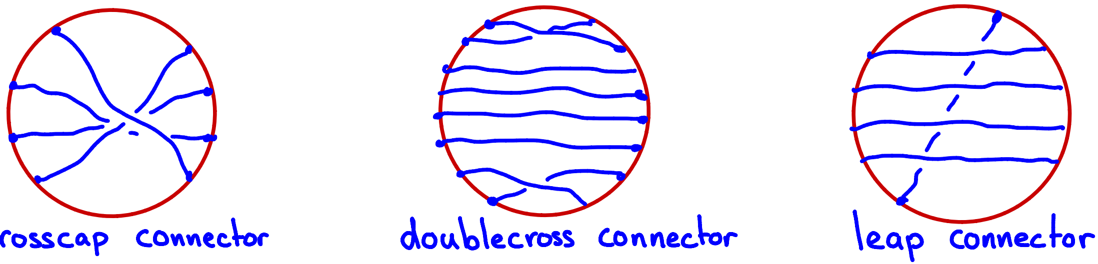
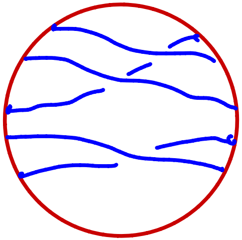
Examples.



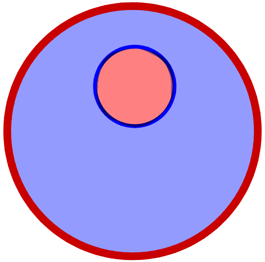
Lemma. If C has a crooked connector with $5k^2$ paths, then C has a crosscap, doublecross, or leap connector with k paths.

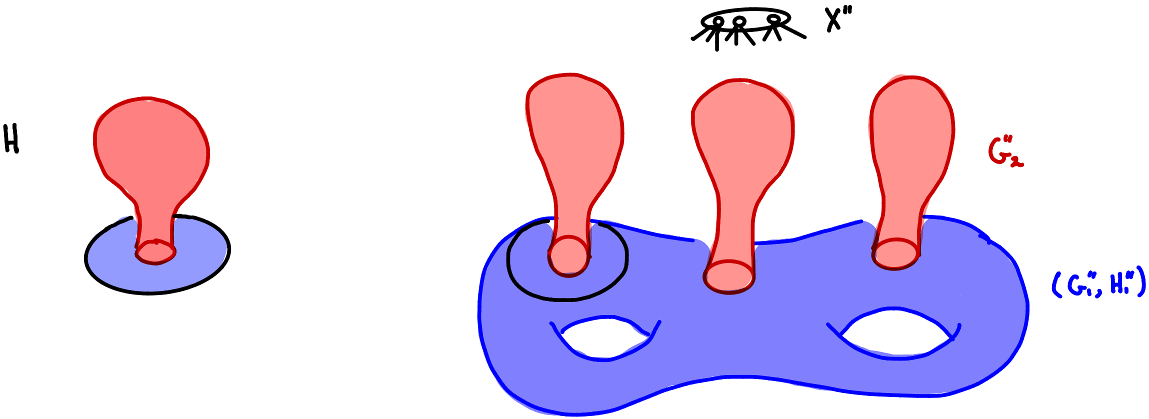
Connector Theorem. Let C be a cycle in a graph G and let $k \geqslant 7$. Then either:

(1) (G,C) has a crooked connector with k paths, or

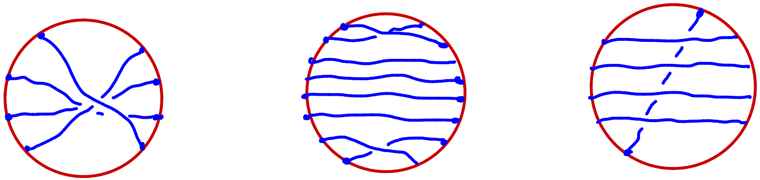


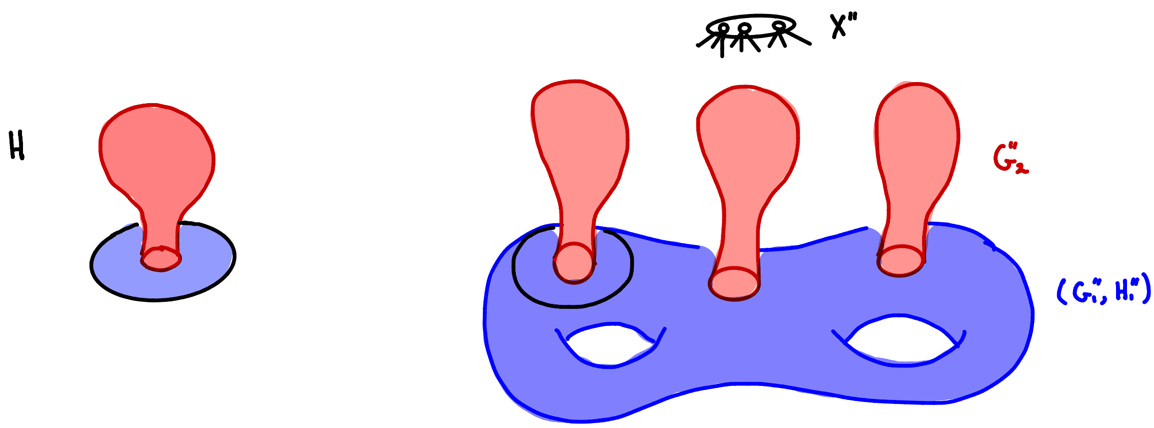
(2) (G,C) has a cylindrical embedding with tension $\leqslant 19k$.





If (H, C) has a crooked connector with $k(t)$ paths, then we get a better (G, J)-structure — contradiction.





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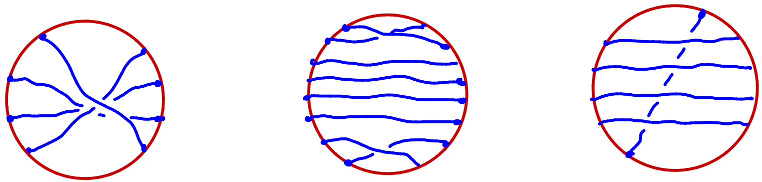

So, by the Connector Theorem, each hole in $(X'', \Sigma, G_1'', G_2'', H_1'', J_1'')$ has tension at most $19k(t)$.