A correction to our paper "Branch-width and well-quasi-ordering in matroids and graphs"*

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ABSTRACT: We correct a minor but disturbing mistake in our article "James F. Geelen, A.M.H. Gerards and Geoff Whittle. Branch-width and well-quasi-ordering in matroids and graphs", *Journal on Combinatorial Theory, Series B* **84** (2002), 270-290".

On page 275 of [1], in the proof of Theorem 2.1, we write: "As the sets displayed by edges in T are pairwise either disjoint or comparable by inclusion, it is straightforward to show that this means that A does not split W." As Matthias Kriesell kindly pointed out to us, the first half of this sentence is, ofcourse, incorrect. It should read: "As any two sets displayed by edges in T are either disjoint, or cover S, or are comparable by inclusion". Although this patch makes the argument correct, we here give the entire proof of Theorem 2.1 with the full argument of why "A does not split W." For notations see [1].

(2.1) THEOREM: *An integer valued symmetric submodular function with branch-width n has a linked branch-decomposition of width n.*

Proof: Let λ be an integer valued symmetric submodular function with branch-width n. For each branch-decomposition T of λ we define T_k to be the forest in T induced by the edges with width at least k. (Edge induced subgraphs have no isolated nodes.) For a graph H we denote by $e(H)$ the number of edges in H and by $c(H)$ the number of components of H. If T and R are two branch-decompositions of λ we write $T < R$ if there exist a number k such that either $e(T_k) < e(R_k)$ or $e(T_k) = e(R_k)$ and $c(T_k) > c(R_k)$, and such that for each $k' > k$: $e(T_{k'}) = e(R_{k'})$ and $c(T_{k'}) = c(R_{k'})$. This defines a partial order on the branch-decompositions of λ. Choose a minimal element T in this partial order. Note that T has width n. We claim that T is linked. Assume not. Choose an unlinked pair of edges f and g in T. Clearly, $f \neq g$. Let F be the set displayed by the component of $T \setminus f$ not containing g,

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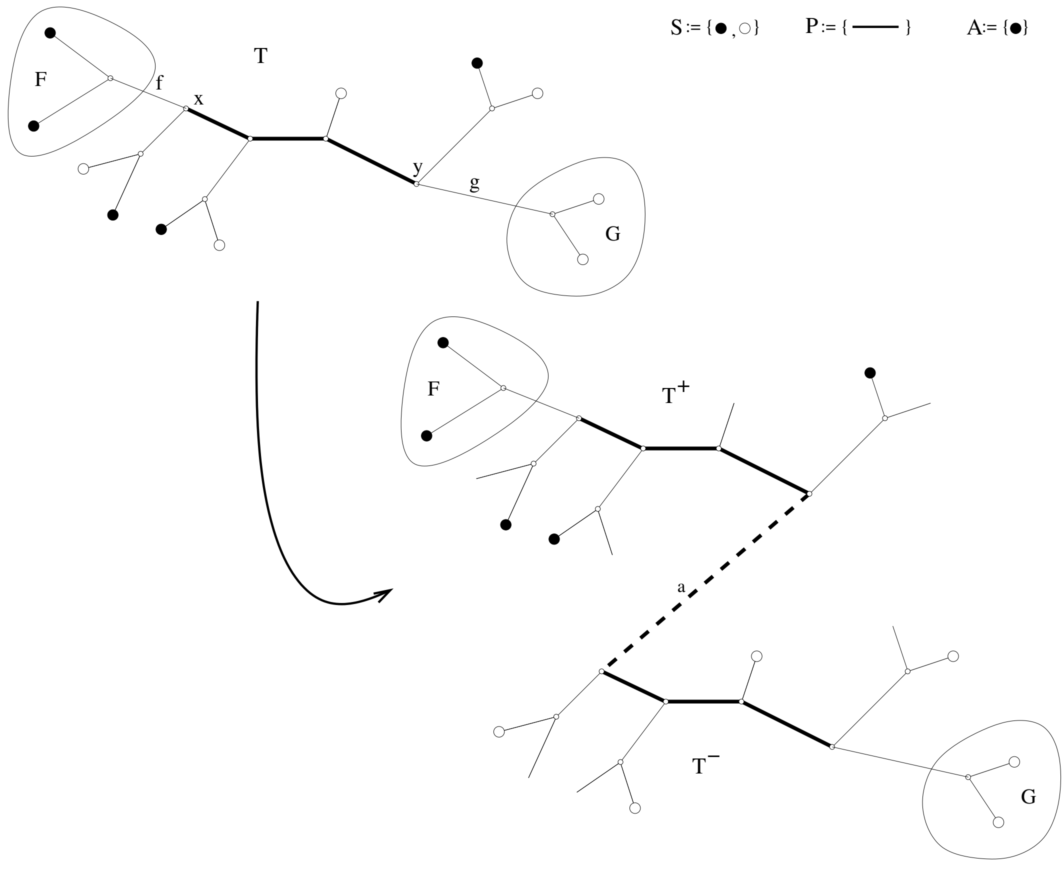

Figure 1: *Proof of Theorem 2.1*

and G the set displayed by the component of $T \setminus g$ not containing f. Let x be the end vertex of f and y be the end vertex of g such that the xy-path P in T does not contain f or g.

We say that a subset X of S *splits* a subset Y of S if $Y \cap X$ and $Y \setminus X$ are both nonempty. Note that X splitting Y does not imply Y splitting X. Choose a subset A of $S \setminus G$ containing F with $\lambda(A) = \lambda(F, G)$ such that A splits as few subsets of S displayed by edges in T as possible. Define a new tree $\widehat{T}$ as follows (see Figure 1): take a copy T^+ of the component of $T \setminus g$ containing f, and a copy T^- of the component of $T \setminus f$ containing g; connect T^+ with T^- by a new edge a joining the copy of y in T^+ to the copy of x in T^-.

We turn $\widehat{T}$ into a branch-decomposition of λ as follows: Each element s of S—which is a leaf of T—is identified with its copy in T^+ if $s \in A$ and with its copy in T^- otherwise.

(2.1.1) *Let e be an edge in T and $\widehat{e}$ be one of its copies in $\widehat{T}$. Then $\lambda(\widehat{e}) \leq \lambda(e)$, with equality only if e has at most one copy in $\widehat{T}_{\lambda(A)+1}$.*

In order to prove this, by symmetry, we may assume that $\widehat{e}$ lies in T^+. Let W be the set displayed by the component of $T \setminus e$ not containing y. Then, $\lambda(e) = \lambda(W)$ and $\lambda(\widehat{e}) = \lambda(W \cap A)$. Combined with submodularity this yields $\lambda(\widehat{e}) + \lambda(W \cup A) \leq \lambda(e) + \lambda(A) = \lambda(e) + \lambda(F, G) \leq \lambda(e) + \lambda(W \cup A)$. Hence $\lambda(\widehat{e}) \leq \lambda(e)$, with equality only if $\lambda(W \cup A) = \lambda(A)$.

Suppose from now on that $\lambda(\widehat{e}) = \lambda(e)$. Then $\lambda(W \cup A) = \lambda(A) = \lambda(F, G)$. ~~As the sets displayed by edges in T are pairwise either disjoint or comparable by inclusion, it is straightforward to show that this means that A does not split W.~~

We prove that A does not split W. Suppose it does. By the choice of A we know that $W \cup A$ splits at least as many sets displayed by edges in T as A does. So as $W \cup A$ does not split W, there exists a set Y displayed by an edge in T that is split by $W \cup A$ but not by A. Then W splits Y and Y does not meet A. As A splits W that means that $W \setminus Y \neq \emptyset$. As Y and W are both displayed by edges in T and as W splits Y that implies that $Y \cup W = E$. So $A \subseteq W$. Hence W contains F. Moreover, as $\hat{e}$ lies in T^+, the choice of W is such that it lies in $S \setminus G$. Hence, as f and g are not linked in T, we have $\lambda(F, G) < \lambda(W) = \lambda(e) = \lambda(\hat{e}) = \lambda(W \cap A) = \lambda(A)$ (as $A \subseteq W$). This contradicts that $\lambda(A) = \lambda(F, G)$. So A does not split W indeed.

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So at least one of $W \cap A$ and $W \setminus A$ is empty. Note that, by combining symmetry and submodularity, $2\lambda(B) = \lambda(B) + \lambda(S \setminus B) \geq \lambda(\emptyset) + \lambda(S) = 2\lambda(\emptyset)$ for each $B \subseteq S$. So either $\lambda(W \cap A) \leq \lambda(A)$ or $\lambda(W \setminus A) \leq \lambda(A)$. Recall that $\lambda(\widehat{e}) = \lambda(W \cap A)$ and note that if e has a second copy e^* in $\widehat{T}$, so in T^-, then $\lambda(e^*) = \lambda(W \cup A) = \lambda(A)$ if $e \in P$ and $\lambda(e^*) = \lambda(W \setminus A)$ if $e \notin P$. Hence, at most one of $\widehat{e}$ and e^* lies in $\widehat{T}_{\lambda(A)+1}$. Thus (2.1.1) follows.

Let p the smallest integer greater than $\lambda(A)$ such that $e(T_k) = e(\widehat{T}_k)$ for $k > p$. For each $k \geq p$, it follows from (2.1.1) that each edge of T_k is copied at most once in $\widehat{T}_k$. Moreover, $\lambda(a) = \lambda(A)$, hence $a \notin \widehat{T}_k$ for $k > \lambda(A)$. So if $k \geq p$, then $e(T_k) \geq e(\widehat{T}_k)$ and $c(T_k) \leq c(\widehat{T}_k)$ whenever $e(T_k) = e(\widehat{T}_k)$. However $\widehat{T} \not< T$, so in fact $e(T_k) = e(\widehat{T}_k)$ and $c(T_k) = c(\widehat{T}_k)$ for $k \geq p$. Thus also T_p and $\widehat{T}_p$ have the same number of edges, which by definition of p implies that $p = \lambda(A)+1$. Moreover, as $c(\widehat{T}_{\lambda(A)+1}) = c(T_{\lambda(A)+1})$, each component of $T_{\lambda(A)+1}$ is copied entirely and as one component in $\widehat{T}_{\lambda(A)+1}$. In particular this is the case for the component of $T_{\lambda(A)+1}$ containing P, which lies entirely in $T_{\lambda(A)+1}$. This is absurd: f has a copy only in T^+, g has a copy only in T^- and a is not in $T_{\lambda(A)+1}$. So T is linked, indeed. □

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References

[1] James F. Geelen, A.M.H. Gerards, and Geoff Whittle, *Branch width and well-quasi-ordering in matroids and graphs*, Journal of Combinatorial Theory, Series B **84** (2002), 270–290.