Available online at www.sciencedirect.com
SCIENCE DIRECT

Journal of Combinatorial Theory, Series B 87 (2003) 264–269

Journal of Combinatorial Theory Series B

http://www.elsevier.com/locate/jctb

Cliques in dense GF(q)-representable matroids ☆

James Geelen[a] and Geoff Whittle[b]

[a] Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada
[b] School of Mathematical and Computing Sciences, Victoria University, Wellington, New Zealand

Received 21 March 2002

Abstract

We prove that, for any finite field $\mathbb{F}$ and positive integer n, there exists an integer λ such that if M is a simple $\mathbb{F}$-representable matroid with no $M(K_n)$-minor, then $|E(M)| \leqslant \lambda r(M)$.

MSC: 05B35

1. Introduction

We prove the following conjecture of Kung [5].

Theorem 1.1. *For any finite field $\mathbb{F}$ and graph G there exists an integer λ such that, if M is a simple $\mathbb{F}$-representable matroid with no $M(G)$-minor, then $|E(M)| \leqslant \lambda r(M)$.*

Note that it suffices to consider the case that G is a clique. Kung [3,4] proved Theorem 1.1 in the case that $G = K_4$ and $\mathbb{F}$ is any finite field and in the case that $G = K_5$ and $\mathbb{F} = GF(2)$.

For the remainder of the introduction we focus primarily on the class of binary matroids. Theorem 1.1 shows that, in the class of simple binary matroids with no $M(K_n)$-minor, the number of elements grows linearly with the rank. Note that, if we consider all (simple) binary matroids this growth rate becomes exponential. Also, if we exclude a non-graphic matroid instead of $M(K_n)$, then the growth rate is at least

☆ This research was partially supported by grants from the Natural Sciences and Engineering Research Council of Canada and the Marsden Fund of New Zealand.

E-mail address: jfgeelen@math.uwaterloo.ca (J. Geelen).

PII: S0095-8956(02)00009-6

quadratic (since the class will contain all graphic matroids); Kung conjectures that this is the correct order of magnitude.

Conjecture 1.2 (Kung [5]). *For any binary matroid N there exists an integer λ such that, if M is a simple binary matroid with no N-minor, then $|E(M)| \leqslant \lambda r(M)^2$.*

To prove this conjecture it would suffice to consider the case that N is a binary projective geometry. Kung has analogous conjectures for other finite fields, but for fields of non-prime order there are complications. For example, all binary matroids are in the class of GF(4)-representable matroids with no $U_{2,4}$-minor.

Restricted to the class of graphic matroids, Theorem 1.1 specializes to the following theorem of Mader [7].

Theorem 1.3 (Mader). *For any graph H there exists an integer λ such that, if G is a simple graph with no H-minor, then $|E(G)| \leqslant \lambda |V(G)|$.*

Mader's theorem readily implies the following theorem of Wagner [9].

Theorem 1.4 (Wagner). *For any positive integer n there exists an integer λ such that, if G is a graph with no K_n-minor, then G has chromatic number at most λ.*

Let M be a simple rank-m GF(q)-representable matroid. Consider a representation of M as a restriction of PG$(m-1, q)$. The *critical exponent* of M is the minimum of $r(M) - r(U)$ among all subspaces U of PG$(m-1, q)$ disjoint with M. The critical exponent of M is not dependent on the particular GF(q)-representation of M; see [6]. The critical exponent of a representable matroid is closely related to the chromatic number of a graph (see, for example, [6]), and there is an analogue of Wagner's theorem for representable matroids. Kung [5] conjectured the following result and showed that it is implied by Theorem 1.1.

Theorem 1.5. *For any finite field $\mathbb{F}$ and integer n there exists an integer λ such that, if M is an $\mathbb{F}$-representable matroid with no $M(K_n)$-minor, then the critical exponent of M is at most λ.*

Kung's argument [5] is short, but non-trivial, so we repeat it here. By Theorem 1.1, there is an integer λ such that, for any simple $\mathbb{F}$-representable matroid M with no $M(K_n)$-minor, $|E(M)| \leqslant \lambda r(M)$. Let M be a simple rank-m $\mathbb{F}$-representable matroid with no $M(K_n)$-minor. Consider M as a restriction of PG$(m-1, q)$, where q denotes the size of $\mathbb{F}$. By our choice of λ, $|X| \leqslant \lambda r_M(X)$ for each set $X \subseteq E(M)$. Then, by the Matroid Partition Theorem [1], $E(M)$ can be partitioned into λ independent sets. Given any one of these independent sets, we can find a hyperplane of PG$(m-1, q)$ disjoint from it. Intersecting all such hyperplanes, we obtain a subspace U of PG$(m-1, q)$ disjoint from M with $r(U) \geqslant r(M) - \lambda$. Hence, the critical exponent of M is at most λ.

2. Matroids with no $U_{2,q+2}$-minor

We assume that the reader is familiar with standard definitions in matroid theory. We use the notation of Oxley [8], with the exception that we denote the simple matroid canonically associated with the matroid M by si(M).

While we are primarily interested in GF(q)-representable matroids, we prove the following extension of Theorem 1.1 that was also conjectured by Kung [5].

Theorem 2.1. *For any positive integers n and q there exists an integer λ such that, if M is a simple matroid with no $U_{2,q+2}$- or $M(K_n)$-minor, then $|E(M)| \leqslant \lambda r(M)$.*

If q is a prime power, then $U_{2,q+2}$ is the shortest line that is not GF(q)-representable. For any positive integer q we define $\mathcal{U}(q)$ to be the class of matroids with no $U_{2,q+2}$-minor. It is well-known that a simple rank-r GF(q)-representable matroid has at most $\frac{q^r-1}{q-1}$ elements; Kung [5] showed that the same bound holds for matroids in $\mathcal{U}(q)$.

Lemma 2.2. *For any integers $r \geqslant 0$ and $q \geqslant 2$, if $M \in \mathcal{U}(q)$ is a simple rank-r matroid, then $|E(M)| \leqslant \frac{q^r-1}{q-1}$.*

The generality gained in extending Theorem 1.1 to Theorem 2.1 comes at the cost of increasing the constant λ. (This is of little concern, since the constants we obtain are tremendously large in either case.) We shall require an upper bound on the number of hyperplanes avoiding an element e of a rank-r matroid M. If M is GF(q)-representable, then, by considering PG$(r-1,q)$, we see that there are at most q^{r-1} such hyperplanes. On the other hand, when $M \in \mathcal{U}(q)$, we cannot prove a comparable bound and settle for the following crude upper bound.

Proposition 2.3. *Let $r \geqslant 1$ and $q \geqslant 2$ be integers and let $M \in \mathcal{U}(q)$ be a simple rank-r matroid. Then, M has at most $q^{r(r-1)}$ hyperplanes.*

Proof. Let $n = |E(M)|$; thus $n \leqslant \frac{q^r-1}{q-1} \leqslant q^r$. Each hyperplane is spanned by a set of $r-1$ points, so the number of hyperplanes is at most $\binom{n}{r-1} \leqslant n^{r-1} \leqslant q^{r(r-1)}$. □

3. Round matroids

We call a matroid M *round* if each cocircuit of M is spanning. Equivalently, M is round if and only if $E(M)$ cannot be written as the union of two proper flats. For a simple graph G, $M(G)$ is round if and only if G is a clique.

Theorem 3.1 (Geelen et al. [2]). *There exists an integer-valued function $f(n,q)$ such that, for any integers $n \geqslant 1$ and $q \geqslant 2$, if $M \in \mathcal{U}(q)$ is a round matroid with rank at least $f(n,q)$, then M contains an $M(K_n)$-minor.*

The following properties are straightforward to check:

1. If M is a round matroid and $e \in E(M)$ then M/e is round.
2. If N is a spanning minor of M and N is round, then M is round.

Let F be a flat of a matroid M. We call F *round* if the restriction of M to F is round. Each rank-one flat is round. Moreover, a rank-two flat is round if and only if it contains at least 3 rank-one flats. We call a rank-two flat with at least 3 rank-one flats a *long line*.

Lemma 3.2. *There exists an integer-valued function $\eta(c,q)$ such that, for any integers $c \geqslant 0$ and $q \geqslant 2$, if $M \in \mathscr{U}(q)$ is a simple matroid with $|E(M)| > \eta(c,q)r(M)$, then there exists a simple minor N of M that contains more than $c|E(N)|$ long lines.*

Proof. Let $\eta(c,q) = c\,q^2$. For each $v \in E$, let $N_v = \mathrm{si}(M/v)$. Inductively, we may assume that $|E(N_v)| \leqslant \eta(c,q)r(N_v)$ for each $v \in E$. Now, $r(N_v) = r(M) - 1$ and $|E(M)| > \eta(c,q)r(M)$, so $|E(M)| - |E(N_v)| \geqslant \eta(c,q) + 1$. Since $M \in \mathscr{U}(q)$, any long line in M has at most $q+1$ points; so when we contract an element the parallel classes contain at most q elements. Thus v is on at least $\eta(c,q)/(q-1)$ long lines. So the number of long lines is at least $\frac{\eta(c,q)}{(q-1)(q+1)}|E(M)| > c|E(M)|$. $\square$

Lemma 3.3. *Let M be a matroid, let F_1 and F_2 be round flats of M such that $r_M(F_1) = r_M(F_2) = k$ and $r_M(F_1 \cup F_2) = k+1$, and let F be the flat of M spanned by $F_1 \cup F_2$. If $F \neq F_1 \cup F_2$ then F is round.*

Proof. Let $e \in F - (F_1 \cup F_2)$; we may assume that $E(M) = F_1 \cup F_2 \cup \{e\}$. Suppose that M is not round, and let C, C' be a pair of disjoint cocircuits of M; we may assume that $e \notin C$. Also, since e is not a coloop, by possibly swapping F_1 and F_2, we may assume that $C' \cap F_1$ is non-empty. Note that, $E(M) - F_1$ is a cocircuit (containing e), so $C \cap F_1$ is nonempty. Let M_1 be the restriction of M to F_1. Then, $C \cap F_1$ and $C' \cap F_1$ both contain cocircuits of M_1, and these cocircuits are disjoint. This contradicts the fact that F_1 is round. $\square$

Let $\mathscr{F}$ be a set of round flats in M. A rank-k flat F is called *$\mathscr{F}$-constructed* if there exist rank-$(k-1)$ flats F_1, $F_2 \in \mathscr{F}$ such that $F = \mathrm{cl}_M(F_1 \cup F_2)$ and $F \neq F_1 \cup F_2$. Thus, the $\mathscr{F}$-constructed flats are round. To facilitate induction, we prove the following technical lemma that readily implies Theorem 2.1.

Lemma 3.4. *There exists an integer-valued function $\lambda(c,n,q)$ such that, for all integers $n \geqslant 2$, $c \geqslant 0$, and $q \geqslant 2$, if $M \in \mathscr{U}(q)$ is a simple matroid with $|E(M)| > \lambda(c,n,q)r(M)$, then there exists a simple minor N of M and a set $\mathscr{F}$ of round rank-$(n-1)$ flats of N such that the number of $\mathscr{F}$-constructed flats is greater than $c|\mathscr{F}|$.*

Proof. Let $\lambda(2,c,q) = \eta(c,q)$, and, for $n \geqslant 2$, we recursively define

$$\lambda(n+1,c,q) = \lambda(n, q^{(n+1)^2}c + q^n, q).$$

The proof is by induction on n. Consider the case that $n = 2$. Now, let $M \in \mathcal{U}(q)$ be a simple matroid with $|E(M)| > \lambda(2, c, q)r(M)$. By Lemma 3.2, there exists a simple minor N of M with more than $c|E(N)|$ long lines. Let $\mathcal{F}$ be the set of rank-one flats. The long lines are $\mathcal{F}$-constructed flats and $c|E(N)| = c|\mathcal{F}|$; as required.

Suppose that the result holds for $n = k$ and consider the case that $n = k + 1$. Now let $M \in \mathcal{U}(q)$ be a simple matroid with $|E(M)| > \lambda(k + 1, c, q)r(M)$. We let c' denote $q^{(k+1)^2}c + q^k$. By the induction hypothesis there exists a simple minor N of M and a set $\mathcal{F}$ of round rank-$(k - 1)$ flats of N such that the number of $\mathcal{F}$-constructed flats is greater than $c'|\mathcal{F}|$; suppose that N is minor-minimal with these properties.

Let $\mathcal{F}^1$ be the set of $\mathcal{F}$-constructed flats in N and let $\mathcal{F}^2$ be the set of $\mathcal{F}^1$-constructed flats in N. Now, for each $v \in E(N)$, let $N_v = \mathrm{si}(N/v)$. Let $\mathcal{F}_v$ denote the set of rank-$(k - 1)$ flats in N_v corresponding to the set of flats in $\mathcal{F}$ in N. That is, if $F \in \mathcal{F}$ and $v \notin F$ then $\mathrm{cl}_{N_v}(F) \in \mathcal{F}_v$. Let $\mathcal{F}^1_v$ be the set of $\mathcal{F}_v$-constructed flats in N_v. By our choice of N, $|\mathcal{F}^1| - c'|\mathcal{F}| > 0$, and, by the minimality of N, $|\mathcal{F}^1_v| - c'|\mathcal{F}_v| \leqslant 0$ for all $v \in E(N)$. Thus,

$$(|\mathcal{F}^1| - |\mathcal{F}^1_v|) - c'(|\mathcal{F}| - |\mathcal{F}_v|) > 0.$$

Let

$$\Delta = \sum(|\mathcal{F}| - |\mathcal{F}_v|\colon\ v \in E(N))$$

and

$$\Delta_1 = \sum(|\mathcal{F}^1| - |\mathcal{F}^1_v|\colon\ v \in E(N)).$$

This proves:

$$\Delta_1 - c'\Delta > 0. \tag{1}$$

Consider a flat $F \in \mathcal{F}^1$. By definition there exist flats $F_1, F_2 \in \mathcal{F}$ such that $F = \mathrm{cl}_N(F_1 \cup F_2)$ and there exists an element $v \in F - (F_1 \cup F_2)$. Now $\mathrm{cl}_{N_v}(F_1) = \mathrm{cl}_{N_v}(F_2)$, so these two flats in $\mathcal{F}$ are reduced to a single flat in $\mathcal{F}_v$. This proves:

$$\Delta \geqslant |\mathcal{F}^1|. \tag{2}$$

Now, for some $v \in E(N)$, compare $\mathcal{F}^1$ with $\mathcal{F}^1_v$. There are two ways to lose constructed flats; we can either contract an element in a flat or we contract two flats onto each other. Firstly, suppose $F \in \mathcal{F}_1$ and $v \in F$. Note that $F - \{v\}$ only has rank $k - 1$ in N/v, so it will not determine a flat in $\mathcal{F}^1_v$. Now F has rank k and, by Lemma 2.2, a rank k flat contains at most $\frac{q^k-1}{q-1} < q^k$ elements; we destroy F if we contract any one of these points. Secondly, consider two flats $F_1, F_2 \in \mathcal{F}^1$ that are contracted onto each other in N_v. Let F be the flat of N spanned by $F_1 \cup F_2$ in N. Since F_1 and F_2 are contracted onto a common rank-k flat in N_v, we see that F has rank $k + 1$ and $v \in F - (F_1 \cup F_2)$. Thus, $F \in \mathcal{F}^2$. Now, F has rank $k + 1$, so it has at most q^{k+1} points. Moreover, by Proposition 2.3, in a flat of rank $k + 1$ there are at most $q^{(k+1)k}$ rank-k flats avoiding a given element. Thus, $F - \{v\}$ contains at most $q^{(k+1)k}$ flats of $\mathcal{F}$;

these flats will be contracted to a single flat in $\mathscr{F}_v^1$. This proves:

$$\Delta_1 \leqslant q^k|\mathscr{F}^1| + q^{(k+1)^2}|\mathscr{F}^2|. \tag{3}$$

Now, combining (1)–(3), we get

$$q^{(k+1)^2}(|\mathscr{F}^2| - c|\mathscr{F}^1|) = q^{(k+1)^2}|\mathscr{F}^2| - (c' - q^k)|\mathscr{F}^1| \geqslant \Delta_1 - c'\Delta > 0.$$

Thus, $|\mathscr{F}^2| > c|\mathscr{F}^1|$. That is, the number of $\mathscr{F}^1$-constructed flats in N is greater than $c|\mathscr{F}^1|$; as required. □

Proof of Theorem 2.1. Let $\lambda = \lambda(f(n,q), 0, q)$ and let $M \in \mathscr{U}(q)$ be a simple matroid with $|E(M)| > \lambda r(M)$. By Lemma 3.4, M contains a simple minor N and a set $\mathscr{F}$ of round rank-$(f(n,q) - 1)$ flats such that the set of $\mathscr{F}$-constructed flats is non-empty. Let F be an $\mathscr{F}$-constructed flat. Then, the restriction of N to F is a round rank-$f(n,q)$ matroid, and hence, by Theorem 3.1, contains an $M(K_n)$-minor. □

References

[1] J. Edmonds, Minimal partition of a matroid into independent sets, J. Res. Nat. Bur. Standards Sect. B. 69 (1965) 67–72.

[2] J.F. Geelen, A.M.H. Gerards, G.P. Whittle, Disjoint cocircuits in matroids with large rank, J. Combin. Theory Ser. B 87 (2003) 270–279.

[3] J.P.S. Kung, Excluding the cycle geometries of the Kuratowski graphs from binary geometries, Proc. London Math. Soc. 55 (1987) 209–242.

[4] J.P.S. Kung, The long-line graph of a combinatorial geometry. I. Excluding $M(K_4)$ and the $(q+2)$-point line as minors, Quart. J. Math. Oxford 39 (1988) 223–234.

[5] J.P.S. Kung, Extremal matroid theory, in: N. Robertson, P.D. Seymour (Eds.), Graph Structure Theory, Amer. Math. Soc., Providence, RI, 1993, pp. 21–61.

[6] J.P.S. Kung, Critical problems, in: Matroid Theory (Seattle, WA, 1995), Contemporary Mathematics, Vol. 197, Amer. Math. Soc., Providence, RI, 1996, pp. 1–127.

[7] W. Mader, Homomorphieeigenschaften und mittlere Kantendichte von Graphen, Math. Ann. 174 (1967) 265–268.

[8] J.G. Oxley, Matroid Theory, Oxford University Press, New York, 1992.

[9] K. Wagner, Beweis einer Abschwächung der Hadwiger-Vermutung, Math. Ann. 153 (1964) 139–141.