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A SPLITTER THEOREM FOR INTERNALLY 4-CONNECTED BINARY MATROIDS*

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Abstract. We prove that if N is an internally 4-connected minor of an internally 4-connected binary matroid M with $E(N) \geq 4$, then there exist matroids $M_0, M_1, \ldots, M_n$ such that $M_0 \cong N$, $M_n = M$, and, for each $i \in \{1, \ldots, i\}$, M_{i-1} is a minor of M_i, $|E(M_{i-1})| \geq |E(M_i)| - 2$, and M_i is 4-connected up to separators of size 5.

Key words. binary matroids, Splitter Theorem, 4-connectivity

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1. Introduction.

We prove the following theorem.

THEOREM 1.1 (main theorem). *Let M be a binary matroid that is 4-connected up to separators of size 5 and let N be an internally 4-connected proper minor of M. If $|E(N)| \geq 10$, then either*

- *there exists $e \in E(M)$ such that $M \setminus e$ or M/e is 4-connected up to separators of size 5 and contains an N-minor, or*
- *M has a fan $(e_1, e_2, e_3, e_4, e_5)$ such that $M/e_3 \setminus e_4$ or $M \setminus e_3/e_4$ is 4-connected up to separators of size 5 and contains an N-minor.*

A matroid M is 4-*connected up to separators of size* k if M is 3-connected and for each 3-separation (A, B) of M either $|A| \leq k$ or $|B| \leq k$. A matroid is *internally* 4-*connected* if it is 4-connected up to separators of size 3. A sequence $(e_1, \ldots, e_i)$ of distinct elements of a matroid M is called a *fan* if the sets $\{e_1, e_2, e_3\}, \{e_2, e_3, e_4\}, \ldots, \{e_{i-2}, e_{i-1}, e_i\}$ are alternately triangles and triads. For other notation and terminology we follow Oxley [6], except we use $\mathrm{si}(M)$ and $\mathrm{co}(M)$ to denote the simplification and cosimplification, respectively, of a matroid M. Recall that M having an *N-minor* means that M has a minor isomorphic to N.

We remark that the bound $|E(N)| \geq 10$ in Theorem 1.1 is included only to simplify the proof; the result holds under the weaker hypothesis that $|E(M)| \geq 7$. (Thus we do not require a lower bound on $|E(N)|$.)

Seymour's Splitter Theorem [7] is a well-known inductive tool for studying 3-connected matroids.

THEOREM 1.2 (the Splitter Theorem). *Let M be a 3-connected matroid with $|E(M)| \geq 4$ and let N be a 3-connected proper minor of M. If M is not a wheel or a whirl, then there exists $e \in E(M)$ such that $M \setminus e$ or M/e is 3-connected and has an N-minor.*

The Splitter Theorem allows a 3-connected matroid to be built one element at a time from a given 3-connected minor so that the intermediate matroids are all 3-connected. Theorem 1.1 provides a similar result for internally 4-connected binary matroids.

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COROLLARY 1.3. *Let N be an internally 4-connected minor of an internally 4-connected binary matroid M, where $|E(N)| \geq 4$. Then there exists a sequence $M_0, M_1, \ldots, M_k$ of matroids such that $M_0 \cong N$, $M_k = M$, and, for each $i \in \{1, \ldots, k\}$, M_{i-1} is a minor of M_i, $|E(M_{i-1})| \geq |E(M_i)| - 2$, and M_i is 4-connected up to separators of size 5.*

We rely heavily on results of Hall [4], who proved the following analogue of Tutte's Wheels and Whirls Theorem.

THEOREM 1.4 (Hall [4]). *If M is 4-connected up to separators of size 5 and $|E(M)| \geq 5$, then either*

- *there exists $e \in E(M)$ such that $M \backslash e$ or M/e is 4-connected up to separators of size 5, or*
- *M has a fan $(e_1, e_2, e_3, e_4, e_5)$ such that $M/e_3 \backslash e_4$ or $M \backslash e_3/e_4$ is 4-connected up to separators of size 5.*

Note that Hall's theorem holds for all matroids, while Theorem 1.1 is only for binary matroids. The main reason is simply that this is what we could prove. There is a very useful lemma (Lemma 4.3) that is particular to binary matroids. We expect that there is a reasonable analogue of the Splitter Theorem for matroids that are 4-connected up to separators of size 5—not just for binary matroids. The applicability of Theorem 1.1 (discussed below) stems from the fact that the class of binary matroids is closed under 3-sums. As there is no reasonable analogue of a 3-sum for general matroids, the proposed generalization may be of only academic interest.

It is a shortcoming of Corollary 1.3 that the intermediate matroids are only 4-connected up to separators of size 5; it would be preferable if this could be strengthened to internally 4-connected. There are, however, numerous obstacles to obtaining such a theorem, even for graphs; see Johnson and Thomas [5]. They proved that if H is an internally 4-connected minor of an internally 4-connected graph G, then either H and G belong to a family of exceptional graphs, or G can be built from H by means of four possible constructions. Their intermediate graphs are "almost" internally 4-connected. Below we give some justification that, other than causing additional case analysis, Corollary 1.3 provides a satisfactory inductive tool for internally 4-connected binary matroids.

First we will outline how one might use Corollary 1.3 to prove Seymour's decomposition of regular matroids [7]. Seymour showed that every regular matroid can be obtained from graphic matroids, cographic matroids, and copies of R_{10} via 1-, 2-, and 3-sums. Equivalently, every internally 4-connected regular matroid is either graphic or cographic or is isomorphic to R_{10}. It would suffice to prove the following claim: *If M is a regular matroid that is 4-connected up to separators of size 5 and M has an $M^*(K_{3,3})$-minor, then either M is graphic or M is isomorphic to R_{10}.* This claim reduces easily to the case that M is internally 4-connected. Therefore, one could attempt to prove the result inductively by using Corollary 1.3. Here we see that relaxing the connectivity condition slightly (from internally 4-connected to 4-connected up to separators of size 5) facilitates the use of induction.

Let $\mathcal{M}$ be a minor-closed class of binary matroids. Recall that a matroid $N \in \mathcal{M}$ is a *splitter* for $\mathcal{M}$ if there is no 3-connected matroid in $\mathcal{M}$ that contains N as a proper minor. Determining whether a 3-connected matroid N is a splitter for $\mathcal{M}$ reduces to a finite case analysis via Seymour's Splitter Theorem. Analogously we could call N a 4-*splitter* if there is no internally 4-connected matroid in $\mathcal{M}$ that contains N as a proper minor. It is a straightforward exercise to prove that, if N is internally 4-connected with $|E(N)| \geq 9$ and N is a 4-splitter for $\mathcal{M}$, then there are only finitely many matroids in $\mathcal{M}$ that are 4-connected up to separators of size 5 and that contain N as a minor. It follows that, using Corollary 1.3, we can test whether or not N is a 4-splitter via a finite case check.

2. Small matroids. When $|E(N)| \geq 10$ it is clear that Theorem 1.1 implies Corollary 1.3. In this section we address the problems that arise for smaller matroids. There are only a few internally 4-connected binary matroids with $|E(N)| \leq 9$. The following result can be easily verified by the reader.

LEMMA 2.1. *If N is an internally 4-connected binary matroid with $|E(N)| \leq 9$, then either N is a uniform matroid with at most three elements or N is isomorphic to one of the following matroids: $M(K_4)$, F_7, F_7^*, $M(K_{3,3})$, or $M^*(K_{3,3})$.*

It follows from Tutte's Wheels and Whirls Theorem that if M is a 3-connected binary matroid with $|E(M)| \geq 4$, then M has an $M(K_4)$-minor. Thus, when $N = M(K_4)$, Corollary 1.3 is an immediate corollary of Theorem 1.4.

Using the Splitter Theorem and "blocking sequences," Zhou [9] studied internally 4-connected binary matroids with an F_7-minor.

LEMMA 2.2 (see Zhou [9]). *If M is an internally 4-connected binary matroid with a proper F_7-minor, then M has an internally 4-connected minor N with an F_7-minor and with $10 \leq |E(N)| \leq 11$.*

Let M be an internally 4-connected matroid with F_7 as a proper minor. By Lemma 2.2, M has an internally 4-connected minor N with $10 \leq |E(N)| \leq 12$ and with an F_7-minor. By the Splitter Theorem, there exists a sequence of 3-connected matroids $M_0, M_1, \ldots, M_j$ such that $M_0 \cong F_7$, $M_j = N$, and, for each $i \in \{1, \ldots, j\}$, there exists $e \in E(M_i)$ such that $M_{i-1} = M_i \setminus e$ or $M_{i-1} = M_i/e$. Since $M_0, \ldots, M_j$ have at most 11 elements, they are 4-connected up to separators of size 5. Now, applying Theorem 1.1 to N, we can prove Corollary 1.3 in the case that $N = F_7$. By duality, Corollary 1.3 holds when $N = F_7^*$.

There are exactly three 10-element binary matroids that are internally 4-connected and that contain an $M(K_{3,3})$-minor; these matroids, named R_{10}, N_{10}, and $\tilde{K_5}^*$, are defined in [7, 9]. The same techniques used by Zhou [9] in proving Lemma 2.2 can be used to prove the following result; we omit the straightforward but lengthy details.

LEMMA 2.3. *Let M be an internally 4-connected binary matroid with a proper $M(K_{3,3})$-minor. Then M has a minor isomorphic to R_{10}, N_{10}, $\tilde{K_5}^*$, or to the cycle matroid of one of the graphs in Figure 1.*

Now, considering each of the graphs in Figure 1, we can prove Corollary 1.3 when $N = M(K_{3,3})$ and $N = M(K_{3,3})^*$.

3. Basic lemmas on separations. In this section, we present some basic lemmas on separations that will be used in later sections.

Let $M = (E, r)$ be a matroid, where r is the rank function. For $A \subseteq E$, we let $\lambda_M(A)$ denote $r(A) + r(E \setminus A) - r(M)$. Then A is k-separating if and only if $\lambda_M(A) \leq k - 1$. We refer to λ_M as the *connectivity function* of M. Tutte [8] proved that the connectivity function is *submodular;* that is, if $X, Y \subseteq E(M)$, then

$$\lambda_M(X) + \lambda_M(Y) \geq \lambda_M(X \cap Y) + \lambda_M(X \cup Y).$$

The next lemma follows easily.

LEMMA 3.1. *Let X and Y be k-separating sets of M. If $X \cap Y$ is not $(k-1)$-separating in M, then $X \cup Y$ is k-separating in M.*

The *coclosure* of a set $X \subseteq E(M)$ is the closure of X in M^*. Clearly, an element $x \in E(M) \setminus X$ belongs to the coclosure of X if and only if x does not belong to the

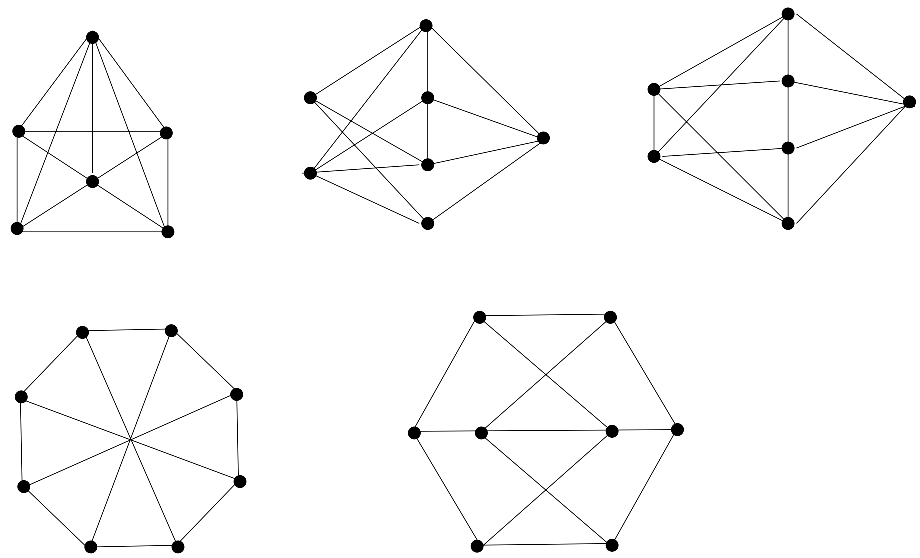

FIG. 1. *Internally 4-connected graphs.*

closure of $E(M)\backslash(X \cup \{x\})$. A set $X \subseteq E(M)$ is *coclosed* if the coclosure of X is the set X itself. We say X is *fully closed* if X is both closed and coclosed.

Let (A, B) be a k-separation of the matroid M. Following the terminology of [3], an element $x \in E(M)$ is in the *guts* of (A, B) if x belongs to the closure of both A and B. Dually, x is in the *coguts* of (A, B) if x belongs to the coclosure of both A and B. We say that (A, B) is an *exact k-separation* or A is *exactly k-separating* if $\lambda_M(A) = k - 1$. The next lemma follows easily from definitions.

LEMMA 3.2. *Let (A, B) be an exact k-separation of matroid M and let $x \in B$. Then*

- *$A \cup \{x\}$ is exactly k-separating if x belongs to either the guts or the coguts of (A, B) but not both;*
- *$A \cup \{x\}$ is exactly $(k-1)$-separating if x belongs to both the guts and the coguts of (A, B);*
- *$A \cup \{x\}$ is exactly $(k+1)$-separating if x belongs to neither the guts nor the coguts of (A, B).*

Suppose x is an element of the matroid M and let (A, B) be a k-separation of $M\backslash x$. Then x *blocks* (A, B) if neither $(A \cup \{x\}, B)$ nor $(A, B \cup \{x\})$ is a k-separation of M. Now let (A, B) be a k-separation of M/x. Then x *coblocks* (A, B) if neither $(A \cup \{x\}, B)$ nor $(A, B \cup \{x\})$ is a k-separation of M. The following lemma also follows easily from definitions.

LEMMA 3.3. *Let M be a matroid and let $\{A, B, \{x\}\}$ be a partition of $E(M)$. Then the following hold:*

- *If (A, B) is an exact k-separation of $M\backslash x$, then x blocks (A, B) if and only if x is not a coloop of M, $x \notin cl_M(A)$, and $x \notin cl_M(B)$.*
- *If (A, B) is an exact k-separation of M/x, then x coblocks (A, B) if and only if x is not a loop, $x \in cl_M(A)$, and $x \in cl_M(B)$.*

Suppose that X_1, X_2, Y_1, and Y_2 are sets. The pairs (X_1, Y_1) and (X_2, Y_2) are said to *cross* if all four sets $X_1 \cap X_2$, $X_1 \cap Y_2$, $Y_1 \cap X_2$, and $Y_1 \cap Y_2$ are nonempty. We omit the proof of the next lemma, which is a standard rank argument.

LEMMA 3.4. *Let e be an element of a 3-connected matroid M. Now, let (X_d, Y_d) be a 3-separation of $M\backslash e$ that is blocked by e, and let (X_c, Y_c) be a 3-separation of M/e that is coblocked by e. Then (X_d, Y_d) and (X_c, Y_c) cross. Moreover,*

- *one of $X_d \cap X_c$ and $Y_d \cap Y_c$ is 3-separating in M, and*
- *one of $X_d \cap Y_c$ and $Y_d \cap X_c$ is 3-separating in M.*

A matroid M is *internally* 3*-connected* if it is connected and, for each 2-separation (A, B) of M, either $|A| = 2$ or $|B| = 2$. The following result is due to Bixby [1].

LEMMA 3.5 (Bixby's lemma). *If e is an element of a 3-connected matroid M, then $M \setminus e$ or M/e is internally 3-connected.*

LEMMA 3.6. *Let (A, B) be a 3-separation of a 3-connected matroid M, where A is coclosed and $|A| \geq 4$. If $e \in A$ is in the guts of the separation (A, B), then $M \setminus e$ is 3-connected.*

Proof. Note that M/e is not internally 3-connected. Therefore, by Bixby's lemma, $M \setminus e$ is internally 3-connected. If $M \setminus e$ is not 3-connected, then there is a triad T of M with $e \in T$. Since $e \in \mathrm{cl}_M(B)$ and $e \in \mathrm{cl}_M(A - \{e\})$, we have $T \cap B \neq \emptyset$ and $T \cap (A - \{e\}) \neq \emptyset$. However, this contradicts the fact that A is coclosed. □

For disjoint sets $X, Y \subseteq E(M)$, we let $\kappa_M(X, Y) = \min\{\lambda_M(S) : X \subseteq S \subseteq E(M) \setminus Y\}$. It is clear that the function κ_M is minor monotone; that is, if N is a minor of M with $X \cup Y \subseteq E(N)$, then $\kappa_N(X, Y) \leq \kappa_M(X, Y)$. The following is due to Tutte [8].

THEOREM 3.7 (Tutte's Linking Theorem). *Let M be a matroid and let X and Y be disjoint subsets of $E(M)$. Then there exists a minor N of M with $E(N) = X \cup Y$ and $\lambda_N(X) = \kappa_M(X, Y)$.*

The next lemma is due to Geelen, Gerards, and Whittle [2, Lemma 4.11].

LEMMA 3.8. *Let M be a matroid and let $X, Y \subseteq E(M)$ be disjoint sets with $\kappa_M(X, Y) \geq k$. If $E(M)\backslash(X \cup Y) \neq \emptyset$, then either*

- *there exists an element $g \in E(M)\backslash(X \cup Y)$ such that $\kappa_{M/g}(X, Y) = \kappa_{M\backslash g}(X, Y) = \kappa_M(X, Y)$, or*
- *$\lambda_M(X) = k$ and there exists an ordering $b_1, b_2, \ldots, b_m$ of elements in $E(M)\backslash(X \cup Y)$ such that for $1 \leq i \leq m$, $\lambda_M(X \cup \{b_1, \ldots, b_i\}) = k$.*

4. Binary matroids and minors.

We require the following lemma.

LEMMA 4.1. *Let (A, B) be a 3-separation of a matroid M, and let $C \subseteq B$ be a circuit of M with $\kappa_M(A, C) = 2$. Then there exists a minor N of M such that $A \subseteq E(N) \subseteq A \cup C$, $C \cap E(N)$ is a triangle of N, and $\lambda_N(A) = 2$.*

Proof. We start with the following claim.

4.2. *There exists a minor M' of M such that $E(M') = A \cup C$, $\lambda_{M'}(A) = 2$, and C is a circuit of M'.*

Subproof. Suppose that M' is a minor of M such that $A \cup C \subseteq E(M)$, $\kappa_{M'}(A, C) = 2$, and C is a circuit of M'. The proof is by induction on $|E(M') - (A \cup C)|$. The result is trivial if $|E(M') - (A \cup C)| = 0$; suppose otherwise, and let $e \in E(M') - (A \cup C)$. If $\kappa_{M'\backslash e}(A, C) = 2$, then the result follows inductively; we may assume otherwise. Therefore, e is in the coguts of a 3-separation (Z_1, Z_2), where $A \subseteq Z_1$ and $C \subseteq Z_2$. It follows that $e \notin \mathrm{cl}_{M'}(C)$ and, hence, that C is a circuit in M'/C. Moreover, by Tutte's Linking Theorem, $\kappa_{M'/e}(A, C) = 2$. Now, considering M'/e, the result follows inductively. □

Let M' be as given in the claim. The proof now proceeds by induction on $|C|$. If $|C| = 3$, then the result is immediate. Thus we may assume that $|C| \geq 4$. Since $\lambda_{M'}(A) = 2 < r_{M'}(C)$, there exists $e \in C - \mathrm{cl}_{M'}(A)$. Thus e is not in the guts of the 3-separation (A, C) of M'. Therefore, $\lambda_{M'/e}(A) = 2$. Moreover, $C - \{e\}$ is a circuit of M'/e; thus the result follows inductively. □

LEMMA 4.3. *Let N be an internally 4-connected minor of a binary matroid M and let (A, B) be a 3-separation of M with $|B \cap E(N)| \leq 3$. If M' is a minor of M with $A \subseteq E(N)$, $|E(M') \cap B| \geq 4$, and $\lambda_{M'}(X) \geq \min(2, |X|)$ for all $X \subseteq E(M') \cap B$, then M' has an N-minor.*

Proof. Let $B' = B \cap E(M')$. By duality, we may assume that either $|E(N) \cap B| \leq 2$ or that $E(N) \cap B$ is a triangle of N. Since M' is binary and $|B'| \geq 4$, B' cannot be a line in M'^*; thus, $r^*_{M'}(B') \geq 3$. Then $B' \not\subseteq \mathrm{cl}^*_{M'}(A)$ and, hence, B' contains a circuit C of M'. By Lemma 4.1, M' has a minor M'' such that $A \subseteq E(M'')$, $\lambda_{M'}(A) = 2$, and $B \cap E(M)$ is a triangle of M''. Evidently N is isomorphic to a minor of M'' and, hence, also of M'. ∎

LEMMA 4.4. *Let N be an internally 4-connected minor of a 3-connected binary matroid M and let (A, B) be a 3-separation of M with $|A|, |B| \geq 5$. If e is in the guts of (A, B), then $M \setminus e$ has an N-minor.*

Proof. By symmetry we may assume that $|E(N) \cap B| \leq 3$. Since e is in the guts of the 3-separation (A, B), M/e is not internally 3-connected. Therefore, by Bixby's lemma, $M \setminus e$ is internally 3-connected. Thus, $\mathrm{co}(M \setminus e)$ is 3-connected. Since $e \in \mathrm{cl}_M(A)$, there is no series-pair of $M \setminus e$ contained in B. Therefore, $\lambda_{M'}(X) \geq \min(2, |X|)$ for all $X \subseteq B - \{e\}$. Then, by Lemma 4.3, $M \setminus e$ has an N-minor. ∎

LEMMA 4.5. *Let N be an internally 4-connected minor of a 3-connected binary matroid M and let (A, B) be a 3-separation of M with $|B| \geq 5$ and $|E(N) \cap B| \leq 3$. If A is fully closed, then there exists $e \in B$ such that $M \setminus e$ and M/e both contain an N-minor.*

Proof. Assume by way of contradiction that the result is false. Let $b \in B$. By duality we may assume that M/b does not have an N-minor. Then, by Lemma 4.3, there exists a 2-separating set $Y \subseteq B - \{b\}$ of M/b with $|Y| \geq 2$. Let $X = Y \cup \{b\}$. Then $X \subseteq B$ is a 3-separating set of M.

By Lemma 3.8 and the fact that A is fully closed, there exists $e \in B - X$ such that $\kappa_{M \setminus e}(A, X) = \kappa_{M/e}(A, X) = 2$. If $|X| \geq 4$, then the result follows easily from Lemma 4.3. Thus we may assume that $|X| = 3$. Since Y is 2-separating in M/b, X is a triangle of M. Let $M' \in \{M \setminus e, M/e\}$. Thus, it suffices to prove that M' has an N-minor. Since A is fully closed in M, $X \not\subseteq \mathrm{cl}_{M'}(A)$. By Tutte's Linking Theorem there exists a partition (D, C) of $E(M) - (A \cup X)$ such that $\lambda_{M' \setminus D/C}(A) = 2$; we choose such D and C so that $|C|$ is minimal. Note that $X \subseteq \mathrm{cl}_{M' \setminus D/C}(A)$ but $X \not\subseteq \mathrm{cl}_{M'}(A)$. Thus $C \neq \emptyset$; choose $f \in C$. Now, let $M'' = M' \setminus D/(C - \{f\})$. By the minimality of C, we have $\lambda_{M'' \setminus f}(A) = 1$ and $\lambda_{M''/f}(A) = 2$. Thus $(A, X \cup \{f\})$ is a 3-separation of M'' consisting of a triangle X with a point f in the coguts. Then, by Lemma 4.3, M'' has an N-minor. Therefore, M' has an N-minor, as required. ∎

5. The internally 4-connected case. The goal of this section is to prove the following theorem.

THEOREM 5.1. *Let N be an internally 4-connected proper minor of an internally 4-connected binary matroid M with $|E(M)| \geq 7$. Then there exists $e \in E(M)$ such that either $M \setminus e$ or M/e is 4-connected up to separators of size 5 and has an N-minor.*

We will make use of the following lemma of Hall [4, Theorem 3.1].

LEMMA 5.2. *Let M be an internally 4-connected binary matroid and $\{a, b, c\}$ be a triangle of M. Then at least one of $M \setminus a$, $M \setminus b$, and $M \setminus c$ is 4-connected up to separators of size 5.*

Note that, by Lemma 5.2, if we find a triangle of M such that each of the three elements can be deleted to keep the N-minor, then Theorem 5.1 holds. Such a triangle will be called *an N-deletable triangle*. Similarly, an *N-contractible triad* is a triad with the property that any one of its elements can be contracted to keep an N-minor.

Suppose M is an internally 4-connected binary matroid and M' is a minor of M. We call M' a *TT-connected minor of* M if the following hold:

- M' is internally 3-connected.
- If (X, Y) is a 3-separation of M', then either $|X| \leq 6$ or $|Y| \leq 6$.
- If (X, Y) is a 3-separation of M' with $\min(|X|, |Y|) = 6$, then one of X and Y can be partitioned into two disjoint subsets of size 3, each of which is a triangle or triad of M.

LEMMA 5.3. *Let M be an internally 4-connected binary matroid and let $e \in E(M)$. Then at least one of $M\backslash e$ and M/e is a TT-connected minor of M.*

Proof. Since M is internally 4-connected, $M\backslash e$ and M/e are both internally 3-connected. Either the lemma holds or there exist 3-separations (X_d, Y_d) and (X_c, Y_c) of $M\backslash e$ and M/e, respectively, such that the four sets X_d, Y_d, X_c, and Y_c all have size at least 6 and none of them is the union of two 3-separating sets in M. By Lemma 3.4, one of $X_d \cap X_c$ and $Y_d \cap Y_c$ is 3-separating in M, and one of $X_d \cap Y_c$ and $Y_d \cap X_c$ is 3-separating in M. By duality, we may assume that $X_d \cap X_c$ and $X_d \cap Y_c$ are 3-separating in M. Therefore X_d is the union of two 3-element 3-separating sets of M, which is a contradiction. □

LEMMA 5.4. *Let M be an internally 4-connected binary matroid and let N be an internally 4-connected minor of M with $E(N) \geq 10$. If $M\backslash e$ is a 3-connected TT-connected minor of M and has an N-minor, then there exists $f \in E(M)$ such that either $M\backslash f$ or M/f is 4-connected up to separators of size 5 and has an N-minor.*

Proof. Assume that $M\backslash e$ is not 4-connected up to separators of size 5. Then there exists a 3-separation (X, Y) of $M\backslash e$ with $|X| = 6$, $|Y| \geq 6$, and X is a disjoint union of two 3-element 3-separating sets, T_1 and T_2 of M. Since N is internally 4-connected and $E(N) \geq 10$, we must have $|E(N) \cap X| \leq 3$. Up to symmetry, we have two cases.

Case 1. T_1 is a triangle, and T_2 is a triad of M.

Since M is internally 4-connected, T_2 is closed in M and, hence, also in $M \backslash e$. Then, since M is binary, we must have $r_M(T_1 \cup T_2) = 5$. So $r^*_{M\backslash e}(T_1 \cup T_2) = 6 - (r(M) - r_M(Y)) = 6 + \lambda_{M\backslash e}(X) - r_M(X) = 3$. Now $T_1 \cup T_2$ is a rank-3 3-separating set in $(M \backslash e)^*$, and T_1 is a triad in $(M \backslash e)^*$. Therefore, $T_2 \subseteq \mathrm{cl}_{(M\backslash e)^*}(Y)$. Thus, by Lemma 4.4, T_2 is an N^*-deletable triangle in $(M \backslash e)^*$. Hence, T_2 is an N-contractible triad in M, proving the result.

Case 2. T_1 and T_2 are both triangles or both triads of M.

Choose $(M', N') \in \{(M\backslash e, N), ((M\backslash e)^*, N^*)\}$ such that T_1 and T_2 are both triads of M'. Since M'^* has no parallel pairs and since M'^* is binary, we have $r_{M'^*}(T_1 \cup T_2) = 4$. It follows that $r_{M'}(T_1 \cup T_2) = 4$. Thus, considering a geometric representation of M', T_1 and T_2 are triads spanning a common line. Now, by Lemma 4.3, we see that T_1 is an N'-contractible triad of M'. Hence, T_1 is either an N-contractible triad or an N-deletable triangle of M, proving the result. □

LEMMA 5.5. *Let M be an internally 4-connected binary matroid and let N be an internally 4-connected minor of M with $E(N) \geq 10$. Let $e \in E(M)$ such that both $M\backslash e$ and M/e have an N-minor. Then there exists $f \in E(M)$ such that either $M\backslash f$ or M/f is 4-connected up to separators of size 5 and has an N-minor.*

Proof. First assume e belongs to a triangle (or a triad) T of M. Since both $M\backslash e$ and M/e have an N-minor, T is an N-deletable triangle (or an N-contractible triad) of M. So the lemma follows from Lemma 5.2. Now we assume that e is not in a triangle or triad of M. Hence both $M\backslash e$ and M/e are 3-connected. So the result follows from Lemmas 5.3 and 5.4. □

Proof of Theorem 5.1. By the discussion in section 2, we may assume that $|E(N)| \geq 10$. By the Splitter Theorem, there exists $e \in E(M)$ and $M' \in \{M \backslash e, M/e\}$ such that M' is 3-connected and has an N-minor. Now, by Lemma 5.4, we can assume that M' is not a TT-connected minor of M. Let (A, B) be a 3-separation of M', where $|A|, |B| \geq 6$ and neither A nor B is a disjoint union of two 3-element 3-separating sets of M'. We may assume that $|E(N) \cap B| \leq 3$. Since $|E(N)| \geq 10$, $|A \cap E(N)| \geq 7$. Now, we may further assume that B is fully closed in M'.

By Lemma 5.5, we may assume that there is no element $f \in B$ such that $M' \setminus f$ and M'/f both have an N-minor. Then, by Lemma 4.5, there exists an element $f \in B$ that is in the closure or the coclosure of A in M'. By duality we may assume that $f \in \mathrm{cl}_{M'}(A)$. By Lemma 4.4, $M' \setminus f$ has a minor N' isomorphic to N.

Let $B' = B \cup \{e\} - \{f\}$. Note that $(A, B - \{f\})$ is a 2-separation of M'/f and, hence, (A, B') is a 3-separation of M/f. By Lemma 3.6, $M' \backslash f$ is 3-connected. Now it is easy to verify that e either blocks or coblocks the 3-separation $(A, B - \{f\})$ in $M' \backslash f$ and, hence, $M \setminus f$ is also 3-connected. By Lemma 5.4, we may assume that $M' \setminus f$ is not a TT-connected minor of M. Therefore, by Lemma 5.3, M'/f is a TT-connected minor of M. Now (A, B') is a 3-separation of M and $|A| \geq 7$. Therefore, $|B'| = 6$ and B' is the union of two 3-separating sets of M. Therefore there exists a triangle or triad $T \subseteq B'$ of M that contains e. First we consider the case that T is a triangle. Then, since M' is 3-connected, we have $M' = M \setminus e$. However, $T - \{e\} \subseteq B$, which contradicts the fact that e blocks the 3-separation (A, B) in M'. Now suppose that T is a triad. Then, since M' is 3-connected, we have $M' = M/e$. However, $T - \{e\} \subseteq B$, which contradicts the fact that e coblocks the 3-separation (A, B) in M'. ∎

6. Proof of the main theorem. In this section we complete the proof of Theorem 1.1. We break the proof into two cases depending on whether or not M is 4-connected up to separators of size 4.

LEMMA 6.1. *Let M be a binary matroid that is 4-connected up to separators of size 4 and let N be an internally 4-connected proper minor of M with $|E(N)| \geq 8$. Then there exists $e \in E(M)$ such that either $M \backslash e$ or M/e is 4-connected up to separators of size 5 and has an N-minor.*

Proof. By Theorem 5.1, we may assume that M has a 4-element 3-separating set $X = \{a, b, c, d\}$. Let $Y = E(M) - X$. By the Splitter Theorem, we may assume that $|E(M)| \geq 13$. Since M is binary, it suffices to consider the following two cases.

Case 1. (a, b, c, d) is a fan of M.

By symmetry we may assume that $\{a, b, c\}$ is a triangle. Note that N is a minor of either $M \setminus a$ or M/d. By duality we may assume that N is a minor of $M \setminus a$. Since M is 4-connected up to separators of size 4, X is fully closed in M. Then, by Lemma 3.6, $M \backslash a$ is 3-connected. Suppose that (A, B) is a 3-separation of $M \backslash a$ with $|A \cap \{b, c, d\}| \geq 2$. Then $A \cup \{b, c, d\}$ is 3-separating in $M \backslash a$ and, since $a \in \mathrm{cl}_M(\{b, c\})$, $A \cup X$ is 3-separating in M. It follows that $M \setminus a$ is 4-connected up to separators of size 5, as required.

Case 2. X is both a circuit and a cocircuit of M.

Since $|E(N)| \geq 8$ and N is internally 4-connected, we have $|E(N) \cap B| \leq 3$. By duality and symmetry, we may assume that N is a minor of $M \setminus a$. We claim that $M \setminus a$ is 4-connected up to separators of size 5. Since X is coclosed in M, $M \setminus a$ is cosimple. Suppose that (A, B) is a 2- or a 3-separation in $M \setminus a$ with $|A \cap \{b, c, d\}| \geq 2$. Then, since $a \in \mathrm{cl}_M(\{b, c, d\})$ and since $\{b, c, d\}$ is a triad in $M \backslash a$, $\lambda_M(B - X) = \lambda_M(A \cup X) = \lambda_{M \backslash a}(A \cup \{b, c, d\}) = \lambda_{M \backslash a}(A)$. Now $|B - X| \geq |B| - 1$. Thus if (A, B) is a 2-separation in $M \setminus a$, then, since M is 3-connected, $|B| \leq 2$.

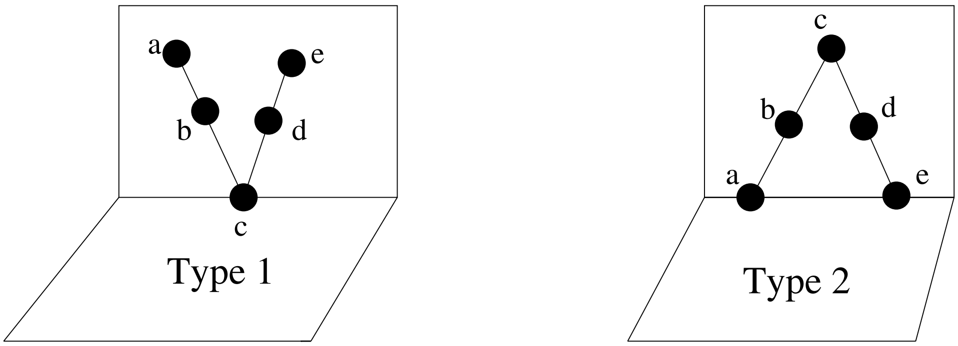

FIG. 2. 3-*separating sets of size* 5.

Since $M \setminus a$ is cosimple, $|B| \leq 1$ and, hence, $M \setminus a$ is 3-connected. Thus if (A, B) is a 3-separation in $M \setminus a$, then, since M is 4-connected up to separators of size 4, $|B| \leq 5$. Thus, $M \setminus a$ is 4-connected up to separators of size 5. ∎

Suppose that M is a binary matroid that is 4-connected up to separators of size 5 and that (X, Y) is a 3-separation of M with $|X| = 5$. Note that $r_M(X) + r^*_M(X) = r_M(X) + |X| - (r(M) - r_M(Y)) = |X| + \lambda_M(X) = 7$. Moreover, since M is binary, $r_M(X), r^*_M(X) \geq 3$. By duality we may assume that $r_M(X) = 3$. Now, since M is 3-connected and binary, there are either one or two elements of X in the guts of (X, Y). Thus, $X = \{a, b, c, d, e\}$ is of one of the following two types:

Type 1. $\{a, b, d, e\}$ is both a circuit and a cocircuit of M, and $\{a, b, c\}$ and $\{c, d, e\}$ are both triangles of M.

Type 2. (a, b, c, d, e) is a fan where $\{a, b, c\}$ is a triangle.

These two types of separations are depicted in Figure 2. The next lemma can be found in Hall [4].

LEMMA 6.2. *Let M be a matroid that is* 4*-connected up to separators of size* 5 *and let (X, Y) be a* 3*-separation of M with $X = \{a, b, c, d, e\}$.*

- *If X is a separation of Type* 1*, then one of $M \backslash a$, $M \backslash b$, and $M \backslash c$ is* 4*-connected up to separators of size* 5.
- *If X is a separation of Type* 2*, then one of $M \setminus a$, $M \setminus e$, and $\mathrm{co}(M \setminus c)$ is* 4*-connected up to separators of size* 5.

LEMMA 6.3. *Let M be a binary matroid that is* 4*-connected up to separators of size* 5 *and let N be an internally* 4*-connected proper minor of M with $|E(N)| \geq 8$. If $X = \{a, b, c, d, e\}$ is a* 3*-separating set of Type* 1*, then there exists $f \in X$ such that $M \setminus f$ is* 4*-connected up to separators of size* 5 *and has an N-minor.*

Proof. Since $|E(N)| \geq 8$, $|E(N) \cap X| \leq 3$. By Lemma 4.3, each of $M \setminus a$, $M \setminus c$, and $M \setminus e$ has an N-minor. So the theorem follows from Lemma 6.2. ∎

LEMMA 6.4. *Let M be a binary matroid that is* 4*-connected up to separators of size* 5 *and let N be an internally* 4*-connected proper minor of N with $|E(N)| \geq 7$. If $X = \{a, b, c, d, e\}$ is a* 3*-separating set of Type* 2*, then one of $M \setminus a$, $M \setminus e$, and $M \setminus c/d$ is* 4*-connected up to separators of size* 5 *and has an N-minor.*

Proof. Since X is a fan and N is internally 4-connected, $|E(N) \cap X| \leq 3$. By Lemma 4.3, both $M \setminus a$ and $M \setminus e$ have an N-minor. So we may assume that neither $M \setminus a$ nor $M \setminus e$ is 4-connected up to separators of size 5. So, by Lemma 6.2, $M \setminus c/d$ is 4-connected up to separators of size 5. Thus we may assume that $M \setminus c/d$ has no N-minor. It follows that $|E(N) \cap X| = 3$, that $E(N) \cap X$ is a triad of N, and that none of M/b, M/c, and M/d has an N-minor.

6.5. *$M \setminus a$ is 3-connected and there exists a 3-separation (A, B) in $M \setminus a$ with $|A|, |B| \geq 6$ and with b or c in its coguts.*

Subproof. By Lemma 3.6, $M \setminus a$ is 3-connected. However, $M \setminus a$ is not 4-connected up to separators of size 5. So there exists a 3-separation (A, B) of $M \setminus a$ with $|A|, |B| \geq 6$. By symmetry we may assume that $|\{b, c, d\} \cap A| \geq 2$. Since $a \in \mathrm{cl}_M(\{b, c\})$ and since a blocks the separation (A, B), we have $|B \cap \{b, c\}| = 1$. Let $f \in \{b, c\} \cap B$. Since $\{b, c, d\}$ is a triad, f is in the coguts of (A, B). ◻

Let (A, B) be the 3-separation of $M \setminus a$ mentioned above and let $f \in \{b, c\}$ be in its coguts. By Lemma 4.4, $M \setminus a/f$ has an N-minor. But this contradicts the fact that M/f has no N-minor. ◻

REFERENCES

[1] R. E. Bixby, *A simple theorem on 3-connectivity*, Linear Algebra Appl., 45 (1982), pp. 123–126.

[2] J. F. Geelen, A. M. H. Gerards, and G. Whittle, *Excluding a Planar Graph from $GF(q)$-Representable Matroids*, manuscript.

[3] J. F. Geelen and G. Whittle, *Matroid 4-connectivity: A deletion-contraction theorem*, J. Combin. Theory Ser. B, 83 (2001), pp. 15–37.

[4] R. Hall, *A chain theorem for 4-connected matroids*, J. Combin. Theory Ser. B, 93 (2005), pp. 45–66.

[5] T. Johnson and R. Thomas, *Generating internally four-connected graphs*, J. Combin. Theory Ser. B, 85 (2002), pp. 21–58.

[6] J. G. Oxley, *Matroid Theory*, Oxford University Press, New York, 1992.

[7] P. D. Seymour, *Decomposition of regular matroids*, J. Combin. Theory Ser. B, 28 (1980), pp. 305–359.

[8] W. T. Tutte, *Connectivity in matroids*, Canad. J. Math., 18 (1966), pp. 1301–1324.

[9] X. Zhou, *On internally 4-connected non-regular binary matroids*, J. Combin. Theory Ser. B, 91 (2004), pp. 327–343.