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Branch-Width and Rota's Conjecture[1]

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For a fixed finite field $\mathbb{F}$ and an integer k there are a finite number of matroids of branch-width k that are excluded minors for $\mathbb{F}$-representability.

1. INTRODUCTION

We prove the following theorem.

THEOREM 1.1. *Let $\mathbb{F}$ be a finite field and k be a positive integer. Then, among the excluded minors for the class of $\mathbb{F}$-representable matroids, there is a finite number matroids with branch-width k.*

We begin by giving some background to this result. A matroid M is an *excluded minor* for a minor-closed class of matroids if M is not in the class but all proper minors of M are. It is natural to attempt to characterize a minor-closed class of matroids by giving a complete list of its excluded minors. Unfortunately, a minor-closed class of matroids can have an infinite number of excluded minors. For example, this is the case for the matroids representable over the rationals; see [7]. This contrasts strikingly with the fundamental theorem of Robertson and Seymour which says that graphs are

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well-quasi-ordered under the minor order. In other words, given any infinite set of graphs there is one that is isomorphic to a minor of another. It follows from this result that any minor-closed class of graphs has a finite number of excluded minors. Combined with Tutte's excluded-minor characterization of graphic matroids [16] this gives the following corollary: *If $\mathcal{M}$ is a minor-closed class of graphic matroids, then $\mathcal{M}$ has a finite number of excluded minors.* Thus, there do exist significant classes of matroids with a finite number of excluded minors. Moreover, in contrast with the situation for infinite fields, Tutte [15] showed that $U_{2,4}$ is the only excluded minor for the class of matroids representable over $GF(2)$. Rota [11] conjectured the following sweeping generalization of Tutte's result.

CONJECTURE 1.2 (*Rota's Conjecture*). If $\mathbb{F}$ is a finite field, then the class matroids representable over $\mathbb{F}$ has a finite number of excluded minors.

Apart from $GF(2)$ Rota's Conjecture is known to be true only for $GF(3)$ (see [1, 5, 12]) and $GF(4)$ (see [4]). Moreover, opinion has been divided on the plausibility of the conjecture in general. In any case, it is widely agreed that the resolution of Rota's Conjecture is one of the two most important problems in matroid theory. (The other is proving that the class of matroids representable over a given finite field is well-quasi-ordered with respect to taking minors.)

Intuitively a matroid (or a graph) has small "width" if it can be decomposed across a number of non-crossing separations of small order into small pieces. There are several ways of making this notion precise. Most readers will be familiar with *tree-width* in graphs. This concept, introduced by Robertson and Seymour [9], has turned out to be extremely fruitful in graph theory. In particular, proving that graphs of bounded tree-width are well-quasi-ordered is a basic step in Robertson and Seymour's proof that graphs are well-quasi-ordered.

We use the related concept of "branch-width" which seems more convenient for matroids. We delay the formal definition of branch-width until Section 3. For the moment it suffices to know that a class of graphs has bounded tree-width if and only if it has bounded branch-width, so that branch-width is genuinely an analogue of tree-width. Moreover, a matroid and its dual have the same branch-width.

In proving Theorem 1.1 we show that Rota's Conjecture holds so long as we restrict attention to excluded minors of bounded branch-width. Theorem 1.1 adds weight to the plausibility of Rota's Conjecture. Indeed, should Rota's Conjecture fail, there would exist excluded minors with arbitrarily large branch-width; which we consider unlikely. Moreover, Theorem 1.1 opens up a possible technique for proving Rota's Conjecture. The next step would be to show that matroids representable over a fixed finite field that have very large branch-width contain a large grid as a minor. (Diestel *et al.*

[2] gave a short proof of the analogous result for graphs.) The final step would be to prove that an excluded minor cannot contain a large grid.

While Theorem 1.1 is the main result of the paper, our techniques prove a more general result. A matroid M is *almost representable* over a field $\mathbb{F}$ if M has an element e such that both $M\backslash e$ and M/e are $\mathbb{F}$-representable. The more general result proved in the paper is the following.

THEOREM 1.3. *Let $\mathbb{F}$ be a finite field and k be a positive integer. Then the class of matroids with branch-width k that are almost representable over $\mathbb{F}$ is well-quasi-ordered under the minor order.*

Excluded minors for $\mathbb{F}$ are almost representable, so Theorem 1.1 is an immediate corollary of Theorem 1.3. But $\mathbb{F}$-representable matroids are also almost representable, so, by Theorem 1.3, the $\mathbb{F}$-representable matroids of branch-width k are well-quasi-ordered with respect to taking minors. (This corollary is the main result of [3].) It is easily checked that an excluded minor for a class of matroids of branch-width at most k has branch-width at most $k+1$. Thus, we also obtain the following theorem as a corollary of Theorem 1.3.

THEOREM 1.4. *Let $\mathbb{F}$ be a finite field and k a positive integer. If $\mathcal{M}$ is a minor-closed class of $\mathbb{F}$-representable matroids with branch-width at most k, then $\mathcal{M}$ has a finite number of excluded minors.*

We conclude the introduction with a brief overview of the proof of Theorem 1.3. Our proof is similar to the proof, in [3], that the $\mathbb{F}$-representable matroids of branch-width k are well-quasi-ordered with respect to taking minors. The proof in [3] works with *represented matroids*, but the matroids that we are dealing with are not necessarily representable. However, if M is an almost $\mathbb{F}$-representable matroid then the 2-polymatroid $M\backslash e + M/e$ is $\mathbb{F}$-representable. (Polymatroids are defined in Section 4.) This idea of associating an excluded minor with a representable 2-polymatroid was suggested by Dirk Vertigan.

Many of the techniques in [3] extend easily to representable polymatroids; these extensions are presented in Section 5. However, the class of $\mathbb{F}$-representable 2-polymatroids with branch-width at most k is *not* well-quasi-ordered with respect to taking minors. The problem is that, while there is a satisfactory analogue of Menger's Theorem for matroids, this result does not extend to polymatroids. Given two disjoint sets of elements in a matroid there is a reasonable notion of how "connected" they are. Essentially, this connectivity is given by the maximum rank of a flat that is spanned by both sets among all minors of the matroid. Tutte proved that this connectivity is given by the minimum order of a separation "between" the given sets; see Theorem 5.1. For polymatroids (indeed, even for the particular class of 2-polymatroids considered here) there can be a gap between the connectivity of the two sets and the minimum order of a separation between them. We find a particular orientation of our branch-decompositions so that we can connect two sets together whenever it is needed in the proof; see Theorem 6.4.

2. A LEMMA ON ORIENTED TERNARY FORESTS

The well-quasi-ordering results of [3] rely crucially on a lemma for oriented binary forests [3, Lemma 3.2]. This lemma is a straightforward consequence of a "lemma on trees" of Robertson and Seymour [10]. For this paper we need a lemma similar to [3, Lemma 3.2] except for the fact that internal vertices of our trees have outdegree 3, rather than 2. The proof of Lemma 2.1 below is an easy modification of the proof of [3, Lemma 3.2] and is omitted.

A *rooted tree* is a finite directed tree where all but one of the vertices have indegree 1. The vertex with indegree 0 is called the *root*, vertices with outdegree 0 are called *leaves* and the remaining vertices are called *internal vertices*. Edges leaving a root are *root edges* and those entering a leaf are *leaf edges*. For a non-root vertex u of a rooted tree we denote the unique edge directed into u by $d(u)$.

A *rooted forest* is a collection of countably many vertex-disjoint rooted trees. The *roots* of the forest are the roots of the trees. Moreover, we extend all the above terminology to rooted forests in a similar obvious way.

An *n-edge labelling* of a graph G is a map from the edges of G to the set $\{0, 1, \ldots, n\}$. Let λ be an n-edge labelling of a rooted forest F and let e and f be edges in F. Then e is *λ-linked* to f if $\lambda(e) = \lambda(f)$, and F contains a directed path P starting with e and ending with f such that $\lambda(g) \geqslant \lambda(e)$ for each directed edge g on P.

A rooted tree is *ternary* if each internal vertex has outdegree 3. An *orientation* of a ternary rooted tree is a function from the edges of the tree to the set $\{l, m, r\}$ that is a bijection whenever it is restricted to the edges leaving an internal vertex. Via this function we have, for each internal vertex u, a labelling $l(u), m(u), r(u)$ of the edges leaving u. An *oriented ternary forest* is a collection of countably many vertex-disjoint oriented ternary rooted trees.

LEMMA 2.1. *Let F be an infinite ternary oriented forest with an n-edge labelling λ. Moreover, let $\preccurlyeq$ be a quasi-order on the edges of F such that*:

(a) *$e \preccurlyeq f$ whenever f is λ-linked to e*;
(b) *$\preccurlyeq$ has no infinite strictly descending sequences*;
(c) *the root edges of F form an antichain with respect to $\preccurlyeq$*;
(d) *the leaf edges are well-quasi-ordered by $\preccurlyeq$.*

Then F contains a sequence $(u_0, u_1, \ldots)$ of internal vertices such that

(i) $\{d(u_0), d(u_1), d(u_2), \ldots,\}$ *is an antichain with respect to* $\preccurlyeq$;
(ii) $l(u_0) \preccurlyeq \cdots \preccurlyeq l(u_{i-1}) \preccurlyeq l(u_i) \preccurlyeq \cdots$;
(iii) $m(u_0) \preccurlyeq \cdots \preccurlyeq m(u_{i-1}) \preccurlyeq m(u_i) \preccurlyeq \cdots$; *and*
(iv) $r(u_0) \preccurlyeq \cdots \preccurlyeq r(u_{i-1}) \preccurlyeq r(u_i) \preccurlyeq \cdots$.

3. SYMMETRIC SUBMODULAR FUNCTIONS AND BRANCH-WIDTH

A function λ defined on the collection of subsets of a finite *ground set* E is called *submodular* if $\lambda(A) + \lambda(B) \geqslant \lambda(A \cap B) + \lambda(A \cup B)$ for each $A, B \subseteq E$. We call λ *symmetric* if $\lambda(A) = \lambda(E - A)$ for each $A \subseteq E$. For disjoint subsets A and B of E we define

$$\lambda(A, B) = \min(\lambda(X) : A \subseteq X \subseteq E - B).$$

The symmetric submodular functions considered in this paper are the connectivity functions of matroids and polymatroids.

A tree is *cubic* if all vertices have degree 1 or 3. (Note the distinction between cubic trees and ternary trees defined in Section 2.) A *branch-decomposition* of a symmetric submodular function λ on a finite set E is a cubic tree such that E labels a set of the leaves of T. The set *displayed* by a given subtree of T is the set of elements of E that label leaves of that subtree. A set of elements of E is *displayed* by an edge e of T if it is displayed by one of the two components of $T \backslash e$. The *width* $\lambda(e)$ of an edge e of T is the λ-value of one of the two sets displayed by e. The *width* of a branch-decomposition is the maximum of the widths of its edges and the *branch-width* of a symmetric submodular function is the minimum among the widths of its branch-decompositions.

Let f and g be two edges in a branch-decomposition T of λ, let F be the set displayed by the component of $T \backslash f$ not containing g, and let G be the set displayed by the component of $T \backslash g$ not containing f. Let P be the shortest path in T containing f and g. Each edge of P displays a subset S that contains F and is disjoint from G. Thus, the widths of the edges of P are upper bounds for $\lambda(F, G)$. We call f and g *linked if* $\lambda(F, G)$ is equal to the minimum of the widths of the edges of P. We call a branch-decomposition *linked* if all its edge pairs are linked. The following result from [3] is an analogue of Thomas' result [13] on linked tree-decompositions of a graph.

THEOREM 3.1. *An integer-valued symmetric submodular function with branch-width n has a linked branch-decomposition of width n.*

The connectivity functions of graphs and matroids are symmetric, submodular and integer valued, so it follows from Theorem 3.1 that these structures have linked branch-decompositions. We shall also apply Theorem 3.1 to the connectivity functions of more general structures, namely polymatroids, and we turn attention to these now.

4. POLYMATROIDS AND ARRANGEMENTS

A *polymatroid on* E is an ordered pair $P = (E, r_P)$, where E is a finite set and r_P is a function from the power set of E into the integers that satisfies the following properties: $r_P(\emptyset) = 0$; if $A \subseteq B \subseteq E$, then $r_P(A) \leqslant r_P(B)$; and if $A, B \subseteq E$, then $r_P(A) + r_P(B) \geqslant r_P(A \cap B) + r_P(A \cup B)$. If k is a positive integer and $r_P(\{x\}) \leqslant k$ for all $x \in E$, then P is a *k-polymatroid.* Of course a matroid is precisely a 1-polymatroid. Moreover, just as matroids abstract the combinatorial properties of a collection of points in a vector space, polymatroids abstract the combinatorial properties of a collection of subspaces in a vector space.

It is straightforward to extend the notion of minor from matroids to polymatroids. If a is an element of the polymatroid P on E, then the *deletion* of a from P, denoted $P \backslash a$ is the polymatroid on $E - \{a\}$ with rank function $r_{P \backslash a}$ defined by $r_{P \backslash a}(X) = r_P(X)$ for all $X \subseteq E - \{a\}$. The *contraction* of a from P, denoted P/a, is the polymatroid on $E - \{a\}$ with rank function defined by $r_{P/a}(X) = r_P(X \cup \{a\}) - r_P(\{a\})$ for all $X \subseteq E - \{a\}$. If P' is obtained from P by a sequence of deletions and contractions, then P' is a *minor* of P.

Let $\mathbb{F}$ be a field. Then a (*subspace*) *arrangement over* $\mathbb{F}$ is a finite set of labelled subspaces of the vector space $V(k, \mathbb{F})$, where all labels are distinct but a subspace may receive more than one label. Two arrangements are *isomorphic* if one can be obtained from the other by relabelling. Formally, an arrangement is a pair $(E, \mathbb{V})$, where E is a finite set and $\mathbb{V} = V(k, \mathbb{F})$, with a function ψ from E to the set of subspaces of $\mathbb{V}$. Let $\mathbb{V}'$ be a vector space over $\mathbb{F}$ and let $\mathscr{L} : \mathbb{V} \to \mathbb{V}'$ be a linear transformation. We let $\mathscr{L}(E, \mathbb{V})$ denote the arrangement obtained by applying $\mathscr{L}$ to $\mathbb{V}$ and relabelling accordingly. That is, an element $e \in E$ labels the subspace $\mathscr{L}(\psi(e))$ in $\mathbb{V}'$. If the subspaces spanned by the image of E in $\mathbb{V}$ and $\mathbb{V}'$ have the same rank then we call $(E, \mathbb{V})$ and $\mathscr{L}(E, \mathbb{V})$ *equivalent*.

It is well known, and easily seen, that if $(E, \mathbb{V})$ is an arrangement over $\mathbb{F}$, then the set function defined for all $A \subseteq E$, by $r(A) = r(\bigcup_{a \in A} a)$ is a polymatroid on E, which we denote by $P(E)$. We say that $P = (E, r)$ is *representable* over a field $\mathbb{F}$ if it is isomorphic to a polymatroid induced by some arrangement over $\mathbb{F}$. Note that equivalent arrangements represent the same polymatroid.

The notion of minors extends naturally to arrangements. Let $(E, \mathbb{V})$ be an arrangement and let $a \in E$. Now, $(E-\{a\}, \mathbb{V})$ is the arrangement obtained from $(E, \mathbb{V})$ by *deleting* a. (Here, the embedding of $E - \{a\}$ in $\mathbb{V}$ is given by the restriction of ϕ to $E - \{a\}$.) Let $\mathscr{L} : \mathbb{V} \to \mathbb{V}$ be a linear transformation whose kernel is equal to the span of a. Now, $\mathscr{L}(E - \{a\}, \mathbb{V})$ is an arrangement obtained from $(E, \mathbb{V})$ by *contracting* a. Note that the operations of deletion and contraction in an arrangement are consistent with the respective operations in a polymatroid. Moreover, it is straightforward to prove that the order in which we apply deletions and contractions is unimportant. Let D and C be disjoint subsets of E and let $\mathscr{L} : \mathbb{V} \to \mathbb{V}$ be a linear transformation whose kernel is the subspace spanned by C. We let $(E, \mathbb{V})\backslash D/C$ denote the arrangement $\mathscr{L}(E - (D \cup C), \mathbb{V})$; any such arrangement is called a *minor* of $(E, \mathbb{V})$. Since $\mathscr{L}$ is not uniquely defined, $(E, \mathbb{V})\backslash D/C$ is not uniquely defined; but any two minors determined by the same sets D and C are equivalent. When $(E', \mathbb{V}')$ is a minor of the arrangement $(E, \mathbb{V})$, to distinguish the particular linear transformation used, we say that $\mathscr{L}$ *projects* $(E, \mathbb{V})$ onto $(E', \mathbb{V}')$.

Henceforth, to avoid cluttering already technical arguments, we will be casual in our discussion of arrangements. In particular, we will refer to an arrangement $(E, \mathbb{V})$ simply by E.

The connectivity function of a polymatroid is defined just as for matroids. Thus, for a polymatroid P on E and $A \subseteq E$, the *connectivity function of* P, denoted λ_P is defined for all $A \subseteq E$, by

$$\lambda_P(A) = r_P(A) + r_P(E - A) - r_P(E) + 1.$$

It is easy to verify that the connectivity function of a polymatroid is symmetric and submodular. A *branch-decomposition* of a polymatroid P is just a branch-decomposition of its connectivity function and the *branch-width* of P is just the branch-width of its connectivity function. We then have the following immediate corollary of Theorem 3.1.

PROPOSITION 4.1. *If P is a polymatroid of branch-width n, then P has a linked branch-decomposition of width n.*

5. POLYMATROIDS AND WELL-QUASI-ORDERING

In light of the fact that matroids of bounded branch-width representable over a fixed finite field are well-quasi-ordered, one might hope that a similar result holds for polymatroids. But this is not the case, even for the class of 2-polymatroids of branch-width 3. To see this consider the following example. Let m be an integer greater than 1, and let C_m be a graph consisting of a single cycle on m vertices. Define the polymatroid P_m on the edges of C_m as follows. For a subset A of edges $r_{P_m}(A)$ is the cardinality of the vertices of C_m incident with at least one edge in A. Note that P_m is representable over any field $\mathbb{F}$. To see this, think of the vertices of C_m as being elements of a basis of $V(m, \mathbb{F})$, and the edges of C_m as being lines joining pairs of elements of this basis. It is immediate that P_m is isomorphic to the polymatroid induced by this arrangement of lines. It is also easily verified that P_m has branch-width 3. Finally, we observe that if $i, j \geqslant 2$ and $i \neq j$, then P_i is not isomorphic to a minor of P_j. To see this consider the effect of polymatroid deletion and contraction on the graph C_m. If we delete an edge we obtain a path of length $m-1$, while if we contract an edge we obtain a path of length $m-3$ with a loop at each end vertex. In summary, the set $\{P_m : m \geqslant 2\}$ is an infinite antichain of polymatroids of branch-width 3, members of which are representable over all fields.

An analogue of Menger's theorem plays a crucial role in the proof that the matroids representable over a finite field with bounded branch-width are well-quasi-ordered. Let A and B be disjoint subsets of a polymatroid P on E. Recall that $\lambda_P(A, B)$ is defined to be the minimum over all subsets X of E with $A \subseteq X$ and $B \subseteq (E - X)$ of $\lambda_P(X)$. The following is a result of Tutte [17] (see also [3, (5.1)]).

Theorem 5.1 (Tutte's Linking Theorem). *Let M be a matroid and A* and *B be disjoint subsets of $E(M)$. Then $\lambda_M(A, B) \geqslant n$ if and only if there exists a minor M' of M with ground set $A \cup B$ such that $\lambda_{M'}(A) \geqslant n$.*

Tutte's Linking Theorem does not extend to polymatroids. This is, however, the only missing ingredient required for a satisfactory well-quasi-ordering theorem (see Theorem 5.2). Consider two distinct edges e and f of P_m, where $m \geqslant 3$. It is readily checked that $\lambda_{P_m}(\{e\}, \{f\}) = 2$, but that P_m has no minor on $\{e, f\}$ with $\lambda(\{e\}) = 2$. Despite this example, polymatroids other than matroids do at times have Menger-like properties in some places. The following definitions enable us to make the distinction.

Let P be a polymatroid on E, and let A and B be disjoint subsets of E. Then the pair (A, B) is *Mengerian* if there exists a partition (X, Y) of $E - (A \cup B)$ such that $\lambda_{P \backslash X / Y}(A) = \lambda_P(A, B)$. Let T be a branch-decomposition of P, let a and b be distinct edges of T, let A be the set displayed by the branch of $T \backslash a$ not containing b, and let B be the set displayed by the branch of $T \backslash b$ not containing a. Then the pair (a, b) is *Mengerian* if (A, B) is Mengerian in P. Finally, we say that T is a *Mengerian branch-decomposition* if it is linked and there exists a vertex v in T such that whenever a and b are distinct edges such that either v is in the branch of $T \backslash a$ not containing b or the branch of $T \backslash b$ not containing a, then (a, b) is Mengerian. Moreover, a vertex with the properties described above is a *Mengerian* vertex. Note that a polymatroid with a Mengerian branch-decomposition may have many pairs of disjoint subsets which are not Mengerian.

In order to obtain any well-quasi-ordering result for polymatroids, it is necessary to restrict our attention to k-polymatroids, for some fixed k. Indeed, for each positive integer i, let Q_i be a polymatroid on a single-element ground set z such that $r_{Q_i}(z) = i$. Then $\{Q_i : i \geqslant 0\}$ is clearly an antichain of polymatroids, each of which is representable over any field. Finally, we can state the main result of this section.

THEOREM 5.2. *Let k and h be positive integers, and $\mathbb{F}$ be a finite field. Let S be an infinite set of $\mathbb{F}$-representable k-polymatroids, each of which has a Mengerian branch-decomposition of width h. Then there exist two members of S such that one is isomorphic to a minor of the other.*

To obtain Theorem 5.2 we prove a somewhat stronger theorem on embedded rooted polymatroids. The technique is a straightforward generalization of that used to prove the special case for matroids [3, Theorem 5.8]. In fact, a number of the arguments are essentially identical to those in [3], the only difference being the observation that the arguments hold in the more general polymatroid case. For the sake of a self-contained exposition we include a full proof here.

We begin by obtaining more information about arrangements. In what follows all arrangements are arrangements over a fixed field $\mathbb{F}$. We denote the span of a collection A of subspaces of $V(r, \mathbb{F})$ by $\langle A \rangle$. A *rooted* arrangement is simply an arrangement with a distinguished element f, and we denote such a rooted arrangement by (E, f), where $f \notin E$, (Although, the subspace labelled by f may also be labelled by elements of E.) Now a rooted arrangement (E', f) is a *minor* of the rooted arrangement (E, f) if $E' \cup \{f\}$ is a minor of $E \cup \{f\}$.

The next proposition enables us to extend the notion of a Mengerian pair of subsets to rooted arrangements. Let E be an arrangement and let $A \subseteq E$. We let $\delta_E(A)$ denote $\langle A \rangle \cap \langle E - A \rangle$; $\delta_E(A)$ is the "sub-space boundary" of A. Thus $\lambda_{P(E)}(A) = r(\delta_E(A)) + 1$. The following proposition is an easy corollary of Tutte's Linking Theorem.

PROPOSITION 5.3. *If A and B are disjoint subsets of an arrangement E such that (A, B) is a Mengerian pair in $P(E)$, and $\lambda_{P(E)}(A) = \lambda_{P(E)}(A, B) = \lambda_{P(E)}(B)$, then $(A, \delta_E(A))$ is a minor of $(E - B,\ \delta_E(B))$.*

Proof. Let (I, J) be a partition of $E - (A \cup B)$ such that $\lambda_{P(E)\backslash I/J}(A) = \lambda_{P(E)}(A, B)$. Note that $\lambda_{P(E)\backslash I/J}(A) = \lambda_{P(E)}(A)$. It follows that the subspaces spanned by A and by J are disjoint in the arrangement. Therefore, $(A, \delta_E(A))$ is equivalent to $(A, \delta_{E\backslash I/J}(A)) = (E - B,\ \delta_E(B))\backslash I/J$. ∎

The next proposition is just a restatement of Proposition 5.3.

PROPOSITION 5.4. *If $A \subseteq B \subseteq E$ with $\lambda_{P(E)}(A) = \lambda_{P(E)}(A, E - B) = \lambda_{P(E)}(B)$, then $(A, \delta_E(A))$ is a minor of $(B, \delta_E(B))$.*

We are now in a position to prove the main result of this section.

THEOREM 5.5. *Let $\mathbb{F}$ be a fixed finite field, and let k and h be positive integers. Let $\mathcal{E}$ be an infinite set of arrangements over $\mathbb{F}$, such that for all $E \in \mathcal{E}$, the polymatroid $P(E)$ is a k-polymatroid of branch-width h that has a Mengerian branch-decomposition. Then $\mathcal{E}$ contains two members, one of which is isomorphic to a minor of another.*

Proof. For each $E \in \mathcal{E}$, let T_E be a Mengerian branch-decomposition of $P(E)$ of width at most h. We now massage each T_E slightly. Choose a Mengerian vertex v of T_E. Add a new vertex r, and a new edge $\{v, r\}$. Now direct the edges of T_E to obtain a rooted tree rooted at r. All internal vertices of this tree, other than v, have outdegree 2; the outdegree of v is at most 3. By adding new leaf edges and leaves at vertices with outdegree less than 2, we may assume that each T_E is an oriented ternary tree.

For an edge a of T_E, let E^a be the set of elements of E displayed by the component of $T_E \backslash a$ not containing the root r of T_E. Moreover, let $X^a = \delta_E(E_a)$ and $\lambda(a) = \lambda_{P(E)}(E^a)$. Thus $\lambda(a)$ is the rank of X^a. We say that (E^a, X^a) is the *rooted arrangement associated with a*. Let F be the oriented ternary forest comprised of the oriented ternary trees $\{T_E : E \in \mathcal{E}\}$. Recall that if u is an internal vertex of T_E, then the edges leaving u are denoted $l(u)$, $m(u)$, and $r(u)$, while the edge directed into u is denoted $d(u)$.

The following claim is straightforward.

5.5.1. *If u is an internal vertex of F, then*

(i) $E^{d(u)} = E^{l(u)} \cup E^{m(u)} \cup E^{r(u)}$,
(ii) $X^{d(u)} \subseteq X^{l(u)} + X^{m(u)} + X^{r(u)}$, *and*
(iii) $X^{l(u)} \cap X^{m(u)} = \langle E^{l(u)} \rangle \cap \langle E^{m(u)} \rangle$,
$X^{l(u)} \cap X^{r(u)} = \langle E^{l(u)} \rangle \cap \langle E^{r(u)} \rangle$,
$X^{m(u)} \cap X^{r(u)} = \langle E^{m(u)} \rangle \cap \langle E^{r(u)} \rangle$.

Now, define a quasi-order $\preccurlyeq$ on the edges of F. Let E_i and E_j be, not necessarily distinct, members of $\mathcal{E}$, and let a and b be edges of T_{E_i} and T_{E_j}, respectively. Then $a \preccurlyeq b$ if (E^a, X^a) is isomorphic to a minor of (E^b, X^b). We have constructed an oriented ternary forest F with an h-edge labelling λ and a quasi-order $\preccurlyeq$ on its edges.

Next we check that all the conditions of Lemma 2.1 are satisfied. Assume that a is λ-linked to b in E. While we massaged T_E somewhat it still corresponds in an obvious way to a Mengerian branch-decomposition with the root as a Mengerian vertex. It follows from this that the pair $(E^b, E - E^a)$ is Mengerian in $P(E)$. Hence, by Proposition 5.3, (E^b, X^b) is a minor of (E^a, X^a). Thus $b \preccurlyeq a$, and part (a) of Lemma 2.1 is satisfied. Clearly $\preccurlyeq$ has no infinite strictly descending sequences so part (b) holds. The root edges clearly form an antichain in $\preccurlyeq$, so part (c) holds. The leaf edges of F correspond to rooted arrangements with at most one element and these are well-quasi-ordered since there are only $k+1$ isomorphism classes of single-element k-polymatroids and part (d) holds. Thus, the hypotheses of Lemma 2.1 are indeed satisfied.

Consequently, there exists an infinite sequence $(u_0, u_1, \ldots)$ of internal vertices of the forest that satisfies the conclusion of Lemma 2.1. To simplify notation, let (E_i^l, X_i^l), (E_i^m, X_i^m), (E_i^r, X_i^r) and (E_i^d, X_i^d) denote the rooted arrangements corresponding to $l(u_i)$, $m(u_i)$, $r(u_i)$ and $d(u_i)$, respectively.

For each nonnegative integer i, the subspace $X_i^l + X_i^m + X_i^r$ has rank at most $3h$. By replacing $(u_0, u_1, \ldots)$ by an appropriate subsequence we may assume that all the subspaces $X_i^l + X_i^m + X_i^r$ have the same rank. By applying appropriate bijective linear transformations we may assume that each $X_i^l + X_i^m + X_i^r$ is equal to the same subspace. As this subspace is a finite set containing each of X_i^l, X_i^m, X_i^r and X_i^d, there is a bounded number of distinct quadruples $(X_i^l, X_i^m, X_i^r, X_i^d)$. Thus some value, say (X^l, X^m, X^r, X^d) is repeated infinitely often. Thus, by replacing $(u_0, u_1, \ldots)$ with an appropriate subsequence we may assume that, for all $i \in \{0, 1, \ldots\}$, we have $X_i^l = X^l$, $X_i^m = X^m$, $X_i^r = X^r$ and $X_i^d = X^d$.

By parts (ii)–(iv) of Lemma 2.1, there exist linear transformations $\mathscr{L}_i$, $\mathscr{M}_i$ and $\mathscr{R}_i$ such that:

5.5.2. *for* $i \geqslant 1$,

(i) $\mathscr{L}_i$ *projects* (E_i^l, X^l) *onto* (E_{i-1}^l, X^l),
(ii) $\mathscr{M}_i$ *projects* (E_i^m, X^m) *onto* (E_{i-1}^m, X^m), *and*
(iii) $\mathscr{R}_i$ *projects* (E_i^r, X^r) *onto* (E_{i-1}^r, X^r).

Moreover, by Lemma 2.1(i), for each $i < j$.

5.5.3. (E_i^d, X^d) *is not isomorphic to a minor of* (E_j^d, X^d).

Consider, for each $i \in \{0, 1, \ldots\}$, the restriction π_i of the product $\mathscr{L}_1 \cdots \mathscr{L}_i$ to X^l, the restriction μ_i of the product $\mathscr{M}_1 \cdots \mathscr{M}_i$ to X^m, and the restriction ρ_i of the product $\mathscr{R}_1 \cdots \mathscr{R}_i$ to X^r. Now π_i, μ_i and ρ_i are permutations of X^l, X^m and X^r, respectively. As X^l, X^m and X^r are finite sets there exists $i < j$ such that the pairs (π_i, μ_i, ρ_i) and (π_j, μ_j, ρ_j) are equal.

Set $\mathscr{L} = \mathscr{L}_{i+1} \cdots \mathscr{L}_j$, $\mathscr{M} = \mathscr{M}_{i+1} \cdots \mathscr{M}_j$, and $\mathscr{R} = \mathscr{R}_{i+1} \cdots \mathscr{R}_j$. Then the restriction of $\mathscr{L}$ to X^l is $\pi_i^{-1}\pi_j = \pi_i^{-1}\pi_i$, the restriction of $\mathscr{M}$ to X^m is $\mu_i^{-1}\mu_i$, and the restriction of $\mathscr{R}$ to X^r is $\rho_i^{-1}\rho_i$. Thus each of these restrictions is the identity map. Clearly, $\mathscr{L}$ projects (E_j^l, X^l) onto (E_i^l, X^l), $\mathscr{M}$ projects (E_j^m, X^m) onto (E_i^m, X^m), and $\mathscr{R}$ projects (E_j^r, X^r) onto (E_i^r, X^r). Now, it is straightforward to see that (E_i^d, X^d) is isomorphic to a minor of (E_j^d, X^d); contrary to 5.5.3. ∎

Theorem 5.2 follows immediately from Theorem 5.5 since the quasi order of rooted arrangements over $\mathbb{F}$ is a refinement of the quasi-order of isomorphism classes of $\mathbb{F}$-representable polymatroids.

6. PROOF OF THE MAIN THEOREM

Let r_1 and r_2 be set functions on a common ground set E. The set function $r_1 + r_2$ is defined by $(r_1 + r_2)(A) = r_1(A) + r_2(A)$ for $A \subseteq E$. If M_1 and M_2 are matroids on a common ground set with rank functions r_1 and r_2, respectively, then $M_1 + M_2$ denotes the set function $r_1 + r_2$. The following straightforward result is well known.

LEMMA 6.1. *Let M_1 and M_2 be matroids on a common ground set E. Then*

(i) *$M_1 + M_2$ is a 2-polymatroid;*
(ii) *if both M_1 and M_2 are F-representable, then so too is $M_1 + M_2$;*
(iii) *if $a \in E$, then $M_1\backslash a + M_2\backslash a = (M_1 + M_2)\backslash a$, and $M_1/a + M_2/a = (M_1 + M_2)/a$.*

LEMMA 6.2. *Let e be an element of the common ground set E of the matroids M_1 and M_2.*

(i) *If $M_1\backslash e + M_1/e = M_2\backslash e + M_2/e$, then $M_1 = M_2$.*
(ii) *If $M_1\backslash e + M_1/e$ is a minor of $M_2\backslash e + M_2/e$, then M_1 is a minor of M_2.*

Proof. Set $P = M\backslash e + M/e$ and let I be a subset of $E - e$. The elementary verification of the following facts is omitted. Both I and $I \cup e$ are independent in M if and only if $r_P(I) = 2|I|$. Moreover I is independent in M and $I \cup e$ is not if an only if $r_P(I) = 2|I| - 1$. Thus, from P we can specify the independent sets of M. Part (i) follows from these observations.

Consider part (ii). By Lemma 6.1, there are sets S and T such that

$$(M_2\backslash S/T)\backslash e + (M_2\backslash S/T)/e = M_1\backslash e + M_1/e.$$

But then by part (i), $M_1 = M_2\backslash S/T$, so that M_1 is a minor of M_2 as required. ∎

Let X be a subset of the ground set of the matroid M. The *coclosure* of X, denoted $\mathrm{cl}^*(X)$ is the closure of X in the dual M^* of M. It is clear that $x \in \mathrm{cl}(X)$ if and only if x is a loop of M/X. Thus $x \in \mathrm{cl}^*(X)$ if and only if x is a coloop of $M\backslash X$.

LEMMA 6.3. *Let e be an element of M and (X, Y) be a partition of $E(M\backslash e)$. Then $e \in \mathrm{cl}(X)$ if and only if $e \notin \mathrm{cl}^*(Y)$.*

The next theorem is the key result of this section. Through it we will be able to apply Theorem 5.2 to a class of representable 2-polymatroids associated with excluded minors.

THEOREM 6.4. *Let e be an element of the matroid M, and set $P = M\backslash e + M/e$. If M has branch-width k, then P has branch-width at most $2k$. Moreover, any branch-decomposition of P is Mengerian.*

Proof. Let $E \cup e$ be the ground set of M, where $e \notin E$; thus E is the ground Set of P. We first show that the branch-width of P is at most $2k$. Take a width-k branch-decomposition of M and remove the label from the vertex labelled by e. We can regard this as the underlying tree for branch-decompositions of $M\backslash e$, M/e and P. Since the width of any edge is clearly at most k for the branch-decompositions of $M\backslash e$ and M/e, it follows immediately from the definitions of P and λ_P that the width of any edge is at most $2k$ for the branch-decomposition of P, so that the branch-width of P is indeed at most $2k$.

The more substantial task is to show that branch-decompositions are Mengerian. The following claim is straightforward.

6.4.1. *Let (X, Y) be a partition of E with $e \in \mathrm{cl}_M(X)$. Then $\lambda_P(Y) = 2\lambda_M(Y)$ if $e \notin \mathrm{cl}_M(Y)$, and $\lambda_P(Y) = 2\lambda_M(Y) - r_M(e)$ if $e \in \mathrm{cl}_M(Y)$.*

We now give sufficient conditions for a pair of subsets to be Mengerian in P.

6.4.2. *Let A and B be disjoint subsets of E. If $e \in \mathrm{cl}_M(A)$ or $e \in \mathrm{cl}^*_M(A)$, then (A, B) is Mengerian in P.*

Proof. Assume that $e \in \mathrm{cl}_M(A)$. If N is a minor of M with $A \cup B \subseteq E(N)$ then, since $e \in \mathrm{cl}_M(A)$, we also have $e \in \mathrm{cl}_N(A)$. So

$$\lambda_{N\backslash e}(A, B) = \lambda_N(A \cup e, B).$$

Also,

$$\lambda_N(A \cup e, B) \geqslant \lambda_{N/e}(A, B) \geqslant \lambda_N(A \cup e, B) - 1.$$

In particular, this holds when $N = M$.

Now assume that $\lambda_{M/e}(A, B) = \lambda_M(A \cup e, B) - 1$. By Tutte's Linking Theorem, there exists a partition (I, J) of $E - (A \cup B)$ such that $\lambda_N(A \cup e, B) = \lambda_M(A \cup e, B)$. It then follows immediately that $\lambda_{N\backslash e}(A, B) = \lambda_{M\backslash e}(A, B)$. Moreover, $\lambda_{M/e}(A, B) \geqslant \lambda_{N/e}(A, B) \geqslant \lambda_N(A \cup e, B) - 1 = \lambda_M(A \cup e, B) - 1 = \lambda_{M/e}(A, B)$. That is, $\lambda_{N/e}(A, B) = \lambda_{M/e}(A, B)$. Hence $\lambda_{P\backslash I/J}(A, B) = \lambda_{N\backslash e}(A, B) + \lambda_{N/e}(A, B) = \lambda_{M\backslash e}(A, B) + \lambda_{M/e}(A, B) = \lambda_P(A, B)$, so that (A, B) is Mengerian in P.

The other possibility is that $\lambda_{M/e}(A,B) = \lambda_M(A \cup e, B)$. By Tutte's Linking Theorem, there exists a partition (I,J) of $E - (A \cup B)$ such that $\lambda_{N/e}(A,B) = \lambda_{M/e}(A,B)$. It then follows that $\lambda_{M\backslash e}(A,B) \geqslant \lambda_{N\backslash e}(A,B) = \lambda_N(A \cup e, B) \geqslant \lambda_{N/e}(A,B) = \lambda_{M/e}(A,B) = \lambda_M(A \cup e, B) = \lambda_{M\backslash e}(A,B)$. Thus, $\lambda_{N\backslash e}(A,B) = \lambda_{M\backslash e}(A,B)$, so that $\lambda_{P\backslash I/J}(A,B) = \lambda_P(A,B)$ in this case too. In either case we conclude that (A,B) is Mengerian.

Now assume that $e \in \mathrm{cl}^*_M(A)$. Then $e \in \mathrm{cl}_{M^*}(A)$. If follows from the above argument that (A,B) is Mengerian in $M^*\backslash e + M^*/e = (M/e)^* + (M\backslash e)^*$. But $\lambda_M = \lambda_{M^*}$. Thus $\lambda_{M^*\backslash e + M^*/e} = \lambda_{M\backslash e + M/e} = \lambda_P$. Hence (A,B) is Mengerian in P in this case too. ∎

We now complete the proof of Theorem 6.4. Let T be a branch-decomposition of P. The task is to show that T is Mengerian, that is, to show that T has a Mengerian vertex. We first direct some of the edges of T. For an edge f of T with incident vertices v_1 and v_2, let (A_1, A_2) be the partition of E displayed by the components of $T{-}f$ containing v_1 and v_2, respectively. Direct f from v_1 to v_2 if $e \in \mathrm{cl}_M(A_2)$ and direct it from v_2 to v_1 if $e \in \mathrm{cl}_M(A_1)$. Denote the resulting bidirected graph by T_{cl}. Edges in T_{cl} can be undirected, directed in one direction, or directed in both directions.

From (6.4.2) we obtain:

6.4.3. *If v is a vertex of T_{cl} with the property that, for every other vertex u, there is a directed path from u to v, then v is a Mengerian vertex of T,*

From the definition of T_{cl} we obtain:

6.4.4. *If $(v_1, v_2, \ldots, v_k)$ is a path in T, and in T_{cl} the edge (v_{k-1}, v_k) is directed from v_{k-1} to v_k, then (v_1, v_2) is directed from v_1 to v_2.*

Assume that T_{cl} has a doubly directed edge. Say that v is one of its incident vertices. Then, by (6.4.4), there is a directed path from any other vertex of T_{cl} to v. So, by (6.4.3), v is a Mengerian vertex.

Now assume that T_{cl} has no doubly directed edges, but that every edge of T_{cl} is directed. In this case (6.4.4) implies that T_{cl} has a unique vertex satisfying the hypotheses of (6.4.3), and again we see that T has a Mengerian vertex, and is therefore Mengerian.

For the final case assume that T_{cl} has an undirected edge. Define the directed graph T_{cl^*} using the same criteria as that for T_{cl}, but replacing cl^* by cl. It follows from Lemma 6.3 and (6.4.2) that an edge is bidirected in T_{cl^*} if and only if it is undirected in T_{cl}. Moreover, both (6.4.3) and (6.4.4) hold for T_{cl^*}. In our case T_{cl^*} has a bidirected edge, so, again we deduce that T is Mengerian. ∎

Recall that if $\mathbb{F}$ is a field, then a matroid is *almost representable* over $\mathbb{F}$ if there is an element e such that both $M\backslash e$ and M/e are $\mathbb{F}$-representable. We are now in a position to prove Theorem 1.3, which, for convenience we restate here.

THEOREM 6.5. *Let $\mathbb{F}$ be a finite field and k be a positive integer. If $M_1, M_2, M_3, \ldots$ is an infinite sequence of matroids of branch-width k that are almost representable over $\mathbb{F}$, then there are two members of $M_1, M_2, M_3, \ldots$ such that one is isomorphic to a minor of another.*

Proof. For each i, let e_i be an element such that both $M_i \backslash e_i$ and M_i/e_i are $\mathbb{F}$-representable, and let P_i denote the polymatroid $M_i \backslash e_i + M_i/e_i$. By Lemma 6.1, P_i is an F-representable 2-polymatroid, and by Theorem 6.4, P_i has branch-width at most $2k$ and a Mengerian branch-decomposition. Thus, by Theorem 5.2, there exist i and j such that P_i is isomorphic to a minor of P_j. But then, by Lemma 6.2(ii), M_i is isomorphic to a minor of M_j as required. ∎

Theorems 1.1 and 1.4 follow immediately.

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REFERENCES

1. R. E. Bixby, On Reid's characterization of the ternary matroids, *J. Combin. Theory Ser. B* **26** (1979), 174–204.
2. R. Diestel, T. R. Jensen, K. Y. Gorbunov, and C. Thomassen, Highly connected sets and the excluded grid theorem, *J. Combin. Theory Ser. B* **75** (1999), 61–73.
3. J. Geelen, A. M. H. Gerards, and G. Whittle, Branch-width and well-quasi-ordering in matroids and graphs, *J. Combin. Theory Ser. B* **84** (2002), 270–290.
4. J. Geelen, A. M. H. Gerards, and A. Kapoor, The excluded minors for $GF(4)$-representable matroids, *J. Combin. Theory Ser. B* **79** (2000), 247–299.
5. J. Kahn and P. D. Seymour, On forbidden minors for $GF(3)$, *Proc. Amer. Math. Soc.* **102** (1988), 437–440.
6. J. Kruskal, Well-quasi-ordering, the tree theorem, and Vázsonyi's conjecture, *Trans. Amer. Math. Soc.* **75** (1960), 210–225.
7. T. Lazarson, The representation problem for independence functions, *J. London Math. Soc.* **33** (1958), 21–25.
8. C. St. J. A. Nash-Williams, On well-quasi-ordering finite trees, *Proc. Camb. Phil. Soc.* **59** (1963), 833–835.
9. N. Robertson and P. D. Seymour, Graph minors. II. Algorithmic aspects of tree-width, *J. Algorithms* **7** (1986), 309–322.
10. N. Robertson and P. D. Seymour, Graph minors. IV. Tree-width and well-quasi-ordering, *J. Combin. Theory Ser. B* **48** (1990), 227–254.
11. G.-C. Rota, Combinatorial theory, old and new, *in* "Proceedings of the International Congress on Mathematics," Nice, September 1970, pp. 229–233, Gauthier, Chichester, 1970.
12. P. D. Seymour, Matroid representation over $GF(3)$, *J. Combin. Theory Ser. B* **26** (1979), 159–173.

13. R. Thomas, A Menger-like property of tree-width: The finite case, *J. Combin. Theory Ser. B* **52** (1991), 67–76.
14. C. Thomassen, A simpler proof of the excluded minor theorem for higher surfaces, *J. Combin. Theory Ser. B* **70** (1997), 306–311.
15. W. T. Tutte, A homotopy theorem for matroids, I, II, *Trans. Amer. Math. Soc.* **88** (1958), 144–174.
16. W. T. Tutte, Matroids and graphs, *Trans. Amer. Math. Soc.* **90** (1959), 527–552.
17. W. T. Tutte, Menger's theorem for matroids, *J. Res. Nat. Bur. Stand.—B. Math.–Math. Phys.* **69B** (1965), 49–53.