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Notes

Characterizing graphic matroids by a system of linear equations ☆

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A B S T R A C T

Given a rank-r binary matroid we construct a system of $O(r^3)$ linear equations in $O(r^2)$ variables that has a solution over GF(2) if and only if the matroid is graphic.

1. Introduction

We prove the following result.

Theorem 1.1. *Let B be a basis in a binary matroid M. Then M is graphic if and only if the following system of linear equations admits a solution over* GF(2).

(G1) $\beta(a,b)+\beta(a,c)=0$, *for each* $(a,b,c)\in B^{(3)}$ *with* $C_b^*\cap C_c^*-C_a^*\neq\emptyset$.
(G2) $\beta(a,b)+\beta(a,c)+\beta(b,a)+\beta(b,c)+\beta(c,a)+\beta(c,b)=1$, *for each* $(a,b,c)\in B^{(3)}$ *with* $C_a^*\cap C_b^*\cap C_c^*\neq\emptyset$.

Here $B^{(k)}$ denotes the set of all ordered k-tuples of distinct elements in B and C_e^* denotes the fundamental cocircuit of e with respect to B; that is, C_e^* is the complement of the hyperplane of M

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spanned by $B - \{e\}$. The variables and equations have a natural interpretation which is revealed in Section 2.

If M is a rank-r binary matroid with n elements, then the system (G1)–(G2) has $O(r^3)$ equations and $O(r^2)$ variables. The system can be easily determined in $O(nr^3)$-time and solved in $O(r^7)$-time. There are faster algorithms for testing graphicness. By analyzing a method proposed by Tutte [7], Bixby and Cunningham [1] gave an $O(r^2n)$-time algorithm. Later, Bixby and Wagner [2] and Fujishige [4], independently, obtained almost linear-time algorithms by using data structures that keep track of 2-separations; these algorithms assume that the binary matroid is given by a matrix in "standard form".

Mighton [5,9] has a closely related characterization of graphic matroids. In fact, it is easy to deduce our main result from Mighton's Theorem which, in turn, can be deduced from Tutte's excluded-minor characterization [8]; we will, however, give a direct proof. We do not know how to deduce either Mighton's or Tutte's characterization from ours; this would be interesting since our characterization has a relatively simple proof.

2. Trees and paths

Let B be a basis of a binary matroid M. For each $f \in E(M) - B$, we define $P_f \subseteq B$ such that $P_f \cup \{f\}$ is the unique circuit contained in $B \cup \{f\}$; that is, $P_f \cup \{f\}$ is the fundamental circuit for f. Note that $e \in P_f$ if and only if $f \in C_e^*$ for each $e \in B$ and $f \in E(M) - B$. To avoid ambiguity, we will refer to the fundamental circuits and cocircuits of (M, B), as they rely on both M and B. Our linear system is motivated by the following well-known result; we include the proof for the sake of completeness.

Lemma 2.1. *If B is a basis of a binary matroid M, then M is graphic if and only if there is a tree T with $E(T) = B$ such that each of the sets $(P_f\colon f \in E(M) - B)$ is a path in T.*

Proof. Suppose that $M = M(G)$ for some graph G; we may assume that G is connected. Then B is a tree and each of the sets $(P_f\colon f \in E(G) - E(T))$ are paths in G.

Conversely, suppose that there is a tree T with $E(T) = B$ such that, for each $f \in E(G) - E(T)$, the set P_f is a path in T. Then there is a graph G such that the fundamental circuits of (M, B) coincide with the fundamental circuits of $(M(G), B)$. Since M and $M(G)$ are both binary, $M = M(G)$. $\square$

Let $\vec{T}$ be an orientation of a tree T. For each $(a, b) \in E(T)^{(2)}$, we define $\beta_{\vec{T}}(a, b) \in \mathrm{GF}(2)$ to be 1 if the head of a is in the same component of $T - a$ as the edge b, and 0 otherwise. Note that, for $(a, b, c) \in E(T)^{(3)}$, the edge b lies between a and c in T if and only if $\beta_{\vec{T}}(b, a) + \beta_{\vec{T}}(b, c) = 1$. The following lemma characterizes paths in T by linear equations.

Lemma 2.2. *Let $\vec{T}$ be an orientation of a tree T and let $P \subseteq E(T)$. Then P is a path in T if and only if*

(H1) $\beta_{\vec{T}}(a, b) + \beta_{\vec{T}}(a, c) = 0$, *for each $(b, c) \in P^{(2)}$ and $a \in E(T) - P$, and*
(H2) $\beta_{\vec{T}}(a, b) + \beta_{\vec{T}}(a, c) + \beta_{\vec{T}}(b, a) + \beta_{\vec{T}}(b, c) + \beta_{\vec{T}}(c, a) + \beta_{\vec{T}}(c, b) = 1$, *for each $(a, b, c) \in P^{(3)}$.*

Proof. Note that P is a path if and only if

(I1) P induces a connected subgraph of T, and
(I2) there is a path of T containing P.

Now (I1) and (H1) are clearly equivalent and (I2) is equivalent to each triple in $P^{(3)}$ being contained in a path of T. Consider $(a, b, c) \in P^{(3)}$. If there is a path of T containing a, b and c, then exactly one of those edges lies between the other two. On the other hand, if a, b and c do not lie on a path, then none of the edges lies between the other two. Thus (I2) is equivalent to (H2). $\square$

The next lemma determines when $\beta : B^{(2)} \to \mathrm{GF}(2)$ encodes a tree.

Lemma 2.3. *Let B be a finite set and let $\beta : B^{(2)} \to \mathrm{GF}(2)$. Then there exists an oriented tree $\vec{T}$ such that $E(\vec{T}) = B$ and $\beta = \beta_{\vec{T}}$ if and only if the following condition is satisfied*:

(T) *for each $(a, b, c) \in B^{(3)}$, either $\beta(b, a) + \beta(b, c) = 0$ or $\beta(a, b) + \beta(a, c) = 0$.*

Proof. If an edge b lies between edges a and c in an oriented tree $\vec{T}$, then a does not lie between b and c. Thus $\beta_{\vec{T}}$ satisfies (T).

Conversely, suppose that $\beta : B^{(2)} \to \mathrm{GF}(2)$ satisfies (T). We may assume that there exists $(a, b, c) \in B^3$ such that $\beta(a, b) + \beta(a, c) = 1$ since otherwise we can readily construct an oriented star $\vec{T}$ satisfying the result. Let β' denote the restriction of β to $(B - \{a\})^{(2)}$. Inductively we may assume that there is an oriented tree $\vec{T}_a$ such that $E(\vec{T}_a) = B - \{a\}$ and $\beta' = \beta_{\vec{T}_a}$. Let $B_0 = \{e \in B - \{a\}\colon \beta(a, e) = 0\}$ and let $B_1 = \{e \in B - \{a\}\colon \beta(a, e) = 1\}$. Since $\beta(a, b) + \beta(a, c) = 1$, the sets B_0 and B_1 are both nonempty. If B_0 and B_1 each form connected subgraphs of $\vec{T}_a$, then it is straightforward to get the desired tree $\vec{T}$. Adding one to each of the values $(\beta(a, e)\colon e \in B - \{a\})$ gives another function satisfying (T) and this change swaps the roles of B_0 and B_1; this change corresponds to the operation of reversing the orientation on an edge in a tree. So we may assume that B_0 does not form a connected subgraph of T and, hence, there exist $(e, f) \in B_0^{(2)}$ and $d \in B_1$ such that d lies between e and f in $\vec{T}_a$. Note that $\beta(d, e) \neq \beta(d, f)$, so, by possibly switching e and f, we may assume that $\beta(d, a) = \beta(d, e)$. Now $\beta(d, a) + \beta(d, f) = 1$ and $\beta(a, d) + \beta(a, f) = 1$, contradicting (T). □

Lemmas 2.1, 2.2, and 2.3 immediately imply the following results.

Lemma 2.4. *If B is a basis of a graphic matroid M, then the linear system* (G1)–(G2) *admits a solution.*

Lemma 2.5. *If B is a basis of a binary matroid M and there is a solution to the system* (G1)–(G2) *that satisfies* (T)*, then M is graphic.*

To complete the proof of Theorem 1.1 we need to prove that, when (G1)–(G2) has a solution, there is a solution satisfying (T). We will prove a stronger result that, when $M(G)$ is 3-connected, every solution of (G1)–(G2) also satisfies (T).

3. Connectivity

The following two results are self-evident.

Lemma 3.1. *Let B be a basis of a matroid M and let (X, Y) be a partition of $E(M)$ into nonempty sets. Then (X, Y) is a separation of M if and only if $P_x \subseteq X$ for each $x \in X - B$ and $P_y \subseteq Y$ for each $y \in Y - B$.*

Lemma 3.2. *Let B be a basis of a binary matroid M, and let (X, Y) be a partition of $E(M)$ with $|X|, |Y| \geqslant 2$. If $C_x^* \subseteq X$, for each $x \in X \cap B$, and there is a set $Z \subseteq X$ such that, for each $y \in Y \cap B$, either $C_y^* \cap X = \emptyset$ or $C_y^* \cap X = Z$, then (X, Y) is a 2-separation of M.*

The next lemma describes solutions to (G1).

Lemma 3.3. *Let B be a basis of a matroid M and let β be a solution to (G1). Then $\beta(b, a) = \beta(b, c)$ for each $(a, b, c) \in B^{(3)}$ where a and c are in the same component of $M \setminus C_b^*$.*

Proof. Suppose that the result fails and let N be the component of $M \setminus C_b^*$ containing a and c. Let $X = \{e \in E(N)\colon \beta(b, e) = \beta(b, a)\}$. By Lemma 3.1, there exists $f \in E(N) - B$ such that $P_f \cap X$ and $P_f - X$ are both nonempty. Let $a' \in P_f \cap X$ and $c' \in P_f - X$. Note that $b \notin P_f$, so, by (G1), $\beta(b, a') = \beta(b, c')$ — contradicting the definition of X. □

Let B be a basis of a matroid M. For $X \subseteq E(M)$, we let $M[B;X]$ denote $M/(B-X)\setminus(E(M)-(X\cup B))$. Note that $B\cap X$ is a basis of $M[B;X]$ and the fundamental cocircuits of $(M[B;X], B\cap X)$ are $(C_x^*\cap X\colon x\in B\cap X)$. Therefore, if β satisfies (G1)–(G2) for M, then the restriction of β to $X^{(2)}$ satisfies (G1)–(G2) for $M[B;X]$.

We now reduce Theorem 1.1 to the 3-connected case.

Lemma 3.4. *Let B be a basis of a matroid M. If M is not graphic, then there exists $Z\subseteq E(M)$ such that $M[B;Z]$ is 3-connected and is not graphic.*

Proof. We may assume that M is not graphic and that, for each proper subset Z of $E(M)$, $M[B;Z]$ is graphic. Then M is connected. We may also assume that M is not 3-connected; let (X,Y) be a 2-separation in M. Note that $r(X)+r(Y)=r(M)+1$, so, up to symmetry, we may assume that $X\cap B$ is a basis of $M|X$. Thus $P_f\subseteq X$ for each $f\in X-B$. Then, by Lemma 3.1, there exists $y\in Y-B$ and $x\in X\cap B$ such that $x\in P_y$. By minimality, $M[B;X\cup\{y\}]$ and $M[B;Y\cup\{x\}]$ are both graphic. However, M is the 2-sum of $M[B;X\cup\{y\}]$ and $M[B;Y\cup\{x\}]$ and, hence, M is graphic. This contradiction completes the proof. $\square$

4. The final step

Combining the following result with Lemmas 2.4, 3.4, and 2.5 completes the proof of Theorem 1.1.

Lemma 4.1. *Let B be a basis of a binary matroid M. If M is 3-connected, then every solution of (G1)–(G2) also satisfies (T).*

Proof. Let β be a solution to (G1)–(G2).

4.1.1. *Let $(a',b',c')\in B^{(3)}$ be such that $\beta(b',a')+\beta(b',c')=1$ and $\beta(a',b')+\beta(a',c')=1$, and let $Z=C_{a'}^*\cap C_{b'}^*$. Then neither a' nor b' is in the same component of $M\setminus Z$ as c'.*

Proof of claim. Let $Z'=(C_{a'}^*-\{a'\})\cup(C_{b'}^*-\{b'\})$ and let N be the component of $M\setminus Z'$ containing c'. Since a' and b' are coloops of $M\setminus Z'$, neither a' nor b' is contained in N. If the claim fails, then N is not a component of $M\setminus Z$ so, by Lemma 3.1, there exists $f\in Z'-Z$ such that $P_f\cap E(N)\neq\emptyset$. Up to symmetry, we may assume that $f\in C_{a'}^*-C_{b'}^*$. Now $P_f\cup\{f\}$ is a circuit in $M\setminus C_{b'}^*$, so there is a component of $M\setminus C_{b'}^*$ containing $E(N)\cup\{a',f\}$. This component contains both a' and c', and $\beta(b',a')\neq\beta(b',c')$, contrary to Lemma 3.3. $\square$

4.1.2. *Let $(a',b',c')\in B^{(3)}$ be such that $\beta(b',a')+\beta(b',c')=1$ and $\beta(a',b')+\beta(a',c')=1$, and let $Z=C_{a'}^*\cap C_{b'}^*$. If $d\in B$ is in the same component of $M\setminus Z$ as c' and $C_d^*\cap Z\neq\emptyset$, then $\beta(b',a')+\beta(b',d)=1$, $\beta(a',b')+\beta(a',d)=1$, $\beta(d,a')+\beta(d,b')=1$, and $Z\subseteq C_d^*$.*

Proof of claim. By 4.1.1, a' is not in the same component of $M\setminus Z$ as c' and d. Now $C_{a'}^*-Z$ is a cocircuit of $M\setminus Z$, and therefore disjoint from the component containing c' and d. So c' and d are in the same component of $M\setminus C_{a'}^*$, and, hence, by Lemma 3.3, $\beta(a',d)=\beta(a',c')$. By symmetry, $\beta(b',d)=\beta(b',c')$. So $\beta(b',a')+\beta(b',d)=1$ and $\beta(a',b')+\beta(a',d)=1$. Note that $C_{a'}^*\cap C_{b'}^*\cap C_d^*\neq\emptyset$, so, by (G2), $\beta(d,a')+\beta(d,b')=1$. Finally, if there were an element $f\in Z-C_d^*$, then, since a' and b' are contained in the circuit $P_f\cup\{f\}$, a' and b' would be in the same component of $M\setminus C_d^*$, contrary to Lemma 3.3. So $Z\subseteq C_d^*$. $\square$

Suppose that β does not satisfy (T) and let $(a,b,c)\in B^{(3)}$ be such that $\beta(b,a)+\beta(b,c)=1$ and $\beta(a,b)+\beta(a,c)=1$. Let $Z=C_a^*\cap C_b^*$. By 4.1.1, neither a nor b is in the same component of $M\setminus Z$ as c. By Lemma 3.1, there exists an element $d\in B$ that is in the same component of $M\setminus Z$ as c and that satisfies $C_d^*\cap Z\neq\emptyset$. By possibly changing our choice of c, we may assume that $C_c^*\cap Z\neq\emptyset$. Now, by 4.1.2, there is symmetry among a, b, and c, and, hence, by 4.1.1, no two of a, b, and c are in the same component of $M \setminus Z$.

Let X_a and X_b be the ground sets of the components of $M \setminus Z$ that contain a and b respectively. Since M is connected, $Z \neq \emptyset$, and, hence, $|X_a \cup X_b|, |E(M) - (X_a \cup X_b)| \geqslant 2$. By 4.1.2, for each $d' \in (X_a \cup X_b) \cap B$, either $C^*_{d'} - (X_a \cup X_b) = \emptyset$ or $C^*_{d'} - (X_a \cup X_b) = Z$. Then, by Lemma 3.2, $(X_a \cup X_b, E(M) - (X_a \cup X_b))$ is a 2-separation of M, contradicting that M is 3-connected. $\square$

5. Planar graphs

Our theorem was motivated by a result of Naji [6] who characterized the class of circle graphs by a system of linear equations over GF(2). Circle graphs are related to graphic matroids through the following two results: De Fraysseix [3] showed that the fundamental graph of a binary matroid M is a circle graph if and only if M is the cycle matroid of a planar graph. Whitney [10] proved that M is the cycle matroid of planar graph if and only if M is both graphic and cographic. By Whitney's theorem, any characterization for the class of graphic matroids immediately gives a characterization for the class of planar graphs; so we obtain the following corollary.

Corollary 5.1. *Let T be a spanning tree in a connected graph G. Then G is planar if and only if the following system of equations has a solution over* GF(2).

(P1) $\beta(a,b) + \beta(a,c) = 0$, *for each* $(a,b,c) \in (E(G) - E(T))^{(3)}$ *with* $P_b \cap P_c - P_a \neq \emptyset$.

(P2) $\beta(a,b) + \beta(a,c) + \beta(b,a) + \beta(b,c) + \beta(c,a) + \beta(c,b) = 1$, *for each* $(a,b,c) \in (E(G) - E(T))^{(3)}$ *with* $P_a \cap P_b \cap P_c \neq \emptyset$.

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