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On the excluded minors for the matroids of branch-width k ☆

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Abstract

We prove that the excluded minors for the class of matroids of branch-width k have size at most $(6^k - 1)/5$.

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1. Introduction

We prove the following theorem.

Theorem 1.1. *If M is an excluded minor for the class of matroids of branch-width at most k and $k \geqslant 2$, then $|E(M)| \leqslant (6^k - 1)/5$.*

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Branch-width is a parameter for graphs and for matroids which was introduced by Robertson and Seymour [4]. Unfortunately, the branch-width of a graph may be larger than that of its cycle matroid; consider, for example, a path of length 3. So, by itself, Theorem 1.1 says little about graphs. We expect, however, that the branch-width of a graph is typically the same as that of its cycle matroid. In particular, it is conjectured that, if G is a graph with a circuit of length at least 2, then the branch-width of G is the same as that of its cycle matroid.

Theorem 1.1 implies that the excluded minors for the class of matroids of branch-width 2 have size at most 7. The matroids of branch-width 2 are precisely the series–parallel matroids; and the excluded minors for this class are known to be $U_{2,4}$ and $M(K_4)$. The excluded minors for the class of matroids of branch-width 3 are studied by Hall, Oxley, Semple, and Whittle [2]; they show that the excluded minors have size at most 14.

2. Branch-width

We assume that the reader is familiar with matroid theory; we use the notation of Oxley [3]. Let M be a matroid. We define the function λ_M by $\lambda_M(X) = r_M(X) + r_M(E(M) - X) - r(M) + 1$ for $X \subseteq E(M)$. This function is *submodular*; that is, $\lambda_M(X \cap Y) + \lambda_M(X \cup Y) \leqslant \lambda_M(X) + \lambda_M(Y)$ for all $X, Y \subseteq E(M)$. Also, $\lambda_M(\cdot)$ is monotone under taking minors; that is, if N is a minor of M with $X \subseteq E(N)$ then $\lambda_N(X) \leqslant \lambda_M(X)$. A partition (X, Y) of $E(M)$ is called a *separation of order* $\lambda_M(X)$ (note that we do not have conditions on the size of X and Y).

A tree is *cubic* if its internal vertices all have degree 3. The *leaves* of a tree are its degree-1 vertices. A *partial branch-decomposition* of M is a cubic tree T whose leaves are labeled by the elements of M. That is, each element of M labels some leaf of T, but leaves may be unlabeled or multiply labeled. A *branch-decomposition* is a partial branch-decomposition without multiply labeled leaves. If T' is a subgraph of T and $X \subseteq E(M)$ is the set of labels of T', then we say that T' *displays* X. The *width* of an edge e of T is defined to be $\lambda_M(X)$ where X is the set displayed by one of the components of $T \backslash e$. The *width* of T, denoted $\varepsilon(T)$, is the maximum among the widths of its edges.

The *branch-width* of M is the minimum among the widths of all branch-decompositions of M. Let (A, B) be a partition of $E(M)$. A *branching* of B is a partial branch-decomposition of M in which there is a leaf that displays A and in which no other leaf is multiply labeled. We say that B is *k-branched* if there is a branching T of B with $\varepsilon(T) \leqslant k$. Note that, if both A and B are k-branched then M has branch-width at most k.

Lemma 2.1. *Let M be a matroid with branch-width k and let (A, B) be a separation of order $\lambda_M(A) \leqslant k$. If B is not k-branched, then there exists a partition (A_1, A_2, A_3) of A such that $\lambda_M(A_i) < \lambda_M(A)$ for all $i \in \{1, 2, 3\}$. (One of A_1, A_2 and A_3 may be empty.)*

Proof. Suppose that for each partition (A_1, A_2, A_3) of A we have $\lambda_M(A_i) \geqslant \lambda_M(A)$ for some $i \in \{1, 2, 3\}$.

2.1.1. *If* (X_1, X_2) *is a separation of order at most* k, *then either* $\lambda_M(B \cap X_1) \leqslant k$ *or* $\lambda_M(B \cap X_2) \leqslant k$.

Subproof. Considering the partition $(A \cap X_1, A \cap X_2, \emptyset)$ of A we see that either $\lambda_M(A \cap X_1) \geqslant \lambda_M(A)$ or $\lambda_M(A \cap X_2) \geqslant \lambda_M(A)$; suppose, $\lambda_M(A \cap X_1) \geqslant \lambda_M(A)$. Hence, by submodularity, we get $\lambda_M(A \cup X_1) \leqslant \lambda_M(X_1) \leqslant k$. So, $\lambda_M(B \cap X_2) = \lambda_M(A \cup X_1) \leqslant k$, as required. □

Let T be a branch-decomposition of M with $\varepsilon(T) = k$. We may assume that T has degree-3 vertices, as otherwise the lemma holds trivially. For the same reason, we may assume $k \geqslant 2$. If v is a vertex of T and e is an edge of T we let X_{ev} denote the set of elements of M displayed by the component of $T \backslash e$ that does not contain v.

2.1.2. *There exists a degree-3 vertex* s *of* T *such that, for each edge* e *of* T, $\lambda_M(X_{es} \cap B) \leqslant k$.

Subproof. We construct an orientation of T. Let e be an edge of T, and let u and v be the ends of e. If $\lambda_M(X_{ev} \cap B) \leqslant k$ then we orient e from u to v, and if $\lambda_M(X_{eu} \cap B) \leqslant k$ then we orient e from v to u. Thus, by 2.1.1, each edge receives at least one orientation, maybe two.

First, assume that there exists a node v of T such that every other vertex can be connected to v by a directed path. As $k \geqslant 2$, each edge incident with a leaf has been oriented away from that leaf. Hence, we may assume that v has degree 3. Then the claim follows with $s = v$.

Next, we assume that there is no vertex reachable from every other vertex. Then there exists a pair of edges e and f and a vertex w on the path connecting e and f such that neither e nor f is oriented toward w. Let $Y_1 = X_{ew}$, $Y_3 = X_{fw}$, and $Y_2 = E(M) - (Y_1 \cup Y_3)$. Since e and f are oriented away from w, $\lambda_M((Y_2 \cup Y_3) \cap B) \leqslant k$ and $\lambda_M(Y_3 \cup A) = \lambda_M((Y_1 \cup Y_2) \cap B) \leqslant k$. The intersection and union of $(Y_2 \cup Y_3) \cap B$ and $Y_3 \cup A$ are $Y_3 \cap B$ and $Y_2 \cup Y_3 \cup A$. So by submodularity, $\lambda_M(Y_3 \cap B) \leqslant k$ or $\lambda_M(Y_1 \cap B) = \lambda_M(Y_2 \cup Y_3 \cup A) \leqslant k$. This contradicts the fact that neither e nor f is oriented towards w. □

Let s be a vertex satisfying 2.1.2, let e_1, e_2, and e_3 be the edges of T incident with s, and let X_i denote $X_{e_i s}$ for each $i \in \{1, 2, 3\}$. Note that $\lambda_M(X_i \cap A) \geqslant \lambda_M(A)$ for some $i \in \{1, 2, 3\}$; suppose that $\lambda_M(X_1 \cap A) \geqslant \lambda_M(A)$. Then, by submodularity, $\lambda_M((X_2 \cup X_3) \cap B) = \lambda_M(X_1 \cup A) \leqslant \lambda_M(X_1) \leqslant k$. Now construct a branching $\hat{T}$ of B by taking a copy of T keeping only the labels from B, subdividing e_1 with a vertex b, adding a new leaf a incident with b, and labeling a with A. It is easy to see that $\varepsilon(\hat{T}) \leqslant k$, so B is k-branched. □

3. (m,f)-connectivity

Let $f:\mathbb{Z}_+\rightarrow\mathbb{Z}_+$ be a function and $m\in\mathbb{Z}_+$. A matroid M is called (m,f)-connected if whenever (A,B) is a separation of order ℓ where $\ell<m$ we have either $|A|\leqslant f(\ell)$ or $|B|\leqslant f(\ell)$.

Lemma 3.1. *Let $f:\mathbb{Z}_+\rightarrow\mathbb{Z}_+$ be a nondecreasing function. If e is an element of an (m,f)-connected matroid M, then $M\backslash e$ or M/e is $(m,2f)$-connected.*

To prove Lemma 3.1, we use the following result stated in [1, (5.2)].

Proposition 3.2. *Let e be an element of a matroid M. If (X_1,X_2) and (Y_1,Y_2) are partitions of $E(M)-\{e\}$ then*

$$\lambda_{M\backslash e}(X_1)+\lambda_{M/e}(Y_1)\geqslant\lambda_M(X_1\cap Y_1)+\lambda_M(X_2\cap Y_2)-1.$$

Proof of Lemma 3.1. Suppose the result is false; let (X_1,X_2) be a separation in $M\backslash e$ of order a with $|X_1|,|X_2|>2f(a)$, and let (Y_1,Y_2) be a separation in M/e of order b with $|Y_1|,|Y_2|>2f(b)$. By symmetry, we may assume that $\lambda_M(X_1\cap Y_1)\leqslant\lambda_M(X_2\cap Y_2)$ and that $\lambda_M(X_1\cap Y_2)\leqslant\lambda_M(X_2\cap Y_1)$. Thus, by Proposition 3.2, $\lambda_M(X_1\cap Y_1)\leqslant\lfloor\frac{a+b+1}{2}\rfloor$ and $\lambda_M(X_1\cap Y_2)\leqslant\lfloor\frac{a+b+1}{2}\rfloor$. Moreover, since $|X_1|>2f(a)$ and M is (m,f)-connected, either $\lambda_M(X_1\cap Y_1)>a$ or $\lambda_M(X_1\cap Y_2)>a$. By symmetry, we may assume that $\lambda_M(X_1\cap Y_1)>a$. Thus, $\lfloor\frac{a+b+1}{2}\rfloor>a$, and so, $\lfloor\frac{a+b+1}{2}\rfloor\leqslant b$. Therefore, $\lambda_M(X_1\cap Y_2)\leqslant b$. Moreover, by Proposition 3.2 and the fact that $\lambda_M(X_1\cap Y_1)\geqslant a+1$, we have $\lambda_M(X_2\cap Y_2)\leqslant b$. Hence, since M is (m,f)-connected, $|Y_2|=|Y_2\cap X_1|+|Y_2\cap X_2|\leqslant 2f(b)$; a contradiction. □

4. Excluded minors

Let $g(n)=(6^{n-1}-1)/5$. Note that, $g(1)=0$ and $g(n)=6g(n-1)+1$ for all $n>1$. The following result is the key lemma in the proof of Theorem 1.1, and is of independent interest.

Lemma 4.1. *If M is an excluded minor for the class of matroids of branch-width $k\geqslant1$ then M is $(k+1,g)$-connected.*

Proof. For $m\in\{1,\dots,k\}$, we will prove that M is $(m+1,g)$-connected by induction on m. It is easy to see that M is $(2,g)$-connected. Suppose then that $m\geqslant2$ and that M is (m,g)-connected. Now, suppose that there exists a separation (A,B) of order m such that $|A|,|B|>g(m)=6g(m-1)+1$. Since M has branch-width greater than k, we may assume that B is not k-branched. Now, let $e\in A$. By Lemma 3.1 and duality, we may assume that M/e is $(m,2g)$-connected. Note that M/e has branch-width k. Consider any partition (A_1,A_2,A_3) of $A-\{e\}$. Since $|A|>6g(m-1)+1$,

$|A_i| > 2g(m-1)$ for some $i \in \{1,2,3\}$. Then, since M/e is $(m,2g)$-connected, $\lambda_{M/e}(A_i) \geqslant m \geqslant \lambda_{M/e}(A)$. Therefore, by Lemma 2.1, B is k-branched in M/e. Since B is not k-branched in M, it must be the case that $e \in \mathrm{cl}_M(B)$. (Indeed, if $X \subseteq B$ and $e \notin \mathrm{cl}_M(B)$ then $\lambda_M(X) = \lambda_{M/e}(X)$.) Therefore, $(A - \{e\}, B)$ is a separation in M/e of order at most $m-1$. However, this contradicts the fact that M/e is $(m,2g)$-connected. □

Proof of Theorem 1.1. Let e be an element of M. By Lemmas 4.1 and 3.1 and by duality, we may assume that M/e is $(k+1,2g)$-connected. Now, M/e has branch-width k; let T be a branch-decomposition of M/e with $\varepsilon(T) = k$. As $k \geqslant 2$, matroid M has at least three elements. Hence T has at least two labeled leaves. As T is cubic, this implies that T has an edge f such that the sets X_1 and X_2 displayed by the two components of $T \backslash f$ each have at least $(|E(M)|-1)/3$ elements. We may assume that $|X_1| \leqslant |X_2|$. Then, since M/e is $(k+1,2g)$-connected, we have $|X_1| \leqslant 2g(k)$. Thus, $|E(M)| \leqslant 6g(k) + 1 = g(k+1)$, as required. □

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