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The Erdös–Pósa property for matroid circuits ☆

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ABSTRACT

The number of disjoint cocircuits in a matroid is bounded by its rank. There are, however, matroids with arbitrarily large rank that do not contain two disjoint cocircuits; consider, for example, $M(K_n)$ and $U_{n,2n}$. Also the bicircular matroids $B(K_n)$ have arbitrarily large rank and have no 3 disjoint cocircuits. We prove that for each k and n there exists a constant c such that, if M is a matroid with rank at least c, then either M has k disjoint cocircuits or M contains a $U_{n,2n}$-, $M(K_n)$-, or $B(K_n)$-minor.

1. Introduction

We prove the following theorem.

Theorem 1.1. *There exists a function $\gamma : \mathbb{N}^2 \to \mathbb{N}$ such that, if M is a matroid with no $U_{n,2n}$-, $M(K_n)$-, or $B(K_n)$-minor and $r(M) \geqslant \gamma(k,n)$, then M has k disjoint cocircuits.*

Here $M(K_n)$ is the cycle matroid of K_n, $B(K_n)$ is the bicircular matroid of K_n (to be defined below), and $\mathbb{N}$ denotes the set of positive integers.

A *circuit-cover* of a graph G is a set $X \subseteq E(G)$ such that $G - X$ has no circuits. Thus the maximum number of (edge-)disjoint circuits in a graph is bounded by the minimum size of a circuit cover. This bound is not tight (consider K_4), but Erdös and Pósa in [3] proved that the maximum number of disjoint circuits is qualitatively related to the minimum size of a circuit cover.

Erdös–Pósa Theorem 1.2. *There is a function $c : \mathbb{N} \to \mathbb{N}$ such that, for any graph G, either G has k disjoint circuits or G has a circuit-cover of size at most $c(k)$.*

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Let M be a matroid. A set $X \subseteq E(M)$ intersects each circuit of M if and only if $E(M) - X$ is independent. So a minimal circuit-cover of M is a basis of M^* and, hence, the minimum size of a circuit-cover is $r(M^*)$. Dually, the minimum size of a "cocircuit-cover" in a matroid M is equal to $r(M)$. The Erdös–Pósa Theorem was generalized to matroids by Geelen, Gerards, and Whittle [4] who proved the following theorem.

Theorem 1.3. *There exists a function $c : \mathbb{N}^2 \to \mathbb{N}$ such that, if M is a matroid with no $U_{2,n}$- or $M(K_n)$-minor and $r(M) \geqslant c(k,n)$, then M has k disjoint cocircuits.*

The result does not extend to all matroids; there exist matroids with arbitrarily large rank that have no two disjoint cocircuits. Matroids with no two disjoint cocircuits are referred to as *round*. Equivalently, a matroid is round if each of its cocircuits is spanning. The matroid $U_{r,n}$, where $n \geqslant 2r-1$ is round. Also, for any positive integer n, $M(K_n)$ is a round matroid. Note that, for a simple graph G, the matroid $M(G)$ is round if and only if G is a complete graph.

Let $G = (V, E)$ be a loopless graph. Define a matroid $\widetilde{B}(G)$ on $V \cup E$ where V is a basis of $\widetilde{B}(G)$ and, for each edge $e = uv$ of G, place e freely on the line spanned by $\{u, v\}$. Now $B(G) := \widetilde{B}(G) \backslash V$ is the *bicircular matroid* of G. Bicircular matroids obtain their name from the graphical description of their circuits; see [8, Prop. 12.1.6]. It is easy to verify that $\widetilde{B}(K_n)$ is round. The bicircular matroid $B(K_n)$ is not round, but it has no three disjoint cocircuits.

Our main theorem, Theorem 1.1, is a generalization of Theorem 1.3 and is, in some sense, best possible. Note that the matroids in each of the classes

$$\{M(K_n)\colon\ n \geqslant 1\}, \qquad \{B(K_n)\colon\ n \geqslant 1\}, \quad \text{and} \quad \{U_{n,2n}\colon\ n \geqslant 1\}$$

have unbounded rank but they have a bounded number of disjoint cocircuits.

We hope that Theorem 1.1 will help in solving the following unpublished conjecture of Johnson, Robertson, and Seymour: *for any positive integer n there is a positive integer k such that, if M is a matroid with branch-width at least k, then either M or M^* has a minor isomorphic to either $U_{n,2n}$ or to the cycle matroid or the bicircular matroid of an $n \times n$ grid.*

Our proof of Theorem 1.1 is based, in part, on the techniques developed in [4]. We follow the notation of Oxley [8].

2. Preliminaries

For a matroid M, we denote by $\Theta(M)$ the maximum number of disjoint cocircuits in M. So, M is round if and only if $\Theta(M) = 1$. The *rank-deficiency* of a set of elements $X \subseteq E(M)$ is $\mathrm{def}_M(X) = r(M) - r_M(X)$. We let $\Gamma(M)$ denote the maximum rank-deficiency among the cocircuits of M. Therefore M is round if and only if $\Gamma(M) = 0$. The two parameters $\Gamma(M)$ and $\Theta(M)$ are related by the inequality

$$\Theta(M) \leqslant \Gamma(M) + 1.$$

The following result lists hereditary properties of the two parameters; we omit the elementary proof.

Lemma 2.1. *Let e be an element of a matroid M. Then*

(i) $\Theta(M/e) \leqslant \Theta(M)$ *and* $\Gamma(M/e) \leqslant \Gamma(M)$.
(ii) *if e is not a coloop, then* $\Theta(M\backslash e) \geqslant \Theta(M)$ *and* $\Gamma(M\backslash e) \geqslant \Gamma(M)$.

The following lemma gives a sufficient condition for equality in (ii).

Lemma 2.2. *Let X be a set of elements in a matroid M such that $M|X$ is uniform and $|X| \geqslant 2r_M(X)$. Then, for any $e \in X$, we have $\Theta(M\backslash e) = \Theta(M)$ and $\Gamma(M\backslash e) = \Gamma(M)$.*

Proof. Let $k = r_M(X)$. Now consider any cocircuit C of $M\backslash e$. Note that either C or $C \cup \{e\}$ is a cocircuit of M. We claim that: *if $C \cup \{e\}$ is a cocircuit of M, then* $|X - (C \cup \{e\})| \leqslant k-1$ and $e \in \mathrm{cl}_M(C)$. Indeed, if

$C \cup \{e\}$ is a cocircuit of M, then $E(M) - (C \cup \{e\})$ is a hyperplane and, hence, it can contain at most $k-1$ elements of X. Therefore $|X \cap C| \geqslant k$ and, hence, $e \in \mathrm{cl}_M(C)$, as claimed.

Suppose that C is a cocircuit of $M\backslash e$ with $\mathrm{def}_{M\backslash e}(C) = \Gamma(M\backslash e)$. Now, there exists $C' \in \{C, C \cup \{e\}\}$ such that C' is a cocircuit of M. By the claim, $r_M(C') = r_M(C)$. Hence $\Gamma(M) \geqslant \mathrm{def}_M(C') = \mathrm{def}_{M\backslash e}(C) = \Gamma(M\backslash e)$. Then, by Lemma 2.1, we have $\Gamma(M) = \Gamma(M\backslash e)$.

Let $(C_1, \dots, C_t)$ be a maximum collection of disjoint cocircuits in $M\backslash e$ and, for each $i \in \{1, \dots, t\}$, let $C'_i \in \{C_i, C_i \cup \{e\}\}$ be a cocircuit of M. By the claim, at most one of the sets $(C'_1, \dots, C'_t)$ contains e. Therefore, $\Theta(M) \geqslant t = \Theta(M\backslash e)$. Then, by Lemma 2.1, we have $\Theta(M) = \Theta(M\backslash e)$, as required. □

A matroid M is called *a-simple* if M is loopless and has no $U_{k,2k}$-restriction for $k = 1, 2, \dots, a$. The following lemma is an immediate consequence of Lemma 2.2.

Lemma 2.3. *Let M be a matroid and let $a \in \mathbb{N}$. There is a spanning a-simple restriction N of M with $\Gamma(N) = \Gamma(M)$ and $\Theta(N) = \Theta(M)$.*

A simple GF(q)-representable rank-r matroid can be realized as a restriction of the projective geometry PG($r-1, q$). Thus, it has at most $\frac{q^r-1}{q-1}$ elements. Kung [6] extended this bound to the class of matroids with no $U_{2,q+2}$-minor (the shortest line not representable over GF(q)).

Theorem 2.4 *(Kung). Let $q > 1$ be an integer and let M be a simple rank-r matroid with no $U_{2,q+2}$-minor. Then*

$$\left|E(M)\right| \leqslant \frac{q^r - 1}{q - 1}.$$

This bound is attained by a projective geometry when q is a prime power. Excluding uniform matroids of larger rank will clearly not yield analogous bounds on the number of elements, so we introduce a new measure of size.

Let a be a positive integer. An *a-covering* of a matroid M is a collection $(X_1, \dots, X_m)$ of subsets of $E(M)$ with $E(M) = X_1 \cup \dots \cup X_m$ and $r_M(X_i) \leqslant a$ for all i. The *size* of the covering is m. The *a-covering number* of M, denoted $\tau_a(M)$, is the minimum size of an a-covering of M. Note that, for a matroid M, $\tau_1(M) = |E(\mathrm{si}(M))|$, where $\mathrm{si}(M)$ denotes the simplification of M. If $r(M) \leqslant a$, then $\tau_a(M) \leqslant 1$. Our first lemma bounds the a-covering number for matroids with rank $a+1$.

Lemma 2.5. *For $a, b \in \mathbb{N}$ with $b > a$, if M is a matroid of rank $a+1$ with no $U_{a+1,b}$-restriction, then*

$$\tau_a(M) \leqslant \binom{b-1}{a}.$$

Proof. Let $X \subseteq E(M)$ be maximal with $M|X \cong U_{a+1,l}$. Then $l \leqslant b-1$ and every point of M is spanned by one of the rank-a flats of $M|X$. Hence $\tau_a(M) \leqslant \binom{l}{a} \leqslant \binom{b-1}{a}$. □

The next result extends Kung's Theorem, although our bound is not sharp.

Theorem 2.6. *For $a, b \in \mathbb{N}$ with $b > a$, if M is a matroid of rank $r \geqslant a$ with no $U_{a+1,b}$-minor, then*

$$\tau_a(M) \leqslant \binom{b-1}{a}^{r-a}.$$

Proof. The proof is by induction on r. The case $r = a$ is trivial since $(E(M))$ is an a-covering of size 1.

Let $r > a$ and assume that the result holds for rank $r-1$. Let x be a non-loop element of M. Then $r(M/x) = r-1$ and by induction $\tau_a(M/x) \leqslant \binom{b-1}{a}^{r-1-a}$. Note that $\tau_{a+1}(M) \leqslant \tau_a(M/x)$ and, by Lemma 2.5, $\tau_a(M) \leqslant \binom{b-1}{a}\tau_{a+1}(M)$. Therefore $\tau_a(M) \leqslant \binom{b-1}{a}^{r-a}$, as required. □

For a-simple matroids, the size is proportional to τ_a:

Lemma 2.7. *There exists an integer-valued function $\sigma(a)$ such that, if $a \geqslant 1$ and M is a-simple, then $|E(M)| \leqslant \sigma(a)\tau_a(M)$.*

Proof. Define σ by

$$\sigma(a) = \prod_{k=2}^{a} \binom{2k-1}{k-1}.$$

Since M has no $U_{k,2k}$-restriction for $k = 2, \dots, a$, Lemma 2.5 gives

$$\tau_{k-1}(M) \leqslant \binom{2k-1}{k-1}\tau_k(M), \quad k = 2, \dots, a.$$

Putting these together, we get $|E(M)| = \tau_1(M) \leqslant \sigma(a)\tau_a(M)$. □

Lemma 2.8. *There exists an integer-valued function $\epsilon(n,r,a,b)$ such that, for $a,b,n,r \in \mathbb{N}$ with $b > a$, if M is a rank-r matroid with no $U_{a+1,b}$-minor and $|E(M)| \geqslant \epsilon(n,r,a,b)$, then there exists an n-element set $X \subseteq E(M)$ such that $r_M(X) \leqslant a$ and $M|X$ is uniform.*

Proof. Let $m_0 = n$ and, for each $i = 1, \dots, n$, let $m_i = \binom{n-1}{i-1}m_{i-1} + 1$. Now let $l = \binom{b-1}{a}^{r-a}$ and let $\epsilon(n,r,a,b) = m_a l$. Let M be a rank-r matroid with no $U_{a+1,b}$-minor and with $|E(M)| \geqslant \epsilon(n,r,a,b)$. By Theorem 2.6, M has an a-cover $(X_1, \dots, X_l)$. Since $|E(M)| \geqslant m_a l$, we may assume that $|X_1| \geqslant m_a$. Let $a' \leqslant a$ be minimum such that there exists a rank-a' set $X \subseteq X_1$ with $|X| = m_{a'}$. If $a' = 0$, then $M|X \cong U_{0,n}$, so we may assume that $a' > 0$. Now $m_{a'} > \binom{n-1}{a'-1}m_{a'-1}$ so, $\tau_{a'-1}(M|X) > \binom{n-1}{a'-1}$. Then, by Lemma 2.5, $M|X$ contains a $U_{a',n}$-restriction. □

3. Building density

The first step in the proof of the main theorem is to show that a matroid of large enough rank has either k disjoint cocircuits or a large minor that is nearly round.

Lemma 3.1. *Let $g : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function. There exists a function $f_g : \mathbb{N} \to \mathbb{N}$ such that, for any $k \in \mathbb{N}$, if M is a matroid with $r(M) \geqslant f_g(k)$, then either*

(a) *M has k disjoint cocircuits or*
(b) *M has a minor $N = M/Y$ with $r(N) \geqslant g(\Gamma(N))$.*

Proof. Let g be given and define f_g as follows: $f_g(0) = f_g(1) = 1$ and

$$f_g(k) = g\big(f_g(k-1)\big), \quad k \geqslant 2.$$

The proof is by induction on k. If $r(M) \geqslant 1$, then M has a cocircuit, so the result holds for $k = 0, 1$. Now let $k \geqslant 2$ and $r(M) \geqslant f_g(k) = g(f_g(k-1))$.

If $\Gamma(M) \geqslant f_g(k-1)$, then let C be a cocircuit of M with $\mathrm{def}_M(C) = \Gamma(M)$. Then $r(M/C) = \mathrm{def}_M(C) \geqslant f_g(k-1)$. If M/C has the desired contraction minor, then we are done. If not, then by induction M/C has $k-1$ disjoint cocircuits. These, together with C, give k disjoint cocircuits of M.

If $\Gamma(M) \leqslant f_g(k-1)$, then as g is non-decreasing, we have $r(M) \geqslant f_g(k) = g(f_g(k-1)) \geqslant g(\Gamma(M))$. □

Lemma 3.2. *Let M be a simple matroid with no $U_{a+1,b}$-minor, where $b > a \geqslant 1$, and let C be a cocircuit of M of minimum size. If $C_1, \dots, C_k$ are disjoint cocircuits of $M \backslash C$ with $|C_1| \leqslant \dots \leqslant |C_k|$, then $|C_i| \geqslant |C|/\big(a\binom{b-1}{a}\big)$ for each $i \in \{a, \dots, k\}$.*

Proof. Let $i \in \{a, \dots, k\}$. There exist sets $(C'_1, \dots, C'_{a-1})$ such that, for each $j \in \{1, \dots, a-1\}$, we have $C'_j \subseteq C_j$ and the set C'_j is a cocircuit of $M \backslash (C \cup C_i \cup C'_1 \cup C'_2 \cup C'_{j-1})$. Let $F = E(M) - (C \cup C_i \cup C'_1 \cup \dots \cup C'_{a-1})$ and $N = M/F$. Deleting a cocircuit of a matroid drops its rank by 1, so $\mathrm{def}_M(F) = a+1$ and, hence, $r(N) = a+1$. By Lemma 2.5, $\tau_a(N) \leqslant \binom{b-1}{a}$. Moreover, C is a cocircuit of minimum size in N, so each rank-a flat of N has size at most $|E(N) - C| \leqslant |C_1 \cup \dots \cup C_{a-1} \cup C_i| \leqslant a|C_i|$. Hence $|C| \leqslant |E(N)| \leqslant a|C_i|\binom{b-1}{a}$, as required. □

The following lemma is the main result of this section.

Lemma 3.3. *There exists an integer-valued function $\delta(\lambda, a, b)$ such that: for any $a, b, \lambda \in \mathbb{N}$ with $b > a$, if M is a matroid with no $U_{a+1,b}$-minor such that $\Gamma(M) \leqslant \frac{1}{2}r(M)$, $r(M) \geqslant \delta(\lambda, a, b)$, then M has a minor N with $\tau_a(N) > \lambda r(N)$.*

Proof. We define the value $\delta(\lambda, a, b)$ using the functions σ and f_{g_n} defined in Lemmas 2.7 and 3.2. Let $\kappa = 2a\binom{b-1}{a}\sigma(a)\lambda + a - 1$. Now define a sequence of functions $g_n : \mathbb{N} \to \mathbb{N}$. Let $g_0(m) = 0$, and for $n \geqslant 1$ define g_n recursively by

$$g_n(m) = \max(2m, \delta_n), \quad \text{where}$$
$$\delta_n = 2\big(f_{g_{n-1}}(\kappa + 1)\big).$$

Finally, let $\delta(\lambda, a, b) = \delta_{n_0}$ where $n_0 = 2\sigma(a)\lambda$.

We first prove the following claim.

Claim. *For any $n \geqslant 0$, if M is a matroid with no $U_{a+1,b}$-minor such that $r(M) \geqslant g_n(\Gamma(M))$, then either*

(i) *M has a minor N with $\tau_a(N) > \lambda r(N)$, or*
(ii) *there is a contraction-minor N of M and a collection of sets $C_1, \dots, C_n \subseteq E(N)$ such that, for each $i \in \{1, \dots, n\}$, the set C_i is a spanning cocircuit of $N \backslash (C_1 \cup \dots \cup C_{i-1})$.*

Proof of Claim. Observe that, we lose no generality in replacing *contraction-minor* with *minor* in outcome (ii). We will prove this weaker version of the claim by induction on n. The case $n = 0$ is trivial, so assume $n \geqslant 1$ and that the result holds for $n - 1$. Note that, by Lemma 2.3 and by possibly deleting elements from M, we may assume that M is a-simple.

Let C_1 be a minimum size cocircuit of M, let Y be a basis of M/C_1, and let $M_1 = M/Y \backslash C_1$. Thus C_1 is a spanning cocircuit of M/Y. Then

$$r(M/Y) = r_M(C_1) \geqslant r(M) - \Gamma(M) \geqslant \frac{1}{2}r(M) \geqslant \frac{1}{2}g_n\big(\Gamma(M)\big) \geqslant \frac{1}{2}\delta_n.$$

Now

$$r(M_1) = r(M/Y) - 1 \geqslant \frac{1}{2}g_n\big(\Gamma(M)\big) - 1 = f_{g_{n-1}}\big(\delta(\lambda, a, b)\kappa\big).$$

So, by Lemma 3.1, we have one of the following two cases.

Case 1. M_1 has κ disjoint cocircuits.

Every cocircuit of $M_1 = M \backslash C_1 / Y$ is a cocircuit of $M \backslash C_1$, so $M \backslash C_1$ has κ disjoint cocircuits, say $C^*_1, \dots, C^*_\kappa$. We may assume that $|C^*_1| \leqslant \dots \leqslant |C^*_\kappa|$. Since $\Gamma(M) \leqslant \frac{1}{2}r(M)$, we have $|C_1| \geqslant r_M(C_1) \geqslant \frac{1}{2}r(M)$. By Lemmas 2.7 and 3.2,

$$\sigma(a)\tau_a(M) \geqslant \big|E(M)\big| > \big|C^*_a\big| + \dots + \big|C^*_\kappa\big| \geqslant (\kappa - a + 1)\frac{r(M)}{2a\binom{b-1}{a}} = \sigma(a)\lambda r(M),$$

as required.

Case 2. M_1 has a contraction-minor M_2 with $r(M_2) \geqslant g_{n-1}(\Gamma(M_2))$.

In this case the claim easily follows by applying the induction hypothesis to M_2. □

We are now ready to prove the lemma. Note that, by Lemma 2.3 and by possibly deleting elements from M, we may assume that M is a-simple. By the claim, either we are done or we find a minor N of M and a sequence of sets $C_1, \dots, C_{n_0} \subseteq E(N)$ such that, for each $i \in \{1, \dots, n_0\}$, the set C_i is a spanning cocircuit of $N \backslash (C_1 \cup \dots \cup C_{i-1})$. We may assume that each of the sets $C_1, \dots, C_{n_0}$ is independent in N and that $r(N) = n_0$. Therefore

$$\left|E(N)\right| = n_0 + (n_0 - 1) + \dots + 1 = \frac{n_0 + 1}{2} r(N).$$

We claim that N is a-simple. Consider any restriction $N|W$ of N, and let i be minimum such that $W \cap C_i$ is non-empty. Note that $C_i \cap W$ contains a cocircuit of $N|W$. Then $N|W$ has an independent cocircuit. However $U_{k,2k}$ does not have an independent cocircuit, and hence N is a-simple as claimed.

Finally, by Lemma 2.7,

$$\sigma(a)\tau_a(N) \geqslant \left|E(N)\right| > \frac{n_0}{2} r(N) = \sigma(a)\lambda r(N),$$

and the result follows. □

4. Arranging circuits

In this section we derive technical “Ramsey-like” results concerning arrangements of low rank sets in a matroid.

A collection $(A_1, \dots, A_n)$ of sets in a matroid M is *skew* if

$$r_M(A_1) + \dots + r_M(A_n) = r_M(A_1 \cup \dots \cup A_n).$$

A *book* in M is a pair $(F, \mathcal{A})$ such that the sets $(A - F\colon\ A \in \mathcal{A})$ are skew in M/F. We are interested in books where $r_M(F)$ is small.

Lemma 4.1. *There exists an integer-valued function $\alpha_1(n, r, a, b)$ such that: for any $a, b, r, n \in \mathbb{N}$ with $b > a$, if M is a matroid with no $U_{a+1,b}$-minor, $\mathcal{F}$ is a collection of sets of rank at most r in $E(M)$, and $|\mathcal{F}| \geqslant \alpha_1(n, r, a, b)$, then M contains a book $(F, \mathcal{A})$ such that $r_M(F) \leqslant ar^2$, $|\mathcal{A}| = n$, and $\mathcal{A} \subseteq \mathcal{F}$.*

Proof. We begin by recalling Ramsey's Theorem (see [9] or [1, 9.1.4]). *There exists an integer-valued function $R(n, c, k)$ such that, for any $n, c, k \in \mathbb{N}$, if X is a set of size $R(n, c, k)$ and we assign each n-element subset of X one of c colours, then there is a k-element subset Y of X such that each n-element subset of Y receives the same colour.*

Let n, r, a, b be as given. Now, let $s_r = 0$, $l_r = n$, and, for $i = r-1, r-2, \dots, 1$, we recursively define

$$s_i = s_{i+1} + l_{i+1}, \qquad u_i = \binom{b-1}{a}^{rs_i - a}, \qquad l_i = n\binom{u_i}{r-i}.$$

Let $m = s_1 + l_1$. So, we have $0 = s_r < s_{r-1} < \dots < s_1 < m$. Next, define numbers $k_0, \dots, k_m$ as follows. Now let $k_m = m$ and, for $i = m, m-1, \dots, 1$, we recursively define $k_{i-1} = R(i, r, k_i)$. Finally, let $\alpha_1(n, r, a, b) = rk_1$.

Let M and $\mathcal{F}$ be as given. Let $\mathcal{F}_0 = \mathcal{F}$ and $a_0 = 0$. We shall iteratively construct sequences

$$\mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \dots \supseteq \mathcal{F}_m \quad \text{and} \quad a_0 < a_1 < a_2 < \dots < a_m$$

such that, for $i = 1, \dots, m$, $|\mathcal{F}_i| = k_i$, and if $\mathcal{F}' \subseteq \mathcal{F}_i$ with $|\mathcal{F}'| = i$, then $r_M(\mathcal{F}') = a_i$. This clearly holds for $\mathcal{F}_0$. Let $i \geqslant 1$, assume that $\mathcal{F}_{i-1}$ and a_{i-1} satisfy the above. Note that $r_M(\mathcal{F}') \in \{a_{i-1} + 1, \dots, a_{i-1} + r\}$, for any i-element subset $\mathcal{F}' \subseteq \mathcal{F}_{i-1}$. This defines an r-colouring of the i-element subsets of $\mathcal{F}_{i-1}$. Since $|\mathcal{F}_{i-1}| = k_{i-1} = R(i, r, k_i)$, there exists $\mathcal{F}_i \subseteq \mathcal{F}_{i-1}$ such that every i-element subset of $\mathcal{F}_i$ has the same rank, say a_i.

For each $i = 1, \dots, m$, let $b_i = a_i - a_{i-1}$. Notice that, by submodularity, this gives a non-increasing sequence

$$r \geqslant b_1 \geqslant b_2 \geqslant \cdots \geqslant b_m \geqslant 1.$$

Hence, by definition of the pairs (s_i, l_i), there exists an $r' \in \{1, \dots, r\}$, such that $b_{s+1} = \cdots = b_{s+l} = r'$, where $s = s_{r'}$ and $l = l_{r'}$. If $r' = b_1$, then any collection of n members $X_1, \dots, X_n \in \mathcal{F}_m$ will be skew and, hence, $(\emptyset, \{X_1, \dots, X_n\})$ is a book. Therefore we may assume that $r' < b_1$.

Choose distinct sets $Z_1, \dots, Z_s$, $X_1, \dots, X_l \in \mathcal{F}_m$ and let $F = Z_1 \cup \cdots \cup Z_s$. Since $b_{s+1} = b_{s+l} = r'$, we have a book $(F, \{X_1, \dots, X_l\})$ of width $\leqslant sr$. For $i = 1, \dots, l$, choose a maximal independent set $\overline{B}_i \subseteq X_i$ that is skew to F, and expand this set to a basis $B_i \cup \overline{B}_i$ of X_i in M. Thus $|\overline{B}_i| = r'$ and $|B_i| = r - r'$.

Let $M' = M/(\overline{B}_1 \cup \cdots \cup \overline{B}_s)$ and $B = B_1 \cup \cdots \cup B_s$. Then $B_i \subseteq \mathrm{cl}_{M'}(F)$, and thus $r_{M'}(B) \leqslant r_{M'}(F) \leqslant sr$. Let $(W_1, \dots, W_u)$ be a minimal a-covering of $M'|B$. By Theorem 2.6, we have

$$u = \tau_a(M'|B) \leqslant \binom{b-1}{a}^{sr-a} = u_{r'}.$$

For each $i \in \{1, \dots, l\}$, choose an r-element subset $I_i \subseteq \{1, \dots, u\}$ such that $B_i \subseteq \cup_{j \in I_i} W_j$. There are $\binom{u}{r} \leqslant \binom{u_{r'}}{r}$ possible choices for I_i, and $l = n\binom{u_{r'}}{r}$. By a majority argument, there exists an n-element set $J \subseteq \{1, \dots, l\}$ and an r-element set $I \subseteq \{1, \dots, u\}$ such that $I_i = I$ for each $i \in J$. By possibly re-ordering the X_i's and the W_j's we can assume that $B_1, \dots, B_n \subseteq W_1 \cup \cdots \cup W_r$. Let $W = W_1 \cup \cdots \cup W_r$. Then $(W, \{X_1, \dots, X_n\})$ is a book in M' and $r_{M'}(W) \leqslant ra$. Let W' be a basis of $M'|W$. Each element in W' is contained in one of the sets $X_1, \dots, X_l$, then there is a set X_0 that is the union of at most $|W'|$ of the sets $(X_1, \dots, X_l)$ such that $W' \subseteq X_0$. Then, it is easy to verify that $(X_0, \{X_1, \dots, X_n\})$ is a book in M and $r_M(X_0) \leqslant ar^2$. □

The following lemma refines the outcome of Lemma 4.1 in the case that $\mathcal{F}$ is a collection circuits.

Lemma 4.2. *There exists an integer-valued function $\alpha_2(l, m)$ such that: for any $l, m, n \in \mathbb{N}$ with $n \geqslant \alpha_2(l, m)$, if $(F, \{C_1 \dots, C_n\})$ is a book in a matroid M where*

(a) *$C_1, \dots C_n$ are circuits of M,*
(b) *$1 \leqslant r_M(F \cup C_i) - r_M(F) < r_M(C_i)$ for all i, and*
(c) *$r_M(F) \leqslant m$,*

then M has an $M(K_{2,l})$-minor where the series classes are contained in distinct sets in $(C_1, \dots, C_n)$.

Proof. Let $\alpha_2(l, m) = m(l+1)$. We prove the result by induction on m. The result is easy when $m = 1$. Then we assume that $m > 1$ and that the result holds for smaller m.

Consider a book $(F, \{C_1, \dots, C_n\})$ satisfying the hypotheses. For each $i \in \{1, \dots, n\}$, by possibly contracting some elements of C_i, we may assume that $r_M(C_i \cup F) = r_M(F) + 1$.

Choose elements $e, f \in C_n - \mathrm{cl}_M(F)$. Then, for each $i \in \{1, \dots, n-1\}$, choose an element $z_i \in C_i - \mathrm{cl}_M(F)$ and let $C'_i \subseteq C_i$ be a circuit of $M/\{e, f\}$ that contains z_i. Note that $n - 1 = l + n'$ where $n' \geqslant \alpha_2(l, m-1)$. Then by possibly reordering $C'_1, \dots, C'_{n-1}$, we have one of the two following cases.

Case 1. $|C'_1| = \cdots = |C'_l| = 2$.

Then the restriction of $M/f\backslash e$ to $C'_1 \cup \cdots \cup C'_n$ is isomorphic to $M(K_{2,l})$.

Case 2. $|C'_1|, \dots, |C'_{n'}| > 2$.

Note that $r_{M/e,f}(F) \leqslant m - 1$, so the result follows by induction. □

The following result is a direct corollary of Lemmas 4.1 and 4.2; we skip the proof.

Lemma 4.3. *There exists an integer-valued function $\alpha_3(s,l,a,b)$ such that: for $a,b,l,s \in \mathbb{N}$ with $b > a$, if M is a matroid with no $U_{a+1,b}$-minor and $\mathcal{C}$ is a set of circuits of M of rank at most $a+1$, with $r_M(\mathcal{C}) \geqslant \alpha_3(s,l,a,b)$, then either*

(i) *there exist s skew circuits $C_1,\dots,C_s \in \mathcal{C}$, or*
(ii) *M has an $M(K_{2,l})$-minor where the series classes are contained in distinct sets in $\mathcal{C}$.*

5. Building a nest

A *point* in a matroid is a rank-1 flat and a *line* is a rank-2 flat; we call a line *long* if it contains at least 3 points. We shall assemble many long lines in a clique-like structure. We first build intermediate structures called *nests*.

Definition 5.1. A matroid M is a *nest* if M has a basis $B = \{b_1,\dots,b_n\}$ such that, for each pair of indices $i,j \in \{1,\dots,n\}$, with $i < j$, the set $\{b_i,b_j\}$ spans a long line in $M/\{b_1,\dots,b_{i-1}\}$. The elements in B are called the *joints* of the nest M.

It is easy to verify that $M(K_n)$ is a nest; take the edges incident to a fixed vertex of K_n as the joints.

For $t \in \mathbb{N}$ we say that M is *t-round* if $\Gamma(M) \leqslant t$. Note that t-roundedness is preserved under contractions. The main result of this section is the following.

Lemma 5.2. *There exists an integer-valued function $\nu(n,t,a,b)$ such that: for any $a,b,n,t \in \mathbb{N}$ with $a > b$, if M is a t-round matroid with no $U_{a+1,b}$-minor and $r(M) \geqslant \nu(n,t,a,b)$, then M has a rank-n nest as a minor.*

We obtain a nest by finding one joint at a time using the next lemma.

Lemma 5.3. *There exists an integer-valued function $\nu_1(m,t,a,b)$ such that: for any $a,b,m,t \in \mathbb{N}$ with $b > a$, if M is a t-round matroid with no $U_{a+1,b}$-minor, $r(M) \geqslant \nu_1(m,t,a,b)$ and B is a basis of M, then M has an $M(K_{2,m})$-minor such that each series class contains an element of B.*

We start by deriving Lemma 5.2 from Lemma 5.3.

Proof of Lemma 5.2. Let t be fixed. Let $\nu(1,t,a,b) = 1$ and for $n \geqslant 2$ define ν recursively by

$$\nu(n,t,a,b) = \nu_1\bigl(\nu(n-1,t,a,b)+2,t,a,b\bigr).$$

To facilitate induction we prove the stronger statement:

If M is a t-round matroid with no $U_{a+1,b}$-minor, $r(M) \geqslant \nu(n,t,a,b)$ and B is a basis of M, then M has a rank-n nest M/Y as a minor, with joints contained in B.

The proof is by induction on n. For $n = 1$ the result is trivial, as any rank-1 matroid is a nest. Let $n \geqslant 2$ and assume the result holds for $n-1$. Let M and B be given as above and let $m = \nu(n-1,t,a,b)+2$.

By Lemma 5.3, M has an $M(K_{2,m})$-minor such that each series class contains an element of B. Let $\{b_1,e\}$ be one of the series classes of $M(K_{2,m})$ with $b_1 \in B$ and let $B' \subseteq B$ be an $(m-1)$-element set containing an element from each of the other series-classes of $M(K_{2,m})$. We may assume that $M(K_{2,m})$ is a spanning restriction of M/Y_1.

Let $N_1 = M/(Y_1 \cup \{e\})$ and $N_1' = N_1/b_1$. Since t-roundness is preserved under contractions, N_1' is t-round. Moreover $r(N_1') = \nu(n-1,t,a,b)$ so, by induction, N_1' has a rank-$(n-1)$ nest N_2 as a minor with joints $B_2 \subseteq B_1 - b_1$.

We may assume that $N_2 = N_1'/Y_2$. Now let $Y = Y_1 \cup Y_2$ and $N = M/Y$, so $N_2 = N/b_1$. It is easy to verify that N is a nest with joints $\{b_1\} \cup B_2$. $\square$

A set $X \subseteq E(M)$ is *connected* if $M|X$ is connected. We denote by $\tau_a^c(M)$ the minimum m such that there exists a collection $(X_1, \dots, X_m)$ of connected sets of rank at most a in M such that $X_1 \cup \cdots \cup X_m$ contains all non-loop elements of M. Clearly $\tau_a^c(M) \geqslant \tau_a(M)$. Note also, that a loopless rank-a matroid M has at most a connected components, so $\tau_a^c(M) \leqslant a\tau_a(M)$. Therefore

$$\tau_a(M) \leqslant \tau_a^c(M) \leqslant a\tau_a(M).$$

We need a technical lemma before we prove Lemma 5.3.

Lemma 5.4. *Let M be a matroid with no $U_{a+1,b}$-minor, where $b > a$, let $e \in E(M)$, and let $\mathcal{F}$ be the collection of all connected rank-$(a+1)$ sets in M containing e. If $r_M(\mathcal{F}) = n$, then*

$$\tau_a^c(M) - \tau_a^c(M/e) \leqslant a^2 \binom{b-1}{a}^{n-a} + 1.$$

Proof. We may assume that M is simple. Let $(X_1, \dots, X_k)$ be a minimal a-covering of M/e by connected sets. We shall construct an a-covering of M by connected sets. For each $i \in X_i$, either $r_M(X_i) = r_{M/e}(X_i)$ or $r_M(X_i) = r_{M/e}(X_i) + 1$. If $r_M(X_i) = r_{M/e}(X_i)$, then X_i is a connected set in M with rank at most a; let $X_i' = X_i$. If $r_M(X_i) = r_{M/e}(X_i) + 1$, then $X_i \cup \{e\}$ is a connected set in M with rank at most $a + 1$; let $X_i' = X_i \cup \{e\}$.

By possibly reordering the sets, we may assume that $X_1, \dots, X_m$ have rank $a + 1$ in M and that $X_{m+1}, \dots, X_k$ have rank at most a. By Lemma 2.5, for $i = 1, \dots, m$, we have

$$\tau_a^c\big(M|X_i'\big) \leqslant a\tau_a\big(M|X_i'\big) \leqslant a\binom{b-1}{a}.$$

Therefore

$$\tau_a^c(M) \leqslant ma\binom{b-1}{a} + (k-m) + 1 \leqslant ma\binom{b-1}{a} + \tau_a^c(M/e) + 1.$$

We may assume that $m \geqslant 1$. Let $M' = (M/e)|(X_1 \cup \cdots \cup X_m)$. Note that $(X_1, \dots, X_m)$ is a minimal a-covering of M' by connected sets and that $r(M') \leqslant n - 1$. Hence, by Theorem 2.6,

$$m = \tau_a^c(M') \leqslant a\tau_a(M') \leqslant a\binom{b-1}{a}^{n-1-a}.$$

Now the result follows by combining the two inequalities displayed above. $\square$

Let M be a matroid, $k \in \mathbb{N}$ and let $B \subseteq E(M)$. We say that B *k-dominates* M, if for any element $x \in E(M)$ there is a set $W \subseteq B$ with $|W| \leqslant k$ such that $x \in \mathrm{cl}_M(W)$. A k-dominating set clearly has to be spanning. It is easily verified that: if $B, Y \subseteq E(M)$ and B k-dominates M, then $B - Y$ k-dominates M/Y.

Proof of Lemma 5.3. Let m, t, a and b be given, and define the following constants,

$$r_4 = \alpha_3(m+1, m, a, b), \qquad l = m + r_4, \qquad r_3 = \alpha_3(2, l+1, a, b),$$
$$\lambda = a^2\binom{b-1}{a}^{r_3-a} + 1, \qquad r_1 = \max\big(2t, \delta(\lambda, a, b)\big),$$

and let us define $\nu_1(m, t, a, b) = \nu_1 = \sigma(a)\binom{b-1}{a}^{r_1-a}$. Let M and B be as given.

By Lemma 2.2 and the fact that B is a basis, there is an a-simple spanning restriction of M that contains B. Let N_1 be a minimal minor of M such that N_1 is t-round and a-simple, and $B \subseteq E(N_1)$.

We claim that B $(a+1)$-dominates N_1. Consider any element $f \in E(N_1) - B$. Since N_1 is t-round, N_1/f is too. By our choice of N_1, the minor $(N_1/f)|B$ cannot be a-simple. Since N_1 is simple, N_1/f is loopless. Then, since $(N_1/f)|B$ is not a-simple, there is a set $W \subseteq B$, with $(N_1/f)|W \cong U_{k,2k}$, for some $k \in \{1,\dots,a\}$. Then $r_{N_1}(W) \leqslant a+1$. However, since N_1 is a-simple, $N_1|W \neq (N_1/e)|W$ and, hence, $e \in \mathrm{cl}_{N_1}(W)$. Thus B $(a+1)$-dominates N_1 as claimed.

By Lemma 2.7, we have

$$\sigma(a)\tau_a(N_1) \geqslant \left|E(N_1)\right| \geqslant |B| = r(M) \geqslant \nu_1,$$

and so, $\tau_a(N_1) \geqslant \binom{b-1}{a}^{r_1-a}$. Then, by Theorem 2.6, we have $r(N_1) \geqslant r_1$.

By the definition of r_1, we have $\Gamma(N_1) \leqslant t \leqslant \frac{1}{2}r(N_1)$ and $r(N_1) \geqslant \delta(\lambda,a,b)$. Then, by Lemma 3.3, there is a minor N_2 of N_1 with $\tau_a(N_2) > \lambda r(N_2)$. We may assume that $N_2 = N_1/Y_1$. Thus $\tau_a^c(N_2) > \lambda r(N_2)$. Let $Y_2 \subseteq E(N_2)$ be maximal such that

$$\tau_a^c(N_2/Y_2) > \lambda r(N_2/Y_2),$$

and let $N_3 = N_2/Y_2$. Since Y_2 was chosen to be maximal, N_3 is loopless. Choose any element $e \in E(N_3)$. Then,

$$\tau_a^c(N_3) - \tau_a^c(N_3/e) > \lambda r(N_3) - \lambda r(N_3/e) = \lambda.$$

Let $\mathcal{F}$ denote the collection of all connected rank-$(a+1)$ sets in N_3 containing e, and let $n = r_{N_3}(\mathcal{F})$. By Lemma 5.4, we have $\lambda < a^2\binom{b-1}{a}^{n-a} + 1$, and, by the definition of λ, this yields $n \geqslant r_3$.

Denote by $\mathcal{C}$ the collection of all circuits of N_3 of rank at most $a+1$ containing e. For each $X \in \mathcal{F}$ and non-loop $y \in X - \{e\}$, since X is connected, there exists a circuit $C \subseteq X$ containing e and y, so $C \in \mathcal{C}$. Hence, $r_{N_3}(\mathcal{C}) \geqslant n$.

Note that $n \geqslant r_3 = \alpha_3(2, l+1, a, b)$ and that no two circuits in $\mathcal{C}$ are skew. Then, by Lemma 4.3, there is an $M(K_{2,l+1})$-minor of N_3. Let $\{e,f\}, \{h_1,h_1'\}, \dots, \{h_l,h_l'\}$ be the series classes of $M(K_{2,l})$. Since $l = m + r_4$, by possibly reordering, we may assume that none of $h_1,\dots,h_{r_4}$ is contained in B.

We may assume that $M(K_{2,l})$ is a spanning restriction of N_3/Y_3; let $N_4 = N_3/(Y_3 \cup \{f\})$. By the remark preceding the proof, $B \cap E(N_4)$ $(a+1)$-dominates N_4. So, for each $i \in \{1,\dots,r_4\}$, h_i is in the closure of a subset of B of rank at most $a+1$. Choose a circuit C_i of N_4 containing h_i, with $r_{N_4}(C_i) \leqslant a+1$ and $C_i \subseteq B \cup \{h_i\}$. Since $\{h_1,\dots,h_{r_4}\}$ is independent, $r_{N_4}(\cup_i C_i) \geqslant r_4 = \alpha_3(m+1, m, a, b)$. Then, by Lemma 4.3, we get one of the following two cases.

Case 1. There are $m+1$ skew circuits among $C_1 \dots, C_{r_4}$ in N_4.

The union of some m of these $m+1$ circuits is skew to the set $\{e\}$ in N_4. After possibly reordering we may assume $C_1,\dots,C_m,\{e\}$ are skew. Now N_4 restricted to the union of the sets $((C_i - \{h_i\}) \cup \{h_i'\}$: $i = 1,\dots,m)$ is isomorpic to the cycle matroid of a subdivision of $M(K_{2,m})$; moreover each of the series classes contains an element of B. Thus we obtain the required $M(K_{2,m})$-minor.

Case 2. There is an $M(K_{2,m})$ minor of N_4 such that each series class of $M(K_{2,m})$ is contained in one of $C_1 \dots, C_{r_4}$.

In this case we are done since each of $C_1,\dots,C_{r_4}$ contains at most one element not in B. □

6. Cleaning a nest

The goal of this section is to further refine nests. A *Dowling clique* is a matroid M with ground set $\{b_1,\dots,b_n\} \cup \{e_{ij}\colon\ 1 \leqslant i < j \leqslant n\}$ such that $\{b_1,\dots,b_n\}$ is a basis and, for each $1 \leqslant i < j \leqslant n$, the set $\{b_i,b_j,e_{ij}\}$ is a triangle. We call the elements $b_1,\dots,b_n$ the *joints* of M. These matroids are related to Dowling Geometries [2].

The proof of the following theorem is based on ideas introduced by Kung [5].

Lemma 6.1. *There exists an integer-valued function $\phi(n,a,b)$ such that: for any $a,b,n \in \mathbb{N}$ with $b > a$, if M is a nest of rank at least $\phi(n,a,b)$ with no $U_{a+1,b}$-minor, then M contains a rank-n Dowling clique as a minor.*

Proof. Recall that α_1 and ϵ are defined in Lemmas 4.1 and 2.8. Let $t_2 = (2a)^n n$, $t_1 = t_2 + a$, $s_1 = n + a^3$, and $s = \alpha_1(s_1, a, a, b)$. Now, let $w_s = t_2$ and, for each $i = s, \dots, 1$, let $w_{i-1} = \epsilon(w_i, s, a, b)$. Let $t_0 = w_0$ and, finally, let $\phi(n,a,b) = s + t_0$.

Let M be a rank-$(s+t_0)$ nest. We start the proof by moving to a different structure whose labeling is disassociated with that of the nest; this will allow us the freedom to relabel later.

Claim 1. *There is a restriction M_0 of M and a partition $(A, C_1^*, \dots, C_{t_0}^*)$ such that $r_{M_0}(A) = s$ and, for each $i \in \{1,\dots,t_0\}$, the set C_i^* is an independent $(s+1)$-element cocircuit of M_0 that spans A.*

Proof of Claim. Let the joints be $a_1,\dots,a_s,b_1,\dots,b_{t_0}$. For each $i \in \{1,\dots,s\}$ and $j \in \{1,\dots,t_0\}$, there is an element $e_{ij} \in E(M)$ and a circuit C_{ij} of M such that $a_i, b_j, e_{ij} \in C_{ij} \subseteq \{a_1,\dots,a_i\} \cup \{b_j, e_{ij}\}$. Now let $A = \{a_1,\dots,a_s\}$ and, for each $j \in \{1,\dots,t_0\}$, let $C_j^* = \{e_{1j},\dots,e_{sj}\} \cup \{b_j\}$. Finally let M_0 be the restriction of M to the union of $(A, C_1^*, \dots, C_s^*)$. It is straightforward to verify that the claim is satisfied by M_0, A, and $C_1^*,\dots,C_{t_0}^*$. $\square$

For each $i \in \{1,\dots,t_0\}$, let $C_i^* = \{e_{1j},\dots,e_{sj}\} \cup \{b_j\}$. Let $B = \{b_1,\dots,b_{t_0}\}$ and let $M_1 = M/B$. Note that A is a basis of M_1 and, hence, $r(M_1) = s$. For each $i \in \{1,\dots,s\}$ and $j \in \{1,\dots,t_0\}$, let $S_{ij} = \{e_{ik}\colon k = 1,\dots,j\}$. Note that $t_0 = w_0$ and, for each $i \in \{1,\dots,s\}$, $w_{i-1} = \epsilon(w_i, s, a, b)$. Therefore, by Lemma 2.8 and possibly reordering, we may assume that: *for each $i = 1,\dots,s$, $r_{M_1}(S_{iw_i}) \leqslant a$ and $M_1|S_{iw_i}$ is uniform.* Now $t_1 = w_s$, so:

Claim 2. *For each $i = 1,\dots,s$, $r_{M_1}(S_{it_1}) \leqslant a$ and $M_1|S_{it_1}$ is uniform.*

Recall that $t_1 = t_2 + a$. Let $M_2 = M/\{b_{t_2+1},\dots,b_{t_2}\}$ and, for each $i \in \{1,\dots,a\}$, let $X_i = \{e_{(t_2+i)k}\colon k = 1,\dots,j\}$. Note that, for each $i = 1,\dots,s_0$, $X_i \subseteq \mathrm{cl}_{M_2}(A)$, $r_{M_2}(X_i) \leqslant a$, and X_i spans the uniform matroid $M_1|S_{it_1}$.

By Lemma 4.1 and by possibly reordering, we may assume that:

Claim 3. *There is a set $F \subseteq X_1 \cup \dots \cup X_s$ with $r_{M_2}(F) \leqslant a^3$ such that $(F, \{X_1,\dots,X_{s_1}\})$ is a book in M_2.*

Note that $\{e_{t_1 1},\dots,e_{t_1 s}\}$ is an independent set of M_2 that spans A and $e_{t_1 i} \in X_i$, for each $i \in \{1,\dots,s\}$. Therefore F spans at most a^3 of the sets $(X_1,\dots,X_{s_1})$. Now, $s_1 = n + a^3$, so, by possibly reordering, we may assume that F spans none of $(X_1,\dots,X_n)$. For each $i \in \{1,\dots,n\}$, choose a maximal independent set $Y_i \subseteq X_i$ in M_2/F.

Recall that, for each $i \in \{1,\dots,n\}$, X_i spans $M_1|S_{it_1}$. Moreover, X_i is not spanned by F. Therefore, since $M_1|S_{it_1}$ is a uniform matroid of rank at most a, F spans fewer than a points in $M_1|S_{it_1}$. For each $j \in \{1,\dots,t_2\}$, if $e_{ij} \notin \mathrm{cl}_{M_1}(F)$, then there is a circuit C in M_2/F such that $b_j, e_{ij} \in C \subseteq Y_1 \cup \{b_j, e_{ij}\}$ and $C \cap Y_i \neq \emptyset$. Now $t_2 = (2a)^n n$ (this number is bigger than necessary), so by a majority argument and possibly reordering, we may assume that:

Claim 4. *For each $i \in \{1,\dots,n\}$ there is an element $f_i \in Y_i$ such that, for each $j \in \{1,\dots,n\}$, there is a circuit C in M_2/F with $b_j, e_{ij}, f_i \in C \subseteq X_1 \cup \{b_j, e_{ij}\}$.*

Let $M_3 = M_2/(F \cup (Y_1 - \{f_1\}) \cup \dots \cup (Y_n - \{f_n\}))$. Then $\{f_1,\dots,f_n\} \cup \{b_1,\dots,b_n\}$ is independent and, for each $i,j \in \{1,\dots,n\}$, the set $\{f_i, e_{ij}, b_j\}$ is a triangle of M_3. Then the restriction of $M_3/\{e_{11},\dots,e_{nn}\}$ to $\{b_1,\dots,b_n\} \cup \{e_{ij}\colon 1 \leqslant i < j \leqslant n\}$ is a Dowling clique. $\square$

7. Cliques

It remains to show that any Dowling clique of sufficiently large rank contains either $M(K_n)$ or $B(K_n)$ as a minor. We need the following theorem of Mader [7].

Mader's Theorem 7.1. *There is an integer valued function $\lambda(n)$ such that: for any $n \in \mathbb{N}$, if G is a simple graph with $|E(M)| > \lambda(n)|V(M)|$, then G has a K_n-minor.*

Let M be a matroid and let $G = (V, E)$ be a loopless graph. We call G a *Dowling representation* of M if $E(M) = V \cup E$, V is a basis of M, and, for each $e \in E$ with ends u and v, the set $\{e, u, v\}$ is a triangle of M. The following lemma helps us to recognize graphic matroids. The result is well-known and can easily be derived from a result of Seymour [10], we omit the proof.

Lemma 7.2. *Let $G = (V, E)$ be a simple connected graph and let M be a matroid. If G is a Dowling representation of M and V is a cocircuit of M, then $M|E = M(G)$.*

We also need to recognize bicircular matroids. The following lemma is also well-known, and again, we skip the proof.

Lemma 7.3. *Let $G = (V, E)$ be a loopless graph and let M be a matroid. If G is a Dowling representation of M and, for each circuit C in G, $E(C)$ is independent in M, then $M|E = B(G)$ (in fact $M = \widetilde{B}(G)$).*

We are ready for the final step in the proof of the main theorem.

Lemma 7.4. *There exists an integer-valued function $\psi(n)$ such that, if M is a Dowling clique with rank at least $\psi(n)$, then M contains an $M(K_n)$- or $B(K_n)$-minor.*

Proof. Let $m = n!$, $l = 2m\lambda(n)$, and $\psi(n) = nl$. Let M be a Dowling clique of rank nl and let $G = (V, E)$ be a Dowling representation of M; thus $G \cong K_{nl}$. Let $T_1, \ldots, T_n$ be vertex disjoint trees of G each having l vertices. For each $1 \leqslant i < j \leqslant n$, let E_{ij} denote the set of all edges of G having one end in $V(T_i)$ and the other end in $V(T_j)$, let G_{ij} be the subgraph of G with vertex set $V(T_i) \cup V(T_j)$ and edge set $E_{ij} \cup E(T_i) \cup E(T_j)$, and let $M_{ij} = M|(V(G_{ij}) \cup E(G_{ij}))$. Thus G_{ij} is a Dowling representation M_{ij}. Note that $M_{ij}/(E(T_i) \cup E(T_j))$ is a loopless matroid with rank 2, and that $V(T_i)$ and $V(T_j)$ are both points of $M_{ij}/(E(T_i) \cup E(T_j))$.

Claim 1. *For each $1 \leqslant i < j \leqslant n$, if $M_{ij}/(E(T_i) \cup E(T_j))$ has at most $m + 2$ points, then M_{ij} contains an $M(K_n)$-minor.*

Proof. Let $M' = M_{ij}/(E(T_i) \cup E(T_j))$. If M' has at most $m + 2$ points, then there is a point $X \subseteq E_{ij}$ of M' with $|X| = l^2/m$. Let G'' be the spanning supgraph of G_{ij} with edge set $E(T_1) \cup E(T_2) \cup X$ and let $M'' = M_{ij}|(V(G'') \cup E(G''))$. Now G'' is connected and $V(G'')$ is a cocircuit in M'' (since it is a cocircuit in $M''/(E(T_i) \cup E(T_j))$). Then, by Lemma 7.2, $M''|E(G'') = M(G'')$. Moreover, $|E(G'')| > |X| \geqslant l^2/m \geqslant \lambda(n)2l = \lambda(n)|V(G'')|$, so, by Mader's Theorem, G'' contains a K_n-minor. □

Now consider the matroid $M/(E(T_1) \cup \cdots \cup E(T_n))$. By Claim 1, we may assume that:

Claim 2. *There exists a simple minor N of M with a basis $B = \{b_1, \ldots, b_n\}$ such that, for each $1 \leqslant i < j \leqslant n$, the elements $\{b_i, b_j\}$ spans an $(m + 3)$-point line in N.*

For each $1 \leqslant i < j \leqslant n$, let $W_{ij} = \mathrm{cl}_N(\{b_i, b_j\}) - \{b_i, b_j\}$. Thus $|W_{ij}| \geqslant m+1$. Let W denote the union of the sets W_{ij}. We may assume that $E(N) = B \cup W$. Now let $H = (B, W)$ be a Dowling representation of N. Note that a simple graph on n vertices has at most $n!$ distinct circuits. Therefore we can build a sequence $(H_0, H_1, \dots, H_{\binom{n}{2}})$ of simple spanning subgraphs of H such that, for each $i \in \{1, \dots, \binom{n}{2}\}$, $|E(H_i)| = i$ and, if C is a circuit of H_i, then $E(C)$ is independent in N. Then, by Lemma 7.3, $N|E(H_{\binom{n}{2}})$ is isomorphic to $B(K_n)$. □

Finally, we restate and prove Theorem 1.1.

Theorem 7.5. *There exists an integer-valued function $\gamma(k,n)$ such that: for any $k, n \in \mathbb{N}$, if M is a matroid with $r(M) \geqslant \gamma(k,n)$, then either M has k disjoint cocircuits or M has a minor isomorphic to $U_{n,2n}$, $M(K_n)$ or $B(K_n)$.*

Proof. Since $M(K_1)$ is trivial, we may assume that $n \geqslant 2$. Recall that the functions ψ, ϕ, ν, and f_g are defined in Lemmas 7.4, 6.1, 5.2, and 3.1. Let $m = \phi(\psi(n), n-1, 2n)$. Now define $g : \mathbb{N} \to \mathbb{N}$ by $g(t) = \nu(m, t, n-1, 2n)$. Finally $\gamma(k,n) = f_g(k)$.

Let M be a matroid such that $r(M) \geqslant \gamma(k,n)$, M has no $U_{n,2n}$-minor and M does not have k disjoint cocircuits. By Lemma 3.1, M has a minor N with $r(N) \geqslant g(\Gamma(N))$. Then, by Lemmas 5.2, 6.1 and 7.4, we obtain an $M(K_n)$- or a $B(K_n)$-minor of N. □

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