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Projective geometries in dense matroids ☆

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ABSTRACT

We prove that, given integers $l, q \geqslant 2$ and n there exists an integer α such that, if M is a simple matroid with no $l+2$ point line minor and at least $\alpha q^{r(M)}$ elements, then M contains a $\mathrm{PG}(n-1, q')$-minor, for some prime-power $q' > q$.

1. Introduction

For a matroid M we let $\varepsilon(M)$ denote the number of *points* of M; that is $\varepsilon(M) = |E(\mathrm{si}(M))|$. We prove the following theorem.

Theorem 1.1. *Let $\mathcal{M}$ be a minor-closed class of matroids. Then either*

(1) $\varepsilon(M) \leqslant r(M)^{c_{\mathcal{M}}}$ *for each* $M \in \mathcal{M}$,
(2) *there is a prime-power q such that* $\varepsilon(M) \leqslant c_{\mathcal{M}} q^{r(M)}$ *for each* $M \in \mathcal{M}$, *and $\mathcal{M}$ contains all* $\mathrm{GF}(q)$*-representable matroids, or*
(3) *$\mathcal{M}$ contains arbitrarily long lines.*

Here $c_{\mathcal{M}}$ is an integer constant depending on $\mathcal{M}$. This result is motivated by the following beautiful conjecture of Kung [4].

Conjecture 1.2 *(Kung's Growth Rate Conjecture). Let $\mathcal{M}$ be a minor-closed class of matroids. Then either*

(1) $\varepsilon(M) \leqslant c_{\mathcal{M}} r(M)$ *for each* $M \in \mathcal{M}$,
(2) $\varepsilon(M) \leqslant c_{\mathcal{M}} r(M)^2$ *for each $M \in \mathcal{M}$ and $\mathcal{M}$ contains all graphic matroids,*

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(3) *there is a prime-power q such that $\varepsilon(M) \leqslant \frac{q^{r(M)}-1}{q-1}$ for each $M \in \mathcal{M}$ with sufficiently high rank, and $\mathcal{M}$ contains all* GF(q)*-representable matroids, or*
(4) *$\mathcal{M}$ contains arbitrarily long lines.*

For a prime-power q, a simple GF(q)-representable matroid M of rank-r has at most $\frac{q^r-1}{q-1}$ elements, as M is isomorphic to a restriction of PG($r-1,q$), which has precisely that many elements. Kung [4] showed that this bound extends from the class of GF(q)-representable matroids to the class of matroids with no $U_{2,q+2}$-minor. For any integer l we let $\mathcal{U}(l)$ denote the class of matroids with no $U_{2,l+2}$-minor.

Theorem 1.3. *Let $l \geqslant 2$ be an integer, and let $M \in \mathcal{U}(l)$ be a rank-r matroid. Then*

$$\varepsilon(M) \leqslant \frac{l^r-1}{l-1}.$$

If l is not a prime-power, the bound is not sharp. As an immediate consequence of Theorem 1.1, we get an asymptotic improvement on the bound in that case:

Corollary 1.4. *Let $l \geqslant 2$ be an integer and let q be the largest prime-power with $q \leqslant l$. There exists a constant c, such that if $M \in \mathcal{U}(l)$ is a rank-r matroid, then $\varepsilon(M) \leqslant cq^r$.*

Kung's conjecture, if true, would imply that the exact bound is $\frac{q^r-1}{q-1}$ for sufficiently large r (this easily fails if $r=2$ and $l>q$). This conjecture has only been verified in the first non-prime-power case $l=6$, see Bonin and Kung [2].

We use the notation of Oxley [5], with the exception that we denote the simplification of a matroid M by si(M). For a subset $A \subseteq E(M)$, we write $\varepsilon_M(A)=\varepsilon(M|A)$.

2. Long lines

Theorem 1.1 is implied by the following two results.

Theorem 2.1. *Let l, n, and q be positive integers with $l \geqslant q \geqslant 2$. There exists an integer α such that, if $M \in \mathcal{U}(l)$ satisfies $\varepsilon(M) \geqslant \alpha q^{r(M)}$, then M contains a* PG($n-1,q'$)*-minor, for some prime-power $q' > q$.*

Using the same techniques, we prove the following theorem. For binary matroids it was proved independently by Sauer [6] and Shelah [7].

Theorem 2.2. *Let l and n be positive integers. Then there exist integers a and m such that, if $M \in \mathcal{U}(l)$ satisfies $\varepsilon(M) > ar(M)^m$, then M contains a* PG($n-1,q'$)*-minor, for some prime-power q'.*

Note that a may be omitted in the statement of the theorem, since the constant can be compensated for by raising the exponent; we keep the constant to facilitate the proof.

Let M be a matroid. A line L of M is a rank-2 flat of M. The *length* of L is the number of points on L, that is $\varepsilon_M(L)$. We call a line L of M *long* if it has length at least 3. For $e \in E(M)$ denote by $\delta_M(e)$ the number of long lines in M containing e. For an integer $q \geqslant 2$, we say that a line L is *q-long*, if L has length at least $q+2$. (The following result is tantamount to Lemma 4.1 in [4].)

Lemma 2.3. *Let l, q and λ be integers with $l \geqslant q \geqslant 2$ and $\lambda \geqslant 1$. If $M \in \mathcal{U}(l)$ is minor minimal with $\varepsilon(M) \geqslant \lambda q^{r(M)}$, then*

$$\delta_M(e) \geqslant \frac{\lambda}{2l} q^{r(M)} \quad \text{for each } e \in E(M),$$

and the number of q-long lines of M is at least $\frac{\lambda}{l+1} q^{r(M)}$.

Proof. Note that, by the minor minimality, M is simple. Consider $e \in E(M)$. Let δ^+ denote the number of q-long lines through e, and let $\delta^- = \delta_M(e) - \delta^+$ be the number of long lines through e of length at most $q+1$.

When contracting e, each line L containing e becomes a point, and so $|L|-2$ points on L other than e are lost. The number of points destroyed is

$$\varepsilon(M) - \varepsilon(M/e) \leqslant 1 + \delta^-(q-1) + \delta^+(l-1).$$

By the minimality of M, we have

$$\varepsilon(M) - \varepsilon(M/e) > \lambda q^{r(M)} - \lambda q^{r(M)-1} = \lambda(q-1)q^{r(M)-1}.$$

The above inequalities together yield

$$\delta^-(q-1) + \delta^+(l-1) \geqslant \lambda(q-1)q^{r(M)-1}. \tag{2.1}$$

In particular, inequality (2.1) gives

$$\delta_M(e) = \delta^- + \delta^+ \geqslant \lambda \frac{q-1}{l-1} q^{r(M)-1},$$

which easily implies the first claim of the lemma.

Again, by the minimality of M,

$$\delta^- + \delta^+ \leqslant \varepsilon(M/e) < \lambda q^{r(M)-1}. \tag{2.2}$$

Now notice that if $\delta^+ = 0$, then the inequalities (2.1) and (2.2) contradict. So we must have $\delta^+ > 0$. Since this holds for all $e \in E(M)$ and since lines have at most $l+1$ elements, the number of q-long lines of M is at least $\varepsilon(M)/(l+1)$. This gives the second claim. □

Lemma 2.4. *Let l, q and λ be integers with $l \geqslant q \geqslant 2$ and $\lambda \geqslant 1$. Let $M \in \mathcal{U}(l)$ and let e be a non-loop element of M. If $A \subseteq E(M) - e$ satisfies $\varepsilon_M(A) \geqslant \lambda q^{r_M(A)}$, then there exists $X \subseteq A$ such that $e \notin \mathrm{cl}_M(X)$ and $\varepsilon_M(X) \geqslant \frac{\lambda}{l} q^{r_M(X)}$.*

Proof. We may assume that A is minimal with $\varepsilon_M(A) \geqslant \lambda q^{r_M(A)}$. This implies that $M|A$ is simple. We may also assume that $E(M) = A \cup \{e\}$ and that A spans e, as otherwise we are done.

Choose a flat W not containing e, with $r_M(W) = r(M) - 2$. Let $H_0, H_1, \dots, H_m$ be the hyperplanes of M containing W where $e \in H_0$. It is easy to see that the $(H_0 - W, \dots, H_m - W)$ is a partition of $E(M) - W$. Also, $\mathrm{si}(M/W) \simeq U_{2,m+1}$ and, since $M \in \mathcal{U}(l)$, we have $m \leqslant l$.

By the minimality of A, $|H_0 \cap A| < \lambda q^{r(M)-1}$ and, hence,

$$|A - H_0| > \lambda(q-1)q^{r(M)-1}.$$

The sets $H_1 - H_0, \dots, H_m - H_0$ partition $A - H_0$, so there exists $k \in \{1, \dots, m\}$ with

$$|H_k - H_0| \geqslant \frac{1}{m}|A - H_0| > \frac{\lambda}{l}(q-1)q^{r(M)-1}.$$

Taking $X = H_k$ gives the desired result. □

3. Pyramids

We now define some intermediate structures that we shall build on our way to constructing a projective geometry.

Definition 3.1. If $\{b_1, \dots, b_n\}$ is a basis of a matroid M and, for each $i \in \{2, \dots, n\}$ the point b_i is on a long line with each point of $\mathrm{cl}_M(\{b_1, \dots, b_{i-1}\})$, then we call $(M; b_1, \dots, b_n)$ a *pyramid*; the elements $b_1, \dots, b_n$ are called *joints*. A pyramid is *q-strong* if each pair of joints spans a q-long line.

Definition 3.2. Let M be a matroid with a basis $B \cup \{b_1, \dots, b_n\}$. We call $(M, B; b_1, \dots, b_n)$ an (n, λ, q)-*prepyramid* if

- $F = \mathrm{cl}_M(B)$ satisfies $\varepsilon_M(F) \geqslant \lambda q^{r_M(F)}$, and
- for each $i = 1, \ldots, n$, b_i is on a long line with every point of $\mathrm{cl}_M(B \cup \{b_1, \ldots, b_{i-1}\})$.

Note that any pyramid is 1-strong. A prepyramid is a pyramid "on top of" a dense flat.

Lemma 3.3. *If l, n, q, and λ are integers with $\lambda \geqslant 1$ and $l \geqslant q \geqslant 2$, and $M \in \mathcal{U}(l)$ satisfies $\varepsilon(M) \geqslant \lambda l^{2n} q^{r(M)}$, then M has an (n, λ, q)-prepyramid as a minor.*

Proof. The proof is by induction on n. The case $n = 0$ is trivial, so suppose $n > 0$ and that the result holds for $n - 1$. We may assume that M is minor minimal with $\varepsilon(M) \geqslant \lambda l^{2n} q^{r(M)}$. In particular M is simple.

Choose an element $b_n \in E(M)$, and let $A \subseteq E(M) - b_n$ be the set of elements on long lines through b_n. By Lemma 2.3,

$$|A| \geqslant 2\delta_M(b_n) \geqslant \frac{\lambda l^{2n}}{l} q^{r(M)}.$$

By Lemma 2.4, there exists a set $X \subseteq A$ with $b_n \notin \mathrm{cl}_M(X)$ and

$$|X| \geqslant \frac{\lambda l^{2n}}{l^2} q^{r_M(X)} = \lambda l^{2(n-1)} q^{r_M(X)}.$$

By the induction hypothesis $M|X$ has an $(n - 1, \lambda, q)$-prepyramid as a minor. Thus, M has an (n, λ, q)-prepyramid, as required. □

4. Getting a strong pyramid

For a matroid M, we call sets $A_1, \ldots, A_n \subseteq E(M)$ *skew* if $r_M(\bigcup_i A_i) = \sum_i r_M(A_i)$. This is analogous to subspaces of a vector-space forming a direct sum.

We shall need a limit on the total number of lines of a matroid in $\mathcal{U}(l)$. Let $m_l(n)$ denote the maximum number of lines of a rank-n matroid in $\mathcal{U}(l)$; note that $m_l(n)$ is non-decreasing. From Theorem 1.3 we easily get the following crude upper bound:

$$m_l(n) \leqslant \binom{\frac{l^n-1}{l-1}}{2}.$$

Lemma 4.1. *There exists an integer-valued function $\theta_1(s, \lambda, l)$ such that the following holds: If l, q, s, and λ are positive integers with $l \geqslant q \geqslant 2$, and $M \in \mathcal{U}(l)$ satisfies $\varepsilon(M) \geqslant \theta_1(s, \lambda, l) q^{r(M)}$, then either*

- *M has a minor N with s skew q-long lines, or*
- *M has a minor N with a non-loop element $e \in E(N)$ such that the number of q-long lines through e in N is at least $\lambda q^{r(N)}$.*

Proof. Define $\theta_1(1, \lambda, l) = 1$ and for $s \geqslant 2$,

$$\theta_1(s, \lambda, l) = (l + 1)4(s - 1)m_l(2s - 1)\lambda.$$

We may assume that M is minor minimal with $\varepsilon(M) \geqslant \theta_1(s, \lambda, l) q^{r(M)}$. Let $\mathcal{L}$ denote the collection of q-long lines in M. By Lemma 2.3,

$$|\mathcal{L}| \geqslant \frac{\theta_1(s, \lambda, l)}{l + 1} q^{r(M)}.$$

In the case $s = 1$ we are now done, since $|\mathcal{L}| > 0$, so assume $s \geqslant 2$ in the following.

If $\mathcal{L}$ contains s skew lines, then we are done, so assume this is not the case. Pick a maximal set of skew lines from $\mathcal{L}$ and let F be the flat spanned by these lines in M. Let $t = r_M(F) \leqslant 2(s - 1)$. Let $\mathcal{L}' \subseteq \mathcal{L}$ be the lines not contained in F. Then, by the definition of $\theta_1(s, \lambda, l)$,

$$|\mathcal{L}'| \geqslant |\mathcal{L}| - m_l(t) \geqslant |\mathcal{L}|.$$

Let B be a basis of F in M. For each $L \in \mathcal{L}'$ pick $B_L \subseteq B$ with $|B_L| = t-1$, such that B_L and L are skew (this can be done by expanding a basis of L to a basis of $L \cup B$). By a majority argument, there is a subcollection $\mathcal{L}'' \subseteq \mathcal{L}'$ with the sets $B_L = B_0$ identical for $L \in \mathcal{L}''$ and such that

$$|\mathcal{L}''| \geqslant \frac{1}{t}|\mathcal{L}'|.$$

Let e be the single element in $B - B_0$ and let $N = M/B_0$. Then each line $L \in \mathcal{L}''$ spans a q-long line through e in N. Two lines $L_1, L_2 \in \mathcal{L}''$ give rise to the same line in N if $\mathrm{cl}_M(B_0 \cup L_1) = \mathrm{cl}_M(B_0 \cup L_2)$. Hence, the number of q-long lines through e in N is at least

$$\frac{|\mathcal{L}''|}{m_l(t+1)}.$$

By concatenating the inequalities, we get the desired result. □

We now use the previous lemma to construct a strong pyramid. This is done in exactly the same way as a prepyramid was constructed in Lemma 3.3.

Lemma 4.2. *There exists an integer-valued function $\theta(s,n,l)$ such that the following holds: If l, n, s and q are positive integers with $l \geqslant q \geqslant 2$, and $M \in \mathcal{U}(l)$ satisfies $\varepsilon(M) \geqslant \theta(s,n,l)q^{r(M)}$, then either*

- *M has a minor N with s skew q-long lines, or*
- *M has a rank-n minor N, such that N is a q-strong pyramid.*

Proof. Let $\theta(s,1,l) = 1$, and for $n \geqslant 2$ define θ recursively by

$$\theta(s,n,l) = \theta_1\bigl(s, l\theta(s,n-1,l), l\bigr).$$

The proof is by induction on n, the case $n = 1$ being trivial. Suppose $n \geqslant 2$ and that M does not have a minor with s skew q-long lines.

By Lemma 4.1, M has a minor M' with a non-loop element b_n, such that the number of q-long lines through b_n is at least

$$l\theta(s,n-1,l)q^{r(M')}.$$

Let $A \subseteq E(M') - b_n$ be the set of elements on q-long lines through b_n. Lemma 2.4 gives a set $X \subseteq A$ with $b_n \notin \mathrm{cl}_{M'}(X)$, such that

$$\varepsilon_{M'}(X) \geqslant \theta(s,n-1,l)q^{r_{M'}(X)}.$$

By induction, $M'|X$ has a q-strong rank-$(n-1)$ pyramid as a minor. Thus M' has a q-strong rank-n pyramid-minor. □

Lemma 4.3. *If l, n, q, and λ are positive integers with $l \geqslant q \geqslant 2$, and $\lambda \geqslant \theta(\binom{n}{2}, n, l)$, and $(M, B; b_1, \dots, b_n)$ is an (n,λ,q)-prepyramid with $M \in \mathcal{U}(l)$, then M has a rank-n q-strong pyramid as a minor.*

Proof. Let $F = \mathrm{cl}_M(B)$. We may assume that $M|F$ does not have a rank-n q-strong pyramid minor. Then, by Lemma 4.2, $M|F$ has a contraction-minor $M|F/Y$ containing $\binom{n}{2}$ skew q-long lines. Let $M' = M/Y$, $F' = F - \mathrm{cl}_M(Y)$, and let $\mathcal{L}$ be a collection of $\binom{n}{2}$ skew q-long lines in $M'|F'$; denote these lines by $(L_{ij}\colon\ 1 \leqslant i < j \leqslant n)$. Now, for each $1 \leqslant i < j \leqslant n$, we take two distinct elements e_{ij} and e_{ji} in L_{ij}. By the definition of a prepyramid, there exist elements f_{ij} and f_{ji} such that $\{b_i, f_{ij}, e_{ij}\}$ and $\{b_j, f_{ji}, e_{ji}\}$ are triangles. Let $S_b := \{b_1, \dots, b_n\}$, $S_e = \{e_{ij}\colon\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n,\ i \neq j\}$, and $S_f = \{f_{ij}\colon\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n,\ i \neq j\}$. Recall that, by the definition of a prepyramid, F and S_b are skew. Then, since $Y \subseteq F$, $M'|\mathrm{cl}_{M'}(S_b)$ is a pyramid and F' and S_b are skew in M'. Hence $S_b \cup S_e$ is an independent set in M'. Moreover, each pair $\{e_{ij}, f_{ij}\}$ is in series in $M'|(S_b \cup S_e \cup S_f)$, and hence $S_b \cup S_f$ is independent in M'. Therefore, M'/S_f is a rank-n q-strong pyramid. □

5. Projective geometries

Let $(M; b_1, \dots, b_n)$ be a pyramid, and, for each i, let $H_i = \mathrm{cl}_M(\{b_1, \dots, b_i\})$. We call $(M; b_1, \dots, b_n)$ *modular* if, for each $i \geqslant 2$ and each $x, y \in H_i - H_{i-1}$ with $r_M(\{x, y\}) = 2$, the line through x and y intersects H_{i-1} in a point.

The first step towards getting a projective geometry minor of a pyramid will be to find a modular pyramid.

Lemma 5.1. *If l, q, and m are positive integers with $l \geqslant 2$, and $M \in \mathcal{U}(l)$ is a q-strong pyramid with $r(M) \geqslant ml^{\binom{m}{2}}$, then M has a rank-m modular q-strong pyramid minor N.*

Proof. Let m be a fixed positive integer. To each pyramid in $\mathcal{U}(l)$, $(N; a_1, \dots, a_n)$ of rank $n \geqslant m$, we assign a vector

$$Q(N; a_1, \dots, a_n) = \big(\varepsilon_N(H_2), \dots, \varepsilon_N(H_{m-1})\big) \in \mathbb{Z}^{m-2},$$

where $H_k = \mathrm{cl}_N(\{a_1, \dots, a_k\})$. By Theorem 1.3, the number of values that $Q(N)$ can attain is bounded by

$$\prod_{k=2}^{m-1} \frac{l^k - 1}{l - 1} \leqslant \prod_{k=2}^{m-1} l^k \leqslant l^{\binom{m}{2}}.$$

We shall also consider the lexicographic ordering on $\mathbb{Z}^{m-2}$ defined by: $(a_1, \dots, a_{m-2}) <_{\mathrm{LEX}} (b_1, \dots, b_{m-2})$ if there is $k \in \{1, \dots, m-2\}$, such that $a_i = b_i$ for $i = 1, \dots, k-1$ and $a_k < b_k$. This is a total order.

Let $(N; a_1, \dots, a_n)$ with $n \geqslant 2m$ be a pyramid in $\mathcal{U}(l)$. Assume that the pyramid $(N|H_m; a_1, \dots, a_m)$ is not modular. We now describe an operation that gives a minor of N with an increased value of $Q(\cdot)$ in the above order. There exist $i \leqslant m$ and an element $y \in H_i - H_{i-1}$, such that $\varepsilon_{N/y}(\mathrm{cl}_{N/y}(\{a_1, \dots, a_{i-1}\})) > \varepsilon_N(H_{i-1})$. Let k be the least integer in $\{2, \dots, i-1\}$ such that

$$\varepsilon_{N/y}\big(\mathrm{cl}_{N/y}\big(\{a_1, \dots, a_k\}\big)\big) > \varepsilon_N(H_k).$$

Now let $B' = (a_1, \dots, a_k, a_{i+1}, \dots, a_n)$ and define

$$N' = N/y|\,\mathrm{cl}_{N/y}(B').$$

It is easily verified that $(N'; B')$ is a pyramid. It has a higher value in the order $Q(N; a_1, \dots, a_n) <_{\mathrm{LEX}} Q(N'; B')$, and rank $r(N') \geqslant r(N) - m$. Also, since N is q-strong, N' is q-strong.

Now, let $M \in \mathcal{U}(l)$ be a pyramid, with $r(M) \geqslant ml^{\binom{m}{2}}$. By the bound on the number of possible values of $Q(\cdot)$, the process of repeating the above operation must terminate with a rank-m modular pyramid minor. □

The projective geometries $\mathrm{PG}(n-1, q)$ are examples of *projective spaces*. We shall not define this concept in general, only state that a matroid is a projective space if every line has at least three points, and every pair of coplanar lines intersect.

The following theorem is the finite case of what is known as the Fundamental Theorem of Projective Geometry (see [3, pp. 27, 28] for a detailed account of the theorem and [1, Chapter VII] for a proof). The result does not hold in rank 3.

Theorem 5.2. *Every finite projective space of rank $n \geqslant 4$ is isomorphic to $\mathrm{PG}(n-1, q')$ for some prime-power q'.*

In the next lemma we use the theorem to identify a projective geometry in a modular pyramid.

Lemma 5.3. *There exists an integer-valued function $\psi(n, l)$ such that the following holds: If l, n and q are positive integers with $l \geqslant 2$ and $n \geqslant 4$, and $M \in \mathcal{U}(l)$ is a modular q-strong pyramid with $r(M) \geqslant \psi(n, l)$, then M has a $\mathrm{PG}(n-1, q')$-restriction for some prime-power $q' > q$.*

Proof. Define $\psi(n,l) = (l-1)(n-1)+2$. Let $(M; b_1, \ldots, b_r)$ be a modular pyramid, where $r = r(M) \geqslant \psi(n,l)$. Assume that M is simple. Let $H_i = \mathrm{cl}_M(\{b_1, \ldots, b_i\})$, for $i = 1, \ldots, r$.

Notice first, that every line $L \subseteq H_{r-1}$ has length at least 3, since otherwise, looking at the plane spanned by L and b_r, we find a contradiction to the modularity of M.

Define numbers $m_2, \ldots, m_{r-1}$, by $m_i = \min\{|L|\colon\ L \subseteq H_i\}$, where the minimum is over all lines of M contained in H_i. This sequence is clearly descending,

$$l+1 \geqslant m_2 \geqslant m_3 \geqslant \cdots \geqslant m_{r-1} \geqslant 3.$$

Since $r-2 \geqslant (l-1)(n-1)$, by a majority argument there exists k, such that $m_k = m_{k+n-2}$; let $m = m_k$. Choose a line $L_* \subseteq H_k$ with $|L_*| = m$, and let $p_1, p_2 \in L_*$ be different elements. Let $p_3 = b_{k+1}, \ldots, p_n = b_{k+n-2}$. We define the minor $N = M|\,\mathrm{cl}_M(\{p_1, \ldots, p_n\})$. By construction, N is a modular pyramid. Let $F_i = \mathrm{cl}_N(\{p_1, \ldots, p_i\})$ for each i.

We claim that every line in N has length m. Clearly, there are no shorter lines. Suppose the claim fails and let i be minimal, such that there is a line $L \subseteq F_i$ with $|L| > m$. We must have $i > 2$, since $F_2 = L_*$ has length m. Choose an element $x \in F_i - F_{i-1}$, not on L. Now, by modularity each element in L is on a line through x that intersects F_{i-1} in a point. This gives $|L|$ colinear points in F_{i-1}, contradicting the minimality of i.

Observe, that as M is a q-strong pyramid, $m \geqslant q+2$, since N contains the line spanned by b_{k+1} and b_{k+2} which is a q-long line of M.

To prove that N is a projective space, we show that coplanar lines intersect. Let L_1 and L_2 be coplanar lines of N and let $P = \mathrm{cl}_N(L_1 \cup L_2)$. Let i be minimal with $P \subseteq F_i$. If L_1 is contained in F_{i-1}, then L_2 must intersect L_1 by the modularity of N. The case that L_2 is contained in F_{i-1} is similar. Suppose $L_1, L_2 \nsubseteq F_{i-1}$, and assume that L_1 and L_2 do not intersect. Let $x \in L_2 - F_{i-1}$. Each point on L_1 is on a line through x than intersects F_{i-1} in a point. These, together with the point of intersection of L_2 and F_{i-1} account for $m+1$ points of $P \cap F_{i-1}$, a contradiction.

Finally by Theorem 5.2, N is isomorphic to $\mathrm{PG}(n-1, q')$, and we must have $m = q'+1$. $\square$

Theorem 2.1 is now proved by applying Lemmas 3.3, 4.3, 5.1 and 5.3 in succession. The bound α in the theorem, depending on n and l becomes

$$\alpha = \lambda l^{2n'},$$

where $\lambda = \theta(\binom{n'}{2}, n', l)$, $n' = ml^{\binom{m}{2}}$, and $m = \psi(\max(n,4), l)$.

6. Proof of the polynomial bound

We now turn to Theorem 2.2. To prove the theorem, by the previous results, we just need to get a large pyramid. This is done in the same way that we obtained a prepyramid in Lemma 3.3, the proof of which rested on Lemmas 2.3 and 2.4. The arguments are the same, only the calculations differ. The following result parallels Lemma 2.4.

Lemma 6.1. *Let l, λ, and n be positive integers with $l \geqslant 2$. Let $M \in \mathcal{U}(l)$ and let e be a non-loop element of M. If $A \subseteq E(M) - e$ satisfies $\varepsilon_M(A) > \lambda r_M(A)^n$, then there exists $X \subseteq A$ such that $e \notin \mathrm{cl}_M(X)$ and $\varepsilon_M(X) > \frac{\lambda n}{l} r_M(X)^{n-1}$.*

Proof. The proof mimics the proof of Lemma 2.4. We may assume that A is minimal with $\varepsilon_M(A) > \lambda r_M(A)^n$, implying that $M|A$ is simple. We also assume that $E(M) = A \cup \{e\}$ and that A spans e, as otherwise we are done.

Choose a flat W not containing e, with $r_M(W) = r(M) - 2$, and let $H_0, H_1, \ldots, H_m$ be the hyperplanes of M containing W where $e \in H_0$. Then $\mathrm{si}(M/e) \simeq U_{2,m+1}$ and so $m \leqslant l$, since $M \in \mathcal{U}(l)$.

By the minimality of A, $|H_0 \cap A| \leqslant \lambda(r(M)-1)^n$ and thus

$$|A - H_0| > \lambda\big(r(M)^n - \big(r(M)-1\big)^n\big) \geqslant \lambda n\big(r(M)-1\big)^{n-1},$$

where we have used the inequality $(x+1)^n - x^n \geqslant nx^{n-1}$ for a non-negative number x. Since $(H_1 - H_0, \dots, H_m - H_0)$ is a partition of $A - H_0$, there exists $i \geqslant 1$ such that

$$|H_i - A| \geqslant \frac{1}{m}|A - H_0| > \frac{\lambda n}{l}\big(r(M) - 1\big)^{n-1}.$$

Taking $X = H_i$ gives the desired result. □

In the following lemma a pyramid is constructed.

Lemma 6.2. *There exists an integer-valued function $\phi(n, l)$ such that the following holds: If l and n are positive integers with $l \geqslant 2$, and $M \in \mathcal{U}(l)$ has $\varepsilon(M) > \phi(n,l)r(M)^{2(n-1)}$, then M has a rank-n pyramid minor.*

Proof. Let $\phi(1,l) = 1$, and for $n \geqslant 2$ define

$$\phi(n,l) = \frac{l^2\phi(n-1,l)}{4n-6}.$$

The proof is by induction on n. The case $n = 1$ is trivial, so assume $n \geqslant 2$, and that the result holds for $n - 1$. We write $\phi = \phi(n,l)$ for brevity.

Let $r = r(M)$, and let $k = 2(n-1)$. We may assume that M is minimal with $\varepsilon(M) > \phi r^k$. Choose an element e of M. Then $\varepsilon(M/e) \leqslant \phi(r-1)^k$ and

$$\varepsilon(M) - \varepsilon(M/e) > \phi\big(r^k - (r-1)^k\big) \geqslant \phi r^{k-1}.$$

When contracting e, $|L| - 2$ points other than e are lost from each line L containing e. Hence $\varepsilon(M) - \varepsilon(M/e) \leqslant 1 + (l-1)\delta_M(e)$ and

$$(l-1)\delta_M(e) \geqslant \phi r^{k-1}.$$

Let $A \subseteq E(M) - e$ be the set of points on long lines through e. Then $|A| \geqslant 2\delta_M(e) > \frac{2\phi}{l}r^{k-1}$. The previous lemma now gives a set $X \subseteq A$, with $e \notin \mathrm{cl}_M(X)$ and

$$|X| > \frac{2\phi(k-1)}{l^2}r_M(X)^{k-2} = \phi(n-1,l)r_M(X)^{2(n-2)}.$$

Applying the induction hypothesis to $M|X$ we get a minor of $M|X$ that is a rank-$(n-1)$ pyramid. Thus, M has a rank-n pyramid minor. □

When $l \geqslant 2$, Theorem 2.2 now follows from Lemmas 6.2, 5.1 and 5.3. For the case $l = 1$, note that a simple matroid M in $\mathcal{U}(1)$ has no circuits, and thus $|E(M)| = r(M)$. So, taking $a = m = 1$, the condition of the theorem is never satisfied.

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