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Matroid 4-Connectivity: A Deletion–Contraction Theorem

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A 3-separation (A, B), in a matroid M, is called *sequential* if the elements of A can be ordered $(a_1, ..., a_k)$ such that, for $i = 3, ..., k$, $(\{a_1, ..., a_i\}, \{a_{i+1}, ..., a_k\} \cup B)$ is a 3-separation. A matroid M is *sequentially* 4-*connected* if M is 3-connected and, for every 3-separation (A, B) of M, either (A, B) or (B, A) is sequential. We prove that, if M is a sequentially 4-connected matroid that is neither a wheel nor a whirl, then there exists an element x of M such that either $M \backslash x$ or M/x is sequentially 4-connected.

1. INTRODUCTION

In this, paper we find an analogue of Tutte's Wheels and Whirls Theorem [10] for 4-connected matroids. We begin by recalling Tutte's theorem.

THEOREM 1.1 (Wheels and Whirls Theorem). *If M is a 3-connected matroid that is neither a wheel nor a whirl, then M has an element x such that either $M \backslash x$ or M/x is 3-connected.*

While there is general agreement on an appropriate definition for matroid 3-connectivity, for higher connectivity the situation is more problematic. Recall Tutte's definition of connectivity [10]. Let M be a

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matroid with ground set E. A set $X \subseteq E$ is *k-separating* if $r(X) + r(E-X) \leqslant r(E) + k - 1$. Thus, a partition (X, Y) of E is a *k-separation* of M if X is k-separating and $|X|, |Y| \geqslant k$. Now, M is *k-connected* if and only if M has no k'-separation, where $k' < k$.

Tutte's notion of connectivity is attractive, as it is self-dual. Moreover, the definition of matroid 3-connectivity is intimately related to graph 3-connectivity and has proved to be fundamental in matroid representation theory and matroid structure theory. But this is not the case for 4-connectivity. Indeed, from a graph theorist's point of view, complete graphs should be regarded as highly connected but, due to the existence of triangles, are not 4-connected in Tutte's definition. Moreover, from the perspective of matroid representation, we would like to think of projective geometries as being 4-connected, but, due to the existence of long lines, this is not the case. We believe that there will not be a universally accepted definition for 4-connectivity, but that several definitions will emerge based on the intended application. However, note that throughout this paper, by a 4-connected matroid we will always mean one that is 4-connected in the sense defined above. Other notions of 4-connectivity will be qualified with an adjective.

A k-separation (A, B) is called *sequential* if the elements of A can be ordered $(a_1, ..., a_m)$ such that $\{a_1, ..., a_i\}$ is k-separating for $i = 1, ..., m$. A k-separation (A, B) is *non-sequential* if neither (A, B) nor (B, A) is sequential. A matroid M is *sequentially* 4-*connected* if M is 3-connected and has no non-sequential 3-separations.

Sequential 4-connectivity is a self-dual notion; moreover, matroids of complete graphs and projective geometries are sequentially 4-connected. It is readily checked that wheels and whirls are sequentially 4-connected. The main result of this paper is the following.

THEOREM 1.2. *If M is a sequentially* 4-*connected matroid that is neither a wheel nor a whirl, then M has an element x such that either $M \backslash x$ or M/x is sequentially* 4-*connected.*

Our main motivation is for intended applications in matroid representation theory. Kahn [4] conjectured that *for any prime power q there exists an integer n_q such that any* 3-*connected matroid has at most n_q inequivalent representations over $GF(q)$*. Unfortunately this conjecture fails for all $q > 5$; see Oxley, Vertigan, and Whittle [7]. This suggests that 3-connectivity is not enough for substantial progress and that higher connectivity is needed. Our goal is to use Theorem 1.2 to prove the following conjecture.

Conjecture 1.3. For any prime power q there exists an integer n_q such that any sequentially 4-connected matroid has at most n_q inequivalent representations over $GF(q)$.

One may wonder why we have chosen sequential 4-connectivity in the statement of the conjecture over other variations of 4-connectivity. A 3-connected matroid M on E is *vertically* 4-connected if whenever X is 3-separating in M, either $r(X) \leqslant 2$ or $r(E-X) \leqslant 2$. Vertical 4-connectivity is a minimal relaxation of 4-connectivity that holds for projective geometries. Using the operation of "segment–cosegment exchange," introduced by Oxley, Semple, and Vertigan [6], it is not hard to show that representations of a sequentially 4-connected matroid over a field are in one-to-one correspondence with representations of a canonically associated vertically 4-connected matroid. It follows that, for many purposes, sequential 4-connectivity is no weaker than vertical 4-connectivity. However, sequential 4-connectivity enjoys the significant advantage that it is maintained under duality.

For graphs, Johnson and Thomas [3] have proved a theorem on "internal" 4-connectivity that is analogous to Seymour's Splitter Theorem [8]. They show how to build an internally 4-connected graph from an internally 4-connected minor in small steps that keep the intermediate graphs "almost internally" 4-connected. In the light of this it seems natural to look for an analogue to the Splitter Theorem for sequential 4-connectivity. However, from the perspective of our intended applications, we foresee other variations on 4-connectivity providing more useful splitter theorems.

We assume that the reader is familiar with basic notions in matroid theory; see Oxley [5] for an excellent introduction. Also see Truemper [9] for a deep discussion of aspects of higher connectivity. In addition to standard notation, we let $\mathrm{si}(M)$ and $\mathrm{co}(M)$ denote the simplification and cosimplification of M, respectively. A set X of elements is a *segment* of M if $|X| \geqslant 3$ and every 3-element subset of X is a triangle. Dually X is a *cosegment* if $|X| \geqslant 3$ and every 3-element subset of X is a triad.

2. 3-CONNECTIVITY

In this section we review results on 3-connectivity. The purpose of this is to show how standard 3-connectivity theorems can be obtained as a straightforward consequence of a lemma of Bixby and Coullard. This casts a somewhat different light on these results, and it also serves to motivate the approach taken in this paper.

Bixby [1] originally stated the following lemma for the case that $k=3$. Coullard [2] observed that the proof, which is a simple rank argument, works for arbitrary k; see [5, pp. 296–297]. We will use this lemma as a starting point in the proof of Theorem 1.2.

LEMMA 2.1. *Let e be an element of a k-connected matroid M, and let (X_d, Y_d) and (X_c, Y_c) be $(k-1)$-separations in $M \backslash e$ and M/e, respectively.*

Then, (X_d, Y_d) *and* (X_c, Y_c) *cross. Moreover, either* $|X_d \cap X_c| \leqslant k-2$ *or* $|Y_d \cap Y_c| \leqslant k-2$.

The following theorem of Bixby [1] is an easy consequence of Lemma 2.1.

THEOREM 2.2. *If e is an element of a* 3*-connected matroid M, then one of the following holds.*

1. $\mathrm{co}(M\backslash e)$ *is* 3*-connected. Moreover, no series class of* $M\backslash e$ *contains more than* 2 *elements.*

2. $\mathrm{si}(M/e)$ *is* 3*-connected. Moreover, no parallel class of* M/e *contains more than* 2 *elements.*

The proof of Tutte's Wheels and Whirls Theorem can be easily derived from Theorem 2.2 and the following lemma of Tutte [10].

LEMMA 2.3 (Tutte's Triangle Lemma). *Let* $\{t_1, t_2, t_3\}$ *be a triangle in a* 3*-connected matroid. If neither* $M\backslash t_1$ *nor* $M\backslash t_2$ *is* 3*-connected, then there exists a triad using* t_1 *and exactly one of* t_2 *and* t_3.

We now briefly sketch the proof of Tutte's Wheels and Whirls Theorem. If M is a 3-connected matroid without triangles or triads, Theorem 2.2 is considerably stronger than Tutte's Wheels and Whirls Theorem. On the other hand, if M has a triangle, then we search for a fan (that is, an alternating sequence of triads and triangles in which consecutive triads and triangles intersect). If we cannot delete or contract some element in the last triangle or triad, then, by Tutte's Triangle Lemma, we can build a longer fan. If this process of building the fan does not terminate, then M is a wheel or a whirl.

Bixby's theorem also provides an easy proof of the following partial result toward Seymour's Splitter Theorem [8]. (To upgrade the following result to the Splitter Theorem, one uses essentially the same method that is used to obtain the Wheels and Whirls Theorem from Bixby's theorem.)

COROLLARY 2.4. *Let M be a* 3*-connected matroid and let N be a* 3*-connected proper minor of M with at least* 4 *elements. Then, there exists an element e of M such that either* $\mathrm{co}(M\backslash e)$ *or* $\mathrm{si}(M/e)$ *is* 3*-connected with N as a minor.*

Proof. Choose $x \in E(M) - E(N)$. By duality we may assume that M/x has N as a minor. We may also assume that $\mathrm{si}(M/x)$ is not 3-connected. Then there exists a 2-separation (A, B) of M/x such that $r(A), r(B) \geqslant 2$. We may assume that $|E(N) \cap A| \geqslant |E(N)| - 1$ and that A is closed. Consider any element $e \in B$. If M/e does not have an N-minor, then $(A, B - \{e\})$ is a separation in $M/x, e$. However, M/x is connected, so e must be in the closure of A in M. This contradicts the fact that A is closed. Hence M/e has an N-minor. If there exists $e \in B$ such that $M \backslash e$ has an N-minor, then the result follows from Bixby's Theorem. Suppose otherwise. Then, for every $e \in B$, $(A, B - \{e\})$ is a separation in $M/x \backslash e$. Therefore, B is a series class in M/x. But then B is a series class in M. This contradicts the fact that M is 3-connected. ∎

3. BASIC LEMMAS ON SEPARATIONS

Virtually all of the argument in this paper is focused on analysing the behaviour of 3-separations in matroids and we need lemmas giving basic properties of such separations. Many of these properties hold for general k-separations in which case we state the results at this level. Most of the proofs are omitted as they are more-or-less immediate consequences of definitions. Throughout this paper free use will be made of the results of this section.

Let X, Y be sets of elements of a matroid M. We let $\lambda_M(X)$ denote $r(X) + r(E(M) - X) - r(M)$. Note that X is k-separating if and only if $\lambda_M(X) \leqslant k-1$. We refer to λ_M as the *connectivity function* of M. Note that the connectivity function is symmetric; that is $\lambda_M(X) = \lambda_M(E(M) - X)$. Moreover, Tutte [10] proved that the connectivity function is submodular.

Lemma 3.1 (Tutte [10]). *If X and Y are sets of elements of a matroid M then, $\lambda_M(X) + \lambda_M(Y) \geqslant \lambda_M(X \cap Y) + \lambda_M(X \cup Y)$.*

The following are easy corollaries of Lemma 3.1.

Proposition 3.2. *Let X and Y be k-separating in M. If $X \cap Y$ is not $(k-1)$-separating in M, then $X \cup Y$ is k-separating in M.*

Proposition 3.3. *Let X be 4-separating and Y be 3-separating in M. Then either $X \cap Y$ or $X \cup Y$ is 3-separating in M.*

Of course, a set is k-separating if and only if its complement is k-separating. Thus the previous two lemmas have alternative equivalent formulations that we use frequently. For example, if (A_1, A_2) and (B_1, B_2) are k-separations of M and $A_1 \cap B_1$, is not $(k-1)$-separating, then $A_2 \cap B_2$ is k-separating.

The *coclosure* of a set X of elements of a matroid M is the closure of X in M^*. Evidently, an element $x \in E(M) - X$ belongs to the coclosure of X if and only if x is a coloop of $M \backslash X$.

Guts and Coguts. Let (A, B) be a k-separation of the matroid M. The elements that can be moved from one side of the separation to the other maintaining the property of being a k-separation play an important role in this paper. These elements come in two types. The *guts* of (A, B) is the set $\mathrm{cl}(A) \cap \mathrm{cl}(B)$. Dually, the *coguts* of (A, B) is the set of elements in the coclosure of both A and B. The following comment may aid intuition. If (A, B) is a 3-separation in a representable matroid, then the subspaces spanned by A and B meet in a line. Points on this line are points in the guts of (A, B). The easy proof of the next proposition is omitted. Let (A, B) be a partition of the elements of M. We say that (A, B) is an *exact k-separation*, or that A is *exactly k-separating*, if $\lambda_M(A) = k-1$.

Proposition 3.4. *Let (A, B) be an exact k-separation of the matroid M and x be an element of B. Then,*

(i) *$A \cup \{x\}$ is exactly k-separating if x belongs to either the guts or the coguts of (A, B), but not both.*

(ii) *$A \cup \{x\}$ is exactly $(k-1)$-separating if x belongs to both the guts and the coguts of (A, B).*

(iii) *$A \cup \{x\}$ is exactly $(k+1)$-separating if x belongs to neither the guts nor the coguts of (A, B).*

Blocking and Coblocking. Say that x is an element of the matroid M, and let (A, B) be a k-separation of $M \backslash x$. Then x *blocks* (A, B) if neither $(A \cup \{x\}, B)$ nor $(A, B \cup \{x\})$ is a k-separation of M. Now let (A, B) be a k-separation of M/x. Then x *coblocks* (A, B) if neither $(A \cup \{x\}, B)$ nor $(A, B \cup \{x\})$ is a k-separation of M.

Proposition 3.5. *Let $\{A, B, \{x\}\}$ be a partition of the ground set of the matroid M.*

(i) *If (A, B) is an exact k-separation of $M \backslash x$, then the element x blocks (A, B) if and only if x is not a coloop of M, $x \notin \mathrm{cl}(A)$ and $x \notin \mathrm{cl}(B)$.*

(ii) *If (A, B) is an exact k-separation of M/x, then the element x coblocks (A, B) if and only if x is not a loop, $x \in \mathrm{cl}_M(A)$ and $x \in \mathrm{cl}_M(B)$.*

This immediately implies

Proposition 3.6. *Let (A, B) be an exact $(k+1)$-separation of the matroid M and let x be an element in B.*

(i) *A is k-separating in $M \backslash x$ if and only if x is in the coguts of (A, B).*

(ii) *A is k-separating in M/x if and only if x is in the guts of (A, B).*

Non-sequential Separations and Quads. PROPOSITION 3.7. *Let (A, B) be a non-sequential k-separation of the matroid M. Then*

(i) $(A-\mathrm{cl}(B), \mathrm{cl}(B))$ *is a non-sequential k-separation of M. Dually,*

(ii) *if B' denotes the coclosure of B, then $(A-B', B')$ is a non-sequential k-separation of M.*

A 4-element subset of elements of a matroid is a *quad* if it is both a circuit and a cocircuit. Apart from $M(K_4)$, 3-connected graphic matroids do not have quads. Alas this is not the case more generally, and quads are the cause of most of the difficulties in this paper.

PROPOSITION 3.8. *Let M be a 3-connected matroid and (A, B) be a non-sequential 3-separation of M. If $|A|=4$, then A is a quad.*

Note that if (A, B) is non-sequential 3-separation of a 3-connected matroid and A contains a triad or a triangle, then $|A| \geqslant 5$.

4. WEAK 4-CONNECTIVITY

A matroid M is *weakly* 4-*connected* if M is 3-connected and has no 3-separations (A, B) where $|A|$, $|B| \geqslant 5$. The next lemma is an immediate consequence of Proposition 3.8.

LEMMA 4.1. *Let (A, B) be a non-sequential 3-separation of a weakly 4-connected matroid M. Then either A or B is a quad.*

As noted earlier, we use Lemma 2.1 as the starting point for our argument. Applied to 4-connected matroids, it shows that any element in a 4-connected matroid can be either deleted or contracted so as to keep weak 4-connectivity.

LEMMA 4.2. *Let M be a 4-connected matroid and x be an element of M. Then at least one of $M\backslash x$ or M/x is weakly 4-connected. Moreover, if P is a quad in $M\backslash x$ and (A, B) is a 3-separation in M/x with $|A|$, $|B| \geqslant 4$, then $|A \cap P| = |B \cap P| = 2$.*

Proof. Assume that $M\backslash x$ is not weakly 4-connected. Then $M\backslash x$ has a 3-separation (X_d, Y_d), where $|X_d|$, $|Y_d| > 4$. Now consider any 3-separation (X_c, Y_c) of M/x. By Lemma 2.1 we may assume that $|X_d \cap X_c| \leqslant 2$. But then, again by Lemma 2.1, either $|X_c \cap Y_d| \leqslant 2$ or $|X_d \cap Y_c| \leqslant 2$. The latter does not occur since $|X_d| > 4$. Thus $|X_c \cap Y_d| \leqslant 2$ and hence $|X_c| \leqslant 4$. It follows that M/x is weakly 4-connected. A similar easy argument establishes the latter part of the theorem. ∎

As noted earlier, if M is a 3-connected graphic matroid with a quad, then M is isomorphic to $M(K_4)$. Moreover, $M(K_4)$ is sequentially 4-connected. Therefore, by Lemma 4.2, *if M is a* 4-*connected graphic matroid and e is an element of M, then either $M\backslash e$ or M/e is sequentially* 4-*connected*. The task of the next section is to deal with the potential presence of quads in non-graphic matroids.

5. TUTTE 4-CONNECTIVITY

The main result in this section is the following.

THEOREM 5.1. *Let M be a* 4-*connected matroid. Then M has an element z such that either $M\backslash z$ or M/z is sequentially* 4-*connected.*

This theorem will be an immediate consequence of more specific structural results that we now develop. One case that arises is dealt with by the next lemma.

LEMMA 5.2. *Let $\{t_1, t_2, t_3, a_1, a_2, a_3, b_1, b_2, b_3\}$ be distinct elements of the* 4-*connected matroid M. Suppose, for $k=1$, 2, 3, that $M\backslash t_k$ is weakly* 4-*connected and that $\{t_1, t_2, t_3, a_k, b_k\} - \{t_k\}$ is a quad of $M\backslash t_k$. Then M/t_1 is sequentially* 4-*connected.*

Proof. For $k=1$, 2, 3, let $P_k=\{t_1, t_2, t_3, a_k, b_k\}$ and $P'_k=P_k-\{t_k\}$. Now let $P=P_1\cup P_2\cup P_3$.

Suppose that M/t_1 is not sequentially 4-connected, and let (A, B) be a non-sequential 3-separation. Since P'_2 and P'_3 are 4-circuits in M, both $\{a_2, b_2, t_3\}$ and $\{a_3, b_3, t_2\}$ are triangles in M/t_1. We may assume without loss of generality that neither of these triangles cross the 3-separation (A, B). By hypothesis P'_1 is a quad in $M\backslash t_1$, so, by Lemma 4.2, $|P'_1\cap A| = |P'_1\cap B|=2$. Also note that, by Proposition 3.4, $(A\cup\{t_1\}, B)$ is a 4-separation of M with t_1 in the guts.

5.2.1. $|A\cap\{t_2, t_3\}|=1$.

Proof. Suppose otherwise, then we may assume without loss of generality that t_2, $t_3\in A$. However, neither $\{a_2, b_2, t_3\}$ nor $\{a_3, b_3, t_2\}$ crosses (A, B), so a_2, a_3, b_2, $b_3\in A$. Now P_2 is a cocircuit in M, so $t_1\notin \mathrm{cl}(B)$. This is, however, a contradiction, since $(A\cup\{t_1\}, B)$ is a 4-separation in M with t_1 in the guts. This proves the claim. ∎

By possibly relabeling, we may assume that $|A|\leqslant|B|$ and $A\cap P'_1= \{a_1, t_2\}$. Therefore, $A\cap P=\{a_1, t_2, a_3, b_3\}$ and $B\cap P=\{b_1, t_3, a_2, b_2\}$. Note that, since (A, B) is non-sequential and $\{t_2, a_3, b_3\}$ is a triangle in M/t_1, we have $5 \leqslant |A| \leqslant |B|$. Consequently M has at least 11 elements.

Note that P_1 and P_3 are 4-separating in M, moreover $|P_1 \cap P_3| = 3$, so, by Proposition 3.2, $P_1 \cup P_3$ is 4-separating. Also A is 4-separating in M and $|E - (A \cap (P_1 \cup P_3))| \geqslant 3$, so, by Proposition 3.2, $A \cap P = A \cap (P_1 \cup P_3)$ is 4-separating in M. But $|A \cap P| = 4$, so $A \cap P$ is either a circuit or a cocircuit. Since t_2 is the only element of $A \cap P$ that is in the cocircuit P_2, $A \cap P$ is not a circuit. Hence $A \cap P$ is a cocircuit. Because P'_2 is a circuit, t_3 is not in the coclosure of $A \cap P$. Therefore, $A \cap P$ is a cocircuit in $M \backslash t_3$. However, this implies that a_1 is in the coguts of the 3-separation $(P'_3, E - P_3)$ in $M \backslash t_3$. Therefore, $P'_3 \cup \{a_1\}$ is 3-separating in $M \backslash t_3$. This contradicts the fact that $M \backslash t_3$ is weakly 4-connected. This completes the proof. ∎

The following lemma contains some of the finite case checking for the proof of Theorem 5.1. The proof is somewhat terse, so the reader may wish to skip it on first reading.

Lemma 5.3. *Let M be a 4-connected matroid with at most 11 elements, and let x, a, p, b_1, b_2, c_1, c_2 be distinct elements of M such that $M \backslash x$ is weakly 4-connected with a quad $\{a, p, b_1, b_2\}$, and such that $\{b_1, b_2, c_1, c_2\}$ is a quad of $M \backslash p$. Then, there exists an element y of M such that either $M \backslash y$ or M/y is sequentially 4-connected.*

Proof. Let $D = E(M) - \{x, a, p, b_1, b_2, c_1, c_2\}$, $P = \{a, p, b_1, b_2\}$, and $Q = \{b_1, b_2, c_1, c_2\}$. Assume that M has rank 4. If $z \in E(M)$, then M/z is a 3-connected rank-3 matroid. It is easily checked that such a matroid is sequentially 4-connected, so that the lemma holds in this case.

By dualising the above argument we assume that the rank and corank of M are both at least 5. Hence M has either 10 or 11 elements, and D has either 3 or 4 elements. By Proposition 3.2, D is 3-separating in $M \backslash x, p$. Since P is a circuit, and $P \cap D = \emptyset$, we see that D is also 3-separating in $M \backslash x$. As x blocks the 3-separation $(D, Q \cup \{a\})$ but not the 3-separation $(D \cup \{a\}, Q)$ in $M \backslash p, x$, we see that $x \in \mathrm{cl}(D \cup \{a\})$ but $x \notin \mathrm{cl}(D)$. Since P and Q are 4-element circuits, meeting in two elements, $3 \leqslant r(P \cup Q) \leqslant 4$. Since P is a quad of $M \backslash x$, $P \nsubseteq \mathrm{cl}(Q)$. Hence $r(P \cup Q) = 4$. Then, since D is 3-separating in $M \backslash x$, we have $r(M) = r(D) + 2$.

Suppose that M/b_1 is not sequentially 4-connected, and let (R, R') be a non-sequential 3-separation in M/b_1. We may assume that the triangle $\{a, p, b_2\}$ is not crossed by (R, R'), and that $a, p, b_2 \in R$. Since $\{b_2, c_1, c_2\}$ is a triangle of M/b_1, we may further assume that, if R contains either c_1 or c_2, then R contains both c_1 and c_2. Note that, since M is 4-connected, b_1 coblocks (R, R'), so that b_1 is in the closure of R'. But $\{b_1, b_2, c_1, c_2\}$ is a quad in $M \backslash p$, so $b_1 \notin \mathrm{cl}(D \cup \{x\})$. Thus R' must contain one of c_1

and c_2. Consequently R' must contain both c_1 and c_2. Also, since $b_1 \notin \mathrm{cl}(D \cup P)$, we must have $x \in R'$. Moreover, since (R, R') is not sequential, R and R' each contain at least two elements of $D \cup \{x\}$.

Suppose that R contains three elements of D. Then $|D| = 4$ and $|R'| = 4$. Consequently, R' is a quad in M/b_1 and, hence, D cannot be a circuit in M/b_1. Therefore, D is a cosegment and $R \cap D$ is a triad in $M \backslash x$. Since R' is a circuit, $x \in \mathrm{cl}((R' \cup \{b_1\}) - \{x\})$. Therefore, $R \cap D$ is a triad in M, contradicting the fact that M is 4-connected. Hence $R \cap D$ contains 2 elements, say d_1 and d_2, and it follows that $\{a, p, d_1, d_2\}$ is a quad in M/b_1. Thus $\{a, p, d_1, d_2\}$ is a cocircuit in M and $r(\{a, p, d_1, d_2, b_1, b_2\}) = 4$. Since Q is a circuit, $r(\{a, p, d_1, d_2, b_1, b_2, c_1, c_2\}) = 5$. Therefore, since M is 4-connected, $r(M) = 5$ and hence D has rank 3.

Now c_1 and c_2 are not both in the closure of $\{d_1, d_2, a, p\}$ in M, since otherwise (R', R) would be a sequential 3-separation in M/b_1. By possibly relabeling, we may assume that c_1 is not in the closure of $\{d_1, d_2, a, p\}$. Now $\{d_1, d_2, a, p\}$ is not a circuit, since otherwise Q would be a quad in M. Therefore $\{d_1, d_2, a, p, c_1\}$ has rank 5 in M. Now suppose that M/c_1 is not sequentially 3-connected, and let (T, T') be a non-sequential 3-separation. Since M/c_1 has rank 4, both T and T' have rank 3 in M/c_1. It follows that neither T nor T' contain all of $\{d_1, d_2, a, p\}$. Moreover, as $\{d_1, d_2, a, p\}$ is a cocircuit, T and T' must each contain two elements of $\{d_1, d_2, a, p\}$. Note that $\{b_1, b_2, c_2\}$ is a triangle in M/c_1. Thus we may assume that $b_1, b_2, c_2 \in T$. Now, as c_1 coblocks (T, T') we must get c_1 in the closure of T'. Therefore, as Q is a cocircuit of $M \backslash p$, we must have $p \in T'$. Moreover p cannot be in the closure of T (since otherwise we could put it in T), so $a \in T'$. It follows that $d_1, d_2 \in T$. Since T has rank 3 in M/c_1, we see that $\{d_1, d_2, c_1, c_2\}$ has rank 3 in M/b_1. However, in M/b_1, $\{a, p, d_1, d_2\}$ has rank 3, and $\{c_1, c_2, b_2\}$ and $\{a, p, b_2\}$ are triangles. We conclude that c_1 and c_2 are both in the closure of R, and hence (R', R) is a sequential 3-separation in M/b_1. ∎

If x is an element of a 4-connected matroid M, then either $M \backslash x$ or M/x is weakly 4-connected. By duality we may assume that $M \backslash x$ is weakly 4-connected. If $M \backslash x$ is not sequentially 4-connected, then $M \backslash x$ has a quad P. Hence, the following lemma implies Theorem 5.1.

LEMMA 5.4. *Let M be a 4-connected matroid, and let x be an element of M such that $M \backslash x$ is a weakly 4-connected matroid with a quad P. Then at least one of the following holds:*

(i) *M/x is sequentially 4-connected;*

(ii) *there exists $z \in P$ such that $M \backslash z$ is sequentially 4-connected; or*

(iii) *M has at most 12 elements and there exists an element y of M such that either $M \backslash y$ or M/y is sequentially 4-connected.*

Proof. Set $P=\{p, a, b_1, b_2\}$, where p is chosen such that, if possible, $M\backslash p$ is weakly 4-connected. Suppose that the result is false, and let M be a counterexample. Note that every 3-connected matroid of rank 2 or 3 is sequentially 4-connected. Therefore, since neither $M\backslash p$ nor M/x is sequentially 4-connected, M has rank and corank at least 5. Thus M has at least 10 elements.

Now $M\backslash p$ has a non-sequential 3-separation. By removing x we obtain a 3-separation (X_1, X_2) of $M\backslash p, x$. Assume that $\{a, b_1, b_2\}$ is contained in one side of (X_1, X_2). Then neither p nor x blocks this 3-separation and it follows that $\mathrm{cl}_M(X_1)$ is 3-separating in M, contradicting the fact that M is 4-connected.

Thus we may assume that (X_1, X_2) crosses the triad $\{a, b_1, b_2\}$ of $M\backslash p, x$. Without loss of generality assume that $X_1 \cap \{a, b_1, b_2\}=\{a\}$, and, hence, $X_2 \cap \{a, b_1, b_2\}=\{b_1, b_2\}$. Set $C=X_2 \cap (E-\{a, b_1, b_2\})$ and $D=X_1 \cap (E-\{a, b_1, b_2\})$. By possibly moving elements, we may assume that C is closed.

Since X_1 and $\{a, b_1, b_2\}$ are both 3-separating, we deduce from Proposition 3.2 that D is 3-separating in $M\backslash p, x$. Since $p\in \mathrm{cl}(\{a, b_1, b_2\})$, D is also 3-separating in $M\backslash x$. Thus we have

5.4.1. *D is* 3-*separating in both $M\backslash p, x$ and $M\backslash x$.*

We also have

5.4.2. *a is in the coguts of* (X_1, X_2).

Proof. We know that both D and X_1 are 3-separating in $M\backslash p, x$. But $X_1=D\cup\{a\}$, so a is either in the cuts or the coguts of (X_1, X_2). If a is in the guts of (X_1, X_2), then $a\in \mathrm{cl}(X_2)$. But then $p\in \mathrm{cl}(X_2)$, contradicting the fact that p blocks (X_1, X_2). ∎

5.4.3. $x\in \mathrm{cl}(X_1)$ *and* $x\notin \mathrm{cl}(D)$. *Consequently* $a\in \mathrm{cl}(D\cup\{x\})$.

Proof. Say $x\in \mathrm{cl}(D)$. Then, since D is 3-separating in $M\backslash x$, we have that $D\cup\{x\}$ is 3-separating in M. This contradicts the fact that M is 4-connected, so $x\notin \mathrm{cl}(D)$.

Since x does not block (X_1, X_2), either $x\in \mathrm{cl}(X_1)$ or $x\in \mathrm{cl}(X_2)$. Assume the latter. By 5.4.1, D is 3-separating in $M\backslash x$. Since $D\cap X_2=\emptyset$ and $x\in \mathrm{cl}(X_2)$, it follows that D is 3-separating in M. Since M is 4-connected we deduce that $|D|=2$. Then $(X_1, X_2\cup\{x\})$ is a sequential 3-separation in $M\backslash p$, contradicting our definition of X_1 and X_2. We conclude that $x\in \mathrm{cl}(X_1)$. ∎

5.4.4. *C is exactly 4-separating in M and $M\backslash x, p$.*

Proof. Lemma 3.1 shows that C is either 3-separating or 4-separating in $M\backslash x, p$. If C is k-separating in $M\backslash x, p$, then, since $x \in \mathrm{cl}(D \cup \{a\})$ and $p \in \mathrm{cl}(\{a, b_1, b_2\})$, C is k-separating in M. Thus C is either 3-separating or 4-separating in M. Assume, for a contradiction, that C is 3-separating in M. Then, as M is 4-connected, $|C| \leqslant 2$. Now $X_2 = C \cup \{b_1, b_2\}$, and (X_1, X_2) is a non-sequential 3-separation of $M\backslash p$, so $C \cup \{b_1, b_2\}$ is a quad in $M\backslash p$. Moreover, since $M\backslash x$ is weakly 4-connected, $|D| \leqslant 4$. Hence M has at most 11 elements. But now, M satisfies the hypotheses of Lemma 5.3. Hence M has an element y such that $M\backslash y$ or M/y is sequentially 4-connected, and part (iii) of this lemma holds, contradicting the assumption that M is a counterexample to the lemma. ∎

5.4.5. $|D| \leqslant 3$.

Proof. Suppose that $|D| = 4$. Since D is 3-separating in $M\backslash x$, $D \cup \{x\}$ is 4-separating in M. Moreover, by 5.4.3, a is in the guts of this 4-separation. Therefore, $(D \cup \{x\}, X_2 \cup \{p\})$ is a 3-separation in M/a. Therefore M/a is not weakly 4-connected. Hence, by Lemma 4.2, $M\backslash a$ is weakly 4-connected. However, $M\backslash p$ is not weakly 4-connected, contradicting our choice of p. ∎

5.4.6. $b_1 \in \mathrm{cl}(C \cup b_2)$

Proof. Since P is a quad in $M\backslash x$, neither b_1 nor b_2 is in the closure of C. If $b_1 \notin \mathrm{cl}(C \cup \{b_2\})$ then b_1 and b_2 are both in the coguts of the 3-separation (X_1, X_2) in $M\backslash x, p$ contradicting the fact that C is not 3-separating in $M\backslash x, p$. ∎

5.4.7. $r(X_1 \cup \{b_1, b_2\}) = r(X_1) + 2$.

Proof. Assume not. Then $r(X_1 \cup \{b_1, b_2\}) \leqslant r(X_1) + 1$. But now

$$r(X_1) + r(X_2) \geqslant r(X_1 \cup \{b_1, b_2\}) + r(X_2 - \{b_1, b_2\}).$$

So $C = X_2 - \{b_1, b_2\}$ is 3-separating, contradicting the fact that this set is exactly 4-separating. ∎

Consider any 3-separation (Q, Q') of $M\backslash a$.

5.4.8. *Q crosses both $D \cup x$ and $\{p, b_1, b_2\}$.*

Proof. This follows immediately from the facts that $a \in \mathrm{cl}(D \cup x)$, $a \in \mathrm{cl}(\{p, b_1, b_2\})$, and M is 4-connected. ∎

5.4.9. *If* $C \subseteq Q'$, *then* $b_1, b_2 \in \mathrm{cl}(Q')$.

Proof. By symmetry, we need only show that $b_1 \in \mathrm{cl}(Q')$. Suppose, to the contrary, that $b_1 \notin \mathrm{cl}(Q')$. Recall that $b_1 \in \mathrm{cl}(C \cup \{b_2\})$, so $b_2 \notin \mathrm{cl}(Q')$. Therefore $b_1, b_2 \in Q$. Then, since Q crosses $\{p, b_1, b_2\}$, we have $p \in Q'$. Moreover, $(Q \cup \{p\}, Q' - \{p\})$ is not a 3-separation of $M \backslash a$ since it does not cross $\{p, b_1, b_2\}$. Therefore p is in the closure of $Q' - \{p\}$. Therefore, as $P \cup \{x\}$ is a cocircuit, we must have $x \in Q'$. Since $r(X_1 \cup \{b_1, b_2\}) = r(X_1) + 2$, the elements b_1 and b_2 are in the coguts of (Q, Q') in $M \backslash a$. Hence $Q - \{b_1, b_2\}$ is 3-separating in $M \backslash a$. However, since P is a circuit, $Q - \{b_1, b_2\}$ is 3-separating in M. Therefore, $|Q - \{b_1, b_2\}| \leqslant 2$. Then, since $r(X_1 \cup \{b_1, b_2\}) = r(X_1) + 2$, we see that Q is an independent set. Thus Q is a cosegment of $M \backslash a$. So, for $d \in Q - \{b_1, b_2\}$, $\{b_1, b_2, d\}$ is a triad of $M \backslash a$, and $\{a, b_1, b_2, d\}$ is a cocircuit of M. Hence d is in the coclosure of P in M, and indeed, in $M \backslash x$. Therefore d is in the coguts of the 3-separation $(P, E - (P \cup \{x\}))$ in $M \backslash x$. However, this implies that $P \cup \{d\}$ is 3-separating in $M \backslash x$, contradicting the fact that $M \backslash x$ is weakly 4-connected. ∎

5.4.10. *If* $C \subseteq Q'$, *then* $Q - \{b_1, b_2\}$ *is a triad containing* p *and two elements of* D.

Proof. If $|Q| = 3$ then Q must be a triad of $M \backslash a$, in which case $Q \cap \mathrm{cl}(Q')$ is empty. Therefore, if $b_1 \in Q$ then $(Q - \{b_1\}, Q' \cup \{b_1\})$ is a 3-separation of $M \backslash a$. Therefore, we may assume that $b_1, b_2 \in Q'$. Hence, by 5.4.8, we have $p \in Q$. Since p blocks the 3-separation (X_1, X_2) of $M \backslash p$, we see that p is not in the closure of $Q - \{p\}$. Therefore $Q - \{p\}$ is 3-separating in $M \backslash a$. However, since P is a circuit, $Q - \{p\}$ is 3-separating in M. Hence $|Q| = 3$. That is, Q is a triad of $M \backslash a$ that contains p and two elements of $D \cup \{x\}$. Suppose that $x \in Q$ and suppose that $D \cap Q = \{d\}$. Now, $(Q \cup \{a\}) - \{x\}$ is a triad of $M \backslash x$. This means that d is in the coguts of the 3-separation $(P, E - (P \cup \{x\}))$ in $M \backslash x$. Hence $P \cup \{d\}$ is 3-separating in $M \backslash x$. This contradicts the fact that $M \backslash x$ is weakly 4-connected. Therefore $x \notin Q$, and the result follows. ∎

5.4.11. *If* $C \subseteq Q'$, *then* Q *is a triad containing* p *and two elements of* D.

Proof. We know that $Q - \{b_1, b_2\}$ is a triad containing p and two elements of D. So, if the result fails, Q contains b_1 or b_2. Say $b_1 \in Q$. By 5.4.8, $b_2 \notin Q$. Now $Q \cup \{a\}$ is 4-separating in M and also in $M \backslash x$. Moreover, P is 3-separating in $M \backslash x$. Since P is a quad of $M \backslash x$, and $|P \cap (Q \cup \{a\})| = 3$, we see that $P \cap (Q \cup \{a\})$ is not 3-separating in $M \backslash x$. Therefore, by Proposition 3.3, $P \cup (Q \cup \{a\})$ is 3-separating in $M \backslash x$. However, $M \backslash x$ is weakly 4-connected, so $|Q' - (P \cup \{x\})| \leqslant 4$. By 5.4.4

$|C| \geqslant 3$ and C is exactly 4-separating in $M\backslash x$, so $C \neq Q' - (P \cup \{x\})$. Therefore, by 5.4.5, $|D| = 3$ and $|C| = 3$. Let d be the element in $D \cap Q'$. Now C is exactly 4-separating and $C \cup \{d\} = Q' - (P \cup \{x\})$ is 3-separating in $M\backslash x$, which contradicts the fact that C is closed. ∎

5.4.12. *If* (R, R') *is a 3-separation in* $M\backslash a$, *where* $x \in R$ *and* $|R|$, $|R'| \geqslant 4$, *then* $|R \cap \{p, b_1, b_2\}| = 1$ *and* $|R \cap C| \leqslant 2$. *Moreover, if* $|R \cap D| \neq 0$ *then* $|R' \cap C| \leqslant 2$.

Proof. By 5.4.8, both $R \cap \{p, b_1, b_2\}$ and $R' \cap \{p, b_1, b_2\}$ are nonempty. Suppose that $R' \cap \{p, b_1, b_2\}$ contains just one element, say t. Since $P \cup \{x\}$ is a cocircuit of M, and $R' - \{t\}$ is contained in the complementary hyperplane, t is not in the closure of $R' - \{t\}$. Thus $(R' - \{t\}, R \cup \{t\})$ is a 3-separation of $M\backslash a$. However, $R' - \{t\}$ does not cross $\{p, b_1, b_2\}$, which contradicts 5.4.8. Therefore $|R' \cap \{p, b_1, b_2\}| > 1$, and it follows that $|R \cap \{p, b_1, b_2\}| = 1$.

Note that C is 4-separating and R is 3-separating in $M\backslash a$. Therefore, by Proposition 3.3, either $R' - C$ or $R \cap C$ is 3-separating in $M\backslash a$. By 5.4.11, $R' - C$ cannot be 3-separating, so $R \cap C$ is 3-separating in $M\backslash a$. Now, since P is a circuit, $R \cap C$ is 3-separating in M. Therefore $|R \cap C| \leqslant 2$.

Suppose then that we have chosen R so that $|R \cap D| \geqslant 1$. Consequently, $|R - C| \geqslant 3$. Since C is 4-separating and R' is 3-separating in $M\backslash a$, either $R - C$ or $R' \cap C$ is 3-separating in $M\backslash a$. However, by 5.4.11, $R - C$ is not 3-separating in $M\backslash a$. Therefore $R' \cap C$ is 3-separating in $M\backslash a$. Then, because P is a circuit, $R' \cap C$ is 3-separating in M. However, M is 4-connected, so $|R' \cap C| \leqslant 2$. ∎

5.4.13. $|C| \leqslant 4$. *Consequently* M *has at most* 12 *elements.*

Proof. Suppose that $|C| \geqslant 5$. We claim that $M\backslash a$ is weakly 4-connected. Consider any 3-separation (R, R') of $M\backslash a$, where $x \in R$ and $|R|$, $|R'| \geqslant 4$. As $|C| \geqslant 5$, we have $|R' \cap C| \geqslant 3$. Then, by the previous claim, R and D are disjoint. Consequently $|R| \leqslant 4$. Therefore, $M\backslash a$ is weakly 4-connected.

By our choice of p, $M\backslash p$ is weakly 4-connected. Consequently $|D| = 2$ and $X_1 \cup \{x\}$ is a quad in $M\backslash p$. Suppose that $M\backslash a$ is not sequentially 4-connected, and let R be a quad in $M\backslash a$. Note that $x \in R$, and let t be the unique element in $P \cap R$. Assume $t \neq p$; then, since R is a circuit, x is in the guts of the 3-separation $(X_1 \cup \{x\}, X_2)$ in $M\backslash p$. This cannot happen, since $X_1 \cup \{x\}$ is a quad in $M\backslash p$. Hence $t = p$. Now the result follows by Lemma 5.2, where $t_1 = x$, $t_2 = a$, and $t_3 = p$. ∎

Now M has at most 12 elements. (The rest of the proof of the lemma is just a finite case check.) Henceforth, we assume that (R, R') is a non-sequential 3-separation in $M\backslash a$ with $x \in R$.

5.4.14. $|R \cap D| \leqslant 1$.

Proof. Now R is 3-separating and $D \cup x$ is 4-separating in $M \backslash a$. Therefore, by Lemma 3.3, either $(D \cap R) \cup \{x\}$ or $R' - D$ is 3-separating in $M \backslash a$. If $R' - D$ is 3-separating in $M \backslash a$ then, since $a \in \mathrm{cl}(D \cup \{x\})$, $R' - D$ is also 3-separating in M. However, this cannot be the case since M is 4-connected and $|R' - D| \geqslant 3$. Therefore, $(D \cap R) \cup \{x\}$ is 3-separating in $M \backslash a$. However, because P is a circuit, $(D \cap R) \cup \{x\}$ is 3-separating in M. Thus, since M is 4-connected, $|R \cap D| \leqslant 1$. ∎

5.4.15. $|R' \cap D| \leqslant 2$.

Proof. Suppose that $|R' \cap D| \geqslant 3$. Hence $D \subset R'$. Now $R' \cap D$ cannot be 3-separating in M, and, since P is a circuit, $D \cap R'$ cannot be 3-separating in $M \backslash a$. Now R is 3-separating and $D \cup x$ is 4-separating in $M \backslash a$. Therefore, by Lemma 3.3, $R - (D \cup \{x\})$ is 3-separating in $M \backslash a$. However, $a \in \mathrm{cl}(D \cup \{x\})$, so $R - (D \cup \{x\})$ is 3-separating in M. Then, since M is 4-connected, $|R - (D \cup \{x\})| = 2$. Therefore R is a triad in $M \backslash a$, contradicting the assumption that this 3-separation is non-sequential. ∎

5.4.16. $|R' \cap D| = 2$.

Proof. Suppose, to the contrary, that $|R' \cap D| = 1$. Then, by 5.4.14, $|D| \leqslant 2$. Now $(D \cup \{a, x\}, C \cup \{b_1, b_2\})$ is a non-sequential 3-separation in $M \backslash p$. Therefore, $|D| = 2$, $|D \cap R| = 1$ and $D \cup \{a, x\}$ is a quad in $M \backslash p$. Since $|D \cap R| \neq 0$, it follows from 5.4.12 that $|R' \cap C| \leqslant 2$.

Now $|R|$, $|R'| \leqslant 5$ and $|E(M)| \leqslant 11$. From part (iii) of the statement of the lemma and the assumption that M is a counterexample, we see that M/a is not sequentially 4-connected. Let (Q, Q') be a non-sequential 3-separation of M/a with $x \in Q$. We may assume that neither of the triangles, $D \cup \{x\}$ and $\{p, b_1, b_2\}$, crosses (Q, Q') in M/a. In particular $D \cup \{x\} \subseteq Q$. Since M is 4-connected, a coblocks (Q, Q') so that $a \in \mathrm{cl}(Q')$. However, $D \cup \{a, x, p\}$ is a cocircuit, so $p \notin Q$. Hence p, b_1, $b_2 \in Q'$. So neither Q nor Q' is a quad of M/a. Hence, M/a is not weakly 4-connected and $|E(M)| = 11$. Consequently, $M \backslash a$ is weakly 4-connected. However, if $|E(M)| = 11$ then $|R| = |R'| = 5$, which is a contradiction. Therefore $|R' \cap D| = 2$. ∎

If there exists an element y in M such that M/y is weakly 4-connected, then we could dualize and use y in place of x. Therefore, we assume that: *if there exists an element y of M such that M/y is weakly* 4-*connected, then* $r(M) \leqslant r^*(M)$.

5.4.17. *R is disjoint from D.*

Proof. Suppose to the contrary that $|R \cap D| = 1$. In this case $M \backslash p$ is not weakly 4-connected. Therefore, M/p is weakly 4-connected. So, by assumption, we have $r(M) \leqslant r^*(M)$. Now $r(M) \geqslant r(D \cup \{a, b_1, b_2\}) = r(D \cup \{a\}) + 2 = 6$. However M has at most 12 elements, so $r(M) = r^*(M) = 6$. In particular, M has 12 elements, so $|C| = 4$. However, C is 4-separating. Hence, C is either a circuit or a cocircuit. Since $r(M) = r(D \cup \{a, b_1, b_2\})$, C is not a cocircuit. Therefore, C is a circuit.

By 5.4.11, $R - C$ is not 3-separating in $M \backslash a$. However, since C is a circuit, $r(R' \cup C) \leqslant r(R') + 1$. Therefore, $r(R - C) = r(R)$. Thus, as $|R - C| = 3$, we have $r(R) = 3$. So, $R - P$ is a circuit containing x. This contradicts the fact that x blocks the 3-separation $(P, E - (P \cup \{x\}))$ in $M \backslash x$. ▮

Now, P is a quad in $M \backslash x$, $D \cup \{a, x\}$ is a quad in $M \backslash p$, and R is a quad in $M \backslash a$. Moreover, since $D \cup \{a, x, p\}$ is a cocircuit and R is a circuit, we must have $p \in R$. Now the proof follows from Lemma 5.2. ▮

6. INTERNAL 4-CONNECTIVITY

A matroid is M is *internally* 4-*connected* if M is 3-connected and has no 3-separations (A, B) where $|A|, |B| \geqslant 4$. That is, if (A, B) is a 3-separation in an internally 4-connected matroid M, then A or B is a triangle or a triad. In this section we prove that, if T is a triangle in a sufficiently large internally 4-connected matroid, then there exists an element of T whose deletion leaves a sequentially 4-connected matroid. Unfortunately there are small exceptions to this assertion. Consider the two matroids in Fig. 1. Note that $\{a, b, c\}$ is a triangle in both of these matroids; however, none of $M \backslash a$, $M \backslash b$, or $M \backslash c$ is sequentially 4-connected.

The main result in this section is an analogue of Tutte's Triangle Lemma.

THEOREM 6.1 (The Triangle Theorem). *If T is a triangle in an internally* 4-*connected matroid M, then either*

(i) *there exists $t \in T$ such that $M \backslash t$ is sequentially* 4-*connected, or*

(ii) *M has at most* 11 *elements, and there exists an element y of M such that M/y is sequentially* 4-*connected.*

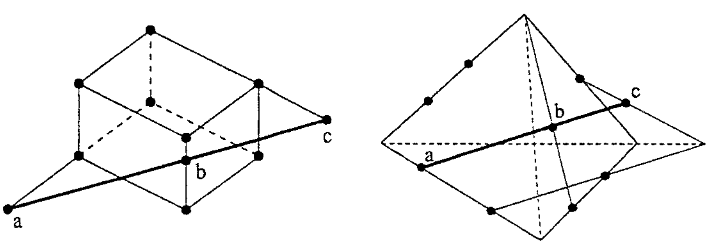

FIG. 1. Nasty examples.

Proof. Let $T=\{a, b, c\}$. We often use the following elementary claim.

6.1.1. *Let (t_1, t_2, t_3) be a permutation of $\{a, b, c\}$, and let (X, Y) be a non-sequential 3-separation of $M\backslash t_3$ such that $t_1 \in X$. Then, $t_2 \in Y$. Moreover, neither t_1 nor t_2 is in either the guts or the coguts of (X, Y).*

Proof. Since M is internally 4-connected, t_3 blocks (X, Y). Therefore, t_3 is not in the closure of X, and hence $t_2 \in Y$. If t_2 were in the guts or the coguts of (X, Y), then $(X \cup \{t_2\}, Y-\{t_2\})$ would be a non-sequential 3-separation of $M\backslash t_3$ that would contradict the first part of the claim. Therefore, t_2 is not in the guts or the coguts of (X, Y), and, by symmetry, neither is t_1. ▌

Suppose that none of $M\backslash a$, $M\backslash b$, and $M\backslash c$ is sequentially 4-connected. Then, let (A_b, A_c), (B_a, B_c) and (C_a, C_b) be non-sequential 3-separations in $M\backslash a$, $M\backslash b$ and $M\backslash c$ respectively, where a is in B_a and C_a, b is in A_b and C_b and c is in A_c and B_c. Note that $(A_b-\{b\}, A_c-\{c\})$, $(B_a-\{a\}, B_c-\{c\})$ and $(C_a-\{a\}, C_b-\{b\})$ are all 3-separations of $M\backslash T$. In this proof it may help to imagine that each of these 3-separations arises from a cube; see Fig. 2. We have symmetries induced by permutations of a, b, c; these symmetries are also indicated in Fig. 2. For example, the two hollow vertices of the cube indicate that $A_b \cap B_c \cap C_a$ and $A_c \cap B_a \cap C_b$ are equivalent sets under some permutation of a, b, c. Also, the three dotted edges indicate that $A_b \cap B_a$, $A_c \cap C_a$, and $B_c \cap C_b$ are equivalent sets under permutations of a, b, c. The solid vertices, the thin edges, and the thick edges indicate three other equivalence classes.

We start by focussing on the pair $\{a, b\}$. However, it is clear that analogous results will hold for any 2-element subset of $\{a, b, c\}$.

6.1.2. *$M\backslash\{a, b\}$ is 3-connected up to series pairs.*

Proof. Say (X, Y) is a 2-separation of $M\backslash a, b$, where $c \in Y$. Now X is 3-separating in $M\backslash a$, and $b \in \mathrm{cl}(A \cup y)$ so X is 3-separating in M. Hence $|X| \leqslant 3$.

Assume that $|X|=3$, so that X is either a triangle or a triad. Consider C_a and C_b. Without loss of generality assume that $|C_a \cap X| \geqslant 2$. If the other element of X is not in $C_a \cap X$, it is in either the closure or coclosure of $C_a \cap X$, so we may assume that $X \subseteq C_a$. But then b is in the coguts of (C_a, C_b) contradicting 6.1.1. Hence $|X|=2$ and it is clear that X is a series pair. ▌

6.1.3. *If $A_b \cap B_a$ contains a series pair $\{a', b'\}$ in $M\backslash a, b$ then $|A_b|$, $|B_a| \geqslant 5$.*

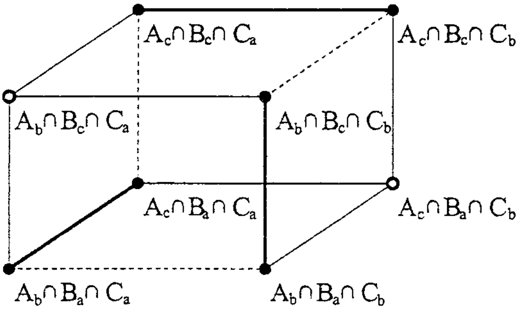

FIG. 2. Three crossing 3-separations.

Proof. Assume that $|A_b| = 4$. Then A_b is a quad of $M\backslash a$. But this quad contains $\{b, a', b'\}$ which is a triad of $M\backslash a$; a contradiction. Thus $|A_b| \geqslant 5$, and similarly $|B_a| \geqslant 5$. ∎

6.1.4. *If $A_b \cap B_c$ (respectively $A_c \cap B_c$ or $A_c \cap B_a$) is k-separating in $M\backslash a, b$, then $A_b \cap B_c$ (respectively $A_c \cap B_c$ or $A_c \cap B_a$) is k-separating in M.*

Proof. We have $a \in \mathrm{cl}(B_a - \{a\})$ and $b \in \mathrm{cl}(\{a, c\})$. Therefore, if $A_b \cap B_c$, is k-separating in $M\backslash a, b$ then $A_b \cap B_c$ is k-separating in M. Similarly, if $A_c \cap B_a$ is k-separating in $M\backslash a, b$, then $A_c \cap B_a$ is k-separating in M. Moreover, since $a \in \mathrm{cl}(B_a - \{a\})$ and $b \in \mathrm{cl}(A_b - \{b\})$, we see that if $A_c \cap B_c$ is k-separating in $M\backslash a, b$ then $A_c \cap B_c$ is k-separating in M. ∎

By the previous claim and Proposition 3.2, we have

6.1.5. *Neither $A_b \cap B_c$, $A_c \cap B_a$ or $A_c \cap B_c$ contains a series pair of $M\backslash a, b$. Moreover,*

1. *If $|A_b \cap B_c| \geqslant 2$, then $A_c \cap B_a$ is 3-separating in M.*
2. *If $|A_c \cap B_a| \geqslant 2$, then $A_b \cap B_c$ is 3-separating in M.*
3. *If $|A_c \cap B_c| \geqslant 2$, then $A_b \cap B_a$ is 3-separating in $M\backslash a, b$.*
4. *If $A_b \cap B_a$ is not 2-separating in $M\backslash a, b$, then $A_c \cap B_c$ is 3-separating in M.*

6.1.6. $|A_b \cap B_c|, |A_c \cap B_a|, |A_c \cap B_c| \geqslant 2$. *Consequently, both $A_b \cap B_c$, and $A_c \cap B_a$ are 3-separating in M, and $A_b \cap B_a$ is 3-separating in $M\backslash a, b$.*

Proof. Suppose that $|A_b \cap B_c| \leqslant 1$. Then, since (A_b, A_c) is a non-sequential 3-separation in $M\backslash a$, $|A_b \cap B_a| \geqslant 2$. Assume that $|A_b \cap B_a| = 2$. Then $|A_b| = 4$, and A_b is a quad. But then, by 6.1.3, $A_b \cap B_a$ is not a series pair, so $A_b \cap B_a$ is not 2-separating in $M\backslash a, b$. On the other hand, if $|A_b \cap B_a| \geqslant 3$, then, since $M\backslash a, b$ is 3-connected up to series pairs, $A_b \cap B_a$ is also not 2-separating in $M\backslash a, b$. In either case it follows from 6.1.5 that $A_c \cap B_c$ is 3-separating in M. However, M is internally 4-connected, so either $|A_c \cap B_c| \leqslant 2$ or $(A_c \cap B_c, E-(A_c \cap B_c))$ is a sequential 3-separation. In either case we deduce that (B_c, B_a) is a sequential 3-separation in $M\backslash b$. This contradiction shows that $|A_b \cap B_c| \geqslant 2$. By symmetry, $|A_c \cap B_a| \geqslant 2$. A similar argument shows that $|A_c \cap B_c| \geqslant 2$. ▮

6.1.7. *If* $|A_c \cap B_c| \geqslant 3$ *then either*

1. $A_c \cap B_c$ *is a triangle and* $|A_c \cap B_c \cap C_a| = |A_c \cap B_c \cap C_b| = 1$, *or*
2. $(A_b \cap B_a) \cup \{a, b\}$ *is a* 4-*element cocircuit.*

Proof. First suppose that $A_b \cap B_a$ is not 2-separating in $M\backslash a, b$. By 6.1.5, $A_c \cap B_c$ is 3-separating in M. Then, since $|A_c \cap B_c| = 3$, $A_c \cap B_c$ is either a triangle or a triad. However, $c \in A_c \cap B_c$ and c is already in the triangle T. Therefore, since M is internally 4-connected, c is not in a triad. Hence $A_c \cap B_c$ is a triangle. Moreover, since c blocks the 3-separation (C_a, C_b) of $M\backslash c$, the triangle $A_c \cap B_c$ must have one element in C_a and another in C_b.

Now suppose that $A_b \cap B_a$ is 2-separating in $M\backslash a, b$. Since $M\backslash a, b$ is 3-connected up to series-pairs, $A_b \cap B_a$ is a series pair of $M\backslash a, b$. Now M is internally 4-connected, and a and b are in a triangle, so neither a nor b is in a triad of M. Hence, $(A_b \cap B_a) \cup \{a, b\}$ is a cocircuit as required. ▮

Note that, since M is internally 4-connected, $|A_b \cap B_c|$, $|A_c \cap B_a| \leqslant 3$. Also, $A_b \cap B_a$ is nonempty, since otherwise $a \in \mathrm{cl}(A_c \cap B_a)$ so that a does not block the non-sequential 3-separation (A_b, A_c).

By 6.1.6 and symmetry between a, b and c, we know that each of the sets $A_b \cap B_c$, $A_c \cap C_b$, $B_c \cap C_a$, $B_a \cap A_c$, $C_a \cap A_b$, and $C_b \cap B_a$ is exactly 3-separating in M, and, hence, any one of these sets contain at most three elements. This shows that M has at most 21 elements. Thus the remainder of the proof is just a finite case check.

Suppose that $|A_b \cap B_c| = 3$. Then, by 6.1.5, $A_b \cap B_c$ is either a triangle or a triad in M. By symmetry we may assume that C_a contains at least two elements of $A_b \cap B_c$. Suppose that $A_b \cap B_c$ is not contained in C_a and let x be the element in $A_b \cap B_c \cap C_b$. Thus x is either in the guts or the coguts of the 3-separation (C_a, C_b) in $M\backslash c$. Therefore $(C_a \cup \{x\}, C_b - \{x\})$ is a nonsequential 3-separation of $M\backslash c$. By changing C_a and C_b, we may assume that $A_b \cap B_c \subset C_a$. Using symmetry we may make the following assumptions.

6.1.8. *We assume, without loss of generality, that*

1. *If* $|A_b \cap B_c| = 3$ *then either* $A_b \cap B_c \subseteq C_a$ *or* $A_b \cap B_c \subseteq C_b$.
2. *If* $|A_c \cap B_a| = 3$ *then either* $A_c \cap B_a \subseteq C_a$ *or* $A_c \cap B_a \subseteq C_b$.
3. *If* $|A_c \cap C_b| = 3$ *then either* $A_c \cap C_b \subseteq B_a$ *or* $A_c \cap C_b \subseteq B_c$.
4. *If* $|A_b \cap C_a| = 3$ *then either* $A_b \cap C_a \subseteq B_a$ *or* $A_b \cap C_a \subseteq B_c$.
5. *If* $|B_c \cap C_a| = 3$ *then either* $B_c \cap C_a \subseteq A_b$ *or* $B_c \cap C_a \subseteq A_c$.
6. *If* $|B_a \cap C_b| = 3$ *then either* $B_a \cap C_b \subseteq A_b$ *or* $B_a \cap C_b \subseteq A_c$.

6.1.9. $|A_b \cap B_a \cap C_a| \leqslant 1$.

Proof. Suppose to the contrary that $|A_b \cap B_a \cap C_a| \geqslant 2$. By 6.1.8 (part 4), $A_b \cap B_c \cap C_a$ is empty. However, by 6.1.6 and symmetry, $|B_c \cap C_a| \geqslant 2$. Therefore, $|A_c \cap B_c \cap C_a| \geqslant 2$. Then, since $c \in A_c \cap B_c$, $|A_c \cap B_c| \geqslant 3$. Therefore, by 6.1.7, $(A_b \cap B_a) \cup \{a, b\}$ a cocircuit of M. Let x and y be the elements of $A_b \cap B_a \cap C_a$. Now, by 6.1.5 and symmetry, no subset of $B_a \cap C_a$ is 2-separating in $M \backslash c$. Thus b is in the coclosure of $\{x, y, a\}$ in $M \backslash c$. Hence b is in the coguts of the 3-separation (C_a, C_b) in $M \backslash c$. This contradiction completes the proof. ∎

6.1.10. $|A_b \cap B_c \cap C_a| \leqslant 1$.

Proof. Suppose to the contrary that $|A_b \cap B_c \cap C_a| \geqslant 2$. By 6.1.8, $A_c \cap B_c \cap C_a$, $A_b \cap B_a \cap C_a$, and $A_b \cap B_c \cap C_b$ are all empty. However, by 6.1.5 (part 3) and symmetry, $|A_c \cap B_c|$, $|A_b \cap C_b|$, $|B_a \cap C_a| \geqslant 2$. Therefore, by 6.1.9 and symmetry, $|A_c \cap B_c \cap C_b| = 1$, $|A_c \cap B_a \cap C_a| = 1$, and $|A_b \cap B_a \cap C_b| = 1$. By 6.1.8 (part 3), $|A_c \cap B_a \cap C_b| \leqslant 1$ and, hence M has at most 10 elements. Since $|A_c| \geqslant 4$, we must have $|A_c \cap B_a \cap C_b| = 1$ and $|A_c| = 4$. Thus A_c is a quad in $M \backslash a$. Similarly, B_a is a quad in $M \backslash b$ and C_b is a quad in $M \backslash c$. Let y be the element in $A_c \cap B_a \cap C_b$. It is straightforward to check that y is not in a triangle. Also, since $A_c - \{y\}$, $B_a - \{y\}$, $C_b - \{y\}$, and $\{a, b, c\}$ are all triangles in M/y, it is straightforward to check that M/y is sequentially 4-connected, as required. ∎

By 6.1.5 and symmetry, $|A_b \cap C_a| \geqslant 2$. Therefore, by 6.1.9 and 6.1.10, $|A_b \cap B_c \cap C_a| = 1$ and $|A_b \cap B_a \cap C_a| = 1$. Then, by symmetry, $|A_b \cap B_c \cap C_b| = 1$, $|A_b \cap B_a \cap C_b| = 1$, $|A_c \cap B_c \cap C_a| = 1$, $|A_c \cap B_c \cap C_b| = 1$, $|A_c \cap B_a \cap C_a| = 1$, and $|A_c \cap B_a \cap C_b| = 1$. Therefore, M has 11 elements. We may assume that there are no elements of $M \backslash a$ in the guts or the coguts of the 3-separation (A_b, A_c) since, otherwise, we could change A_b and A_c and then apply 6.1.9 or 6.1.10. Similarly we may assume that there are no elements of $M \backslash b$ in the guts or the coguts of the 3-separation (B_a, B_c), and there are no elements of $M \backslash c$ in the guts or the coguts of the 3-separation (C_a, C_b).

Let y be the element in $A_b \cap B_c \cap C_a$. By the assumptions above, it is straightforward to check that y is not in a triangle. We will show that M/y is sequentially 4-connected. Consider the case that $A_c \cap B_c$ and $A_b \cap C_b$ are both triangles. Now, since (A_b, A_c) is a nonsequential 4-separation of $M\backslash a, b$, A_b and A_c must both have rank 3. Therefore M has rank 4. However, any 3-connected matroid of rank 3 is sequentially 4-connected, so M/y is sequentially 4-connected, as required. Now, by symmetry, we may assume that $A_c \cap B_c$ is not a triangle.

By 6.1.7, $(A_b \cap B_a) \cup \{a, b\}$ is a cocircuit. Thus $(A_b \cap B_a) \cup \{b\}$ is a triad of $M\backslash a$. So, since (A_c, A_b) is not a sequential 3-separation of $M\backslash a$, $A_b \cap C_b$ is not a triangle. Therefore, by 6.1.7 and symmetry, $(A_c \cap C_a) \cup \{a, c\}$ is a cocircuit. Similarly, $A_c \cap B_c$ is not a triangle and $(B_c \cap C_b) \cup \{b, c\}$ is a cocircuit.

Suppose that M/y is not sequentially 4-connected and let (Q, Q') be a nonsequential 3-separation of M/y. Since $\{a, b, c\}$ is a triangle, we may assume without loss of generality that $a, b, c \in Q$. We may also assume that Q is closed and coclosed in M/y. Therefore, Q does not cross any of $A_b \cap B_a$, $A_b \cap C_b$, or $A_c \cap B_c$. Moreover, $|Q| \geqslant 5$, so, by symmetry, we may assume that $A_b \cap B_a \subseteq Q$. Let z be the element in $A_b \cap B_c \cap C_b$. Now, since there are no elements of $M\backslash a$ in the coguts of (A_b, A_c), z is in the closure of $A_b - \{z\}$. Thus z is in the closure of Q in M/y. However, Q is closed in M/y, so $z \in Q$. Therefore, Q contains $B_c \cap C_b$. Thus, $|Q'| \leqslant 3$. This contradiction completes the proof. ∎

7. SEQUENTIAL 4-CONNECTIVITY

It follows from Theorem 5.1, Theorem 6.1 and the dual of Theorem 6.1 that Theorem 1.2 holds if M is internally 4-connected. The case when M is not internally 4-connected is surprisingly straightforward.

Let M be a sequentially 4-connected matroid with a sequential 3-separation (A, B), where $|A| \geqslant 4$. Assume that the elements of A are ordered $(a_1, ..., a_k)$. Let A_i denote $\{a_1, ..., a_i\}$, and let B_i denote $\{a_i, ..., a_k\} \cup B$. Note that, if $i \geqslant 3$, then a_i is either in the guts or the coguts of the 3-separation (A_i, B_{i+1}).

THEOREM 7.1. (i) *For $i \geqslant 3$, if a_i is in the guts of (A_i, B_{i+1}) and $M\backslash a_i$ is 3-connected, then $M\backslash a_i$ is sequentially 4-connected.*

(ii) *If A is coclosed and a_k is in the guts of (A, B), then $M\backslash a_k$ is sequentially 4-connected.*

(iii) *If A is both closed and coclosed, then either $M\backslash a_k$ or M/a_k is sequentially 4-connected.*

Proof. Consider part (i). Suppose that $M\backslash a_i$ is not sequentially 4-connected, and let (X, Y) be a non-sequential 3-separation in $M\backslash a_i$. We first consider the case that $i>3$. Note that $\{a_1, a_2, a_3\}$ is either a triangle or a triad. Therefore, we may assume that $a_1, a_2, a_3 \in X$. Moreover, for $j=4, ..., i-1$, the element a_j is either in the closure or coclosure of $\{a_1, ..., a_{j-1}\}$. Then, inductively, we may assume that $a_1, ..., a_{i-1} \in X$. However, $a_i \in \mathrm{cl}(\{a_1, ..., a_{i-1}\})$, so that $(X \cup \{a_i\}, Y)$ is a non-sequential 3-separation of M contradicting the fact that M is sequentially 4-connected.

Now consider the case where $i=3$. Since a_i is in the guts of (A_3, B_4), we see that A_3 is a triangle. Moreover, since $M\backslash a_3$ is 3-connected, $\{a_1, a_2, a_4\}$ is either a triangle or a triad of $M\backslash a_3$. Thus we may assume that a_1, a_2, $a_4 \in X$. However, $a_3 \in \mathrm{cl}(\{a_1, a_2\})$, so that $(X \cup \{a_3\}, Y)$ is a non-sequential 3-separation in M. This contradiction proves the claim.

Consider part (ii). By (i), it suffices to show that $M\backslash a_k$ is 3-connected. Suppose otherwise. Since a_k is in the guts of (A, B), (A_{k-1}, B) is a 2-separation in M/a_k. If $\mathrm{si}(M/a_k)$ is not 3-connected, then $\mathrm{co}(M\backslash a_k)$ is 3-connected by Bixby's theorem. If $\mathrm{si}(M/a_k)$ is 3-connected, then A_{k-1} is a parallel class in M/a_k with at least 3 elements, and again, by Bixby's theorem, $\mathrm{co}(M\backslash a_k)$ is 3-connected. Since $M\backslash a_k$ is not 3-connected, a_k is in a triad T. Moreover, as $a_k \in \mathrm{cl}(A_{k-1})$ and $a_k \in \mathrm{cl}(B)$, T contains an element of A_{k-1} and an element, say b, of B. Thus b is in the coclosure of A. This contradiction completes the proof.

Now consider part (iii). Evidently a_k is either in the guts or the coguts of (A, B). In the former case, $M\backslash a_k$ is sequentially 4-connected by part (ii). In the latter case, M/a_k is sequentially 4-connected by the dual of part (ii). ∎

We can now complete the proof of Theorem 1.2.

Proof of Theorem 1.2. We have already noted that if M is internally 4-connected, then the theorem follows from Theorem 5.1, Theorem 6.1, and the dual of Theorem 6.1. Assume that the sequentially 4-connected matroid M is not internally 4-connected and is not a wheel or a whirl. If M has a sequential 3-separation (A, B) where $|A| \geqslant 4$ and A is both closed and coclosed, then it follows from Theorem 7.1(iii) that the theorem holds.

Assume that M has no such 3-separation. Let (A, B) be a sequential 3-separation of M, where $|A| \geqslant 4$. By assumption A is either not closed or not coclosed. This means that we can grow A by taking in elements from the closure or the coclosure of A. Proceeding in this way we shall eventually obtain an ordering $(e_1, ..., e_n)$ of $E(M)$ such that, for $i=1, ..., n$, $\{e_1, ..., e_i\}$ is 3-separating. By Tutte's Wheels and Whirls Theorem, there exists an element e_i, such that either $M\backslash e_i$ or M/e_i is 3-connected. Note that the order of $\{e_1, e_2, e_3\}$ does not matter, and we may reverse the order of $e_1, ..., e_n$. Thus, we may assume that $3 \leqslant i \leqslant n-3$. By duality, we may assume that e_i is in the guts of the 3-separation $(\{e_1, ..., e_i\}, \{e_{i+1}, ..., e_n\})$. Therefore, M/e_i is not 3-connected. Hence, $M\backslash e_i$ is 3-connected. So, by Theorem 7.1(ii), $M\backslash e_i$ is sequentially 4-connected. ∎

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