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Excluding a planar graph from GF(q)-representable matroids ☆

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Abstract

We prove that a binary matroid with huge branch-width contains the cycle matroid of a large grid as a minor. This implies that an infinite antichain of binary matroids cannot contain the cycle matroid of a planar graph. The result also holds for any other finite field.

Keywords: Branch-width; Matroids; Graph minors; Grids

1. Introduction

We prove the following conjecture of Johnson, Robertson, and Seymour [9].

Theorem 1.1. *For any positive integer θ and finite field $\mathbb{F}$, there exists an integer ω such that if M is an $\mathbb{F}$-representable matroid with branch-width at least ω, then M contains a minor isomorphic to the cycle-matroid of the θ by θ grid.*

In fact we prove a stronger theorem (see Theorem 2.2) that does not require representability.

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Let H be a planar graph. For large enough θ, H is a minor of the θ by θ grid. Thus, by Theorem 1.1, there is an integer ω, depending only on H and $\mathbb{F}$, such that any $\mathbb{F}$-representable matroid with no $M(H)$-minor has branch-width at most ω. Combining this with the results in [4] we obtain the following corollary.

Corollary 1.2. *For any planar graph H and finite field $\mathbb{F}$, the class of $\mathbb{F}$-representable matroids with no $M(H)$-minor is well quasi-ordered with respect to taking minors.*

For graphs such results were obtained by Robertson and Seymour [12].

We hope that Theorem 1.1 will lead to progress on Rota's Conjecture, which says that for any finite field $\mathbb{F}$ there are only finitely many excluded minors for the class of $\mathbb{F}$-representable matroids. Combining Theorem 1.1 with results in [2] we obtain the following.

Corollary 1.3. *For any finite field $\mathbb{F}$ and positive integer θ, there are, up to isomorphism, only finitely many minor-minimal non-$\mathbb{F}$-representable matroids that do not contain the cycle matroid of the θ by θ grid as a minor.*

Theorem 1.1 also has interesting algorithmic consequences. Let H be a planar graph and let $\mathbb{F}$ be a finite field. Consider the following problem:

Given an $\mathbb{F}$-represented matroid M, does M have an $M(H)$-minor?

Since H is planar, H is a minor of some grid. So, by Theorem 1.1, there exists an integer ω_H such that every $\mathbb{F}$-representable matroid with branch-width at least ω_H has an $M(H)$-minor. While we cannot determine the branch-width of a matroid efficiently, there is a straightforward polynomial-time algorithm that, given an $\mathbb{F}$-represented matroid M with branch-width at most ω_H, will find a branch-decomposition of M with width at most $3\omega_H$. Thus, it remains to solve the problem for matroids with a given branch-decomposition of width at most $3\omega_H$; this is done by Hliněný [8].

2. Notation

We assume that the reader is familiar with matroid theory; we use the notation of Oxley [11], except that we denote the simplification of a matroid M by $\mathrm{si}(M)$ and the cosimplification by $\mathrm{co}(M)$. We also use different conventions with respect to connectivity. For subsets A and B of $E(M)$ we let $\sqcap_M(A,B) = r_M(A) + r_M(B) - r_M(A \cup B)$. In a representation of M, $\sqcap_M(A,B)$ is the dimension of the intersection of the subspaces spanned by A and B. Now, for a set $A \subseteq E(M)$, we let $\lambda_M(A) = \sqcap_M(A, E(M) - A)$; we call λ_M the *connectivity function* of M. This function is submodular; that is, $\lambda_M(X \cap Y) + \lambda_M(X \cup Y) \leqslant \lambda_M(X) + \lambda_M(Y)$ for all $X, Y \subseteq E(M)$. Also, λ_M is monotone under taking minors; that is, if N is a minor of M with $X \subseteq E(N)$, then $\lambda_N(X) \leqslant \lambda_M(X)$. Finally, λ_M is invariant under duality; that is $\lambda_M(X) = \lambda_{M^*}(X)$ for all $X \subseteq E(M)$. A partition (A, B) of $E(M)$ is called a *separation of order* $\lambda_M(A)$ (note that we do not have conditions on $|A|$ and $|B|$).

The following fact is geometrically intuitive and we frequently use it without reference; we prove it here for completeness.

Lemma 2.1. *Let A, B, and C be sets of elements in a matroid M where A is disjoint from both B and C. Then*

$$\sqcap_{M/A}(B,C)=\sqcap_M(A\cup B,C)-\sqcap_M(A,C).$$

Proof.

$$\begin{aligned}\sqcap_{M/A}(B,C)&=\big(r_M(A\cup B)-r_M(A)\big)+\big(r_M(A\cup C)-r_M(A)\big)\\&\quad-\big(r_M(A\cup B\cup C)-r_M(A)\big)\\&=\big(r_M(A\cup B)+r_M(C)-r_M(A\cup B\cup C)\big)\\&\quad-\big(r_M(A)+r_M(C)-r_M(A\cup C)\big)\\&=\sqcap_M(A\cup B,C)-\sqcap_M(A,C),\end{aligned}$$

as required. □

We mostly use the previous lemma when B and C are disjoint, but it is also interesting in other cases. For example, when $B=C$ it shows that $r_{M/A}(B)=r_M(B)-\sqcap_M(A,B)$.

Two sets A and B of elements of a matroid M are called *skew* if $\sqcap_M(A,B)=0$ or, equivalently, $r_M(A\cup B)=r_M(A)+r_M(B)$. More generally, a collection $\mathcal{S}$ of subsets of $E(M)$ is called *skew* if $r_M(\bigcup(S\colon S\in\mathcal{S}))=\sum(r_M(S)\colon S\in\mathcal{S})$.

Let $\mathcal{S}$ be a collection of subsets of $E(M)$. Where there is no possibility of ambiguity, we shall on occasion associate $\bigcup(S\colon S\in\mathcal{S})$ with $\mathcal{S}$. For example, we may write $M/\mathcal{S}$ in place of $M/\bigcup(S\colon S\in\mathcal{S})$. Also, for a matroid M and set X, we write $M|X$ in place of $M|(E(M)\cap X)$.

For any positive integer q we let $\mathcal{U}(q)$ denote the class of matroids with no $U_{2,q+2}$-minor and we let $\mathcal{U}^*(q)$ denote the class of matroids with no $U_{q,q+2}$-minor. Note that, if q is a prime-power, then $\mathcal{U}(q)\cap\mathcal{U}^*(q)$ contains all GF(q)-representable matroids. We prove a more general version of Theorem 1.1 by extending it to the class $\mathcal{U}(q)\cap\mathcal{U}^*(q)$.

The *θ by θ grid* is the graph, denoted grid_θ, with vertex-set $\{v_{ij}\colon i,j\in\{0,\dots,\theta\}\}$ and edge-set $\{e_{ij}\colon i\in\{0,\dots,\theta\}, j\in\{1,\dots,\theta\}\}\cup\{f_{ij}\colon i\in\{1,\dots,\theta\}, j\in\{0,\dots,\theta\}\}$, where edge e_{ij} is incident with vertices $v_{i,j-1}$ and v_{ij} and edge f_{ij} is incident with vertices $v_{i-1,j}$ and v_{ij}. It is easy to see that $M^*(\mathrm{grid}_\theta)$ contains an $M(\mathrm{grid}_{\theta-1})$-minor. Nevertheless, to facilitate duality, we shall exclude both $M(\mathrm{grid}_\theta)$ and its dual. By $\mathcal{A}(\theta,q)$ we denote the class of matroids in $\mathcal{U}(q)\cap\mathcal{U}^*(q)$ that have neither the cycle matroid of the θ by θ grid nor its dual as a minor. The main result of this paper is as follows.

Theorem 2.2. *There exists an integer-valued function $\omega(\theta,q)$ such that for any positive integers θ and q, if $M\in\mathcal{A}(\theta,q)$, then M has branch-width at most $\omega(\theta,q)$.*

To prove Theorem 2.2, we work toward building a large clique minor (see Lemma 9.2). We start with a very large set of pairwise highly connected circuits (see Lemma 8.1) and then try to disentangle the connectivities (see Lemma 7.1); grids arise explicitly when we cannot disentangle. When we can disentangle, we construct either a large clique minor or the dual of a large clique as a minor (see Section 10); in each case we get a large grid as a minor.

We conclude this section by defining branch-width. In fact, we do not use the definition in this paper, instead we use results that we obtained jointly with Neil Robertson [5,7]; see Lemmas 8.2 and 8.3.

A tree is *cubic* if its internal vertices all have degree 3. The *leaves* of a tree are its degree-1 vertices. A *branch-decomposition* of M is a cubic tree T whose leaves are injectively labeled by the elements of M. That is, each element of M labels some leaf of T, but some leaves may be unlabeled. If T' is a subgraph of T and $X \subseteq E(M)$ is the set of labels of T', then we say that T' *displays* X. The *width* of an edge e of T is defined to be $\lambda_M(X)$ where X is the set displayed by one of the components of $T \setminus e$. The *width* of T is the maximum among the widths of its edges. Finally, the *branch-width* of M is the minimum among the widths of all branch-decompositions of M.

3. Extremal results

For positive integers n and q we let $\mathcal{G}(n,q)$ denote the class of matroids $M \in \mathcal{U}^*(q)$ with no $M^*(K_n)$-minor. We let $\mathcal{G}^*(n,q)$ denote the class of matroids obtained by dualizing each of the matroids in $\mathcal{G}(n,q)$; that is, $\mathcal{G}^*(n,q)$ is the class of matroids $M \in \mathcal{U}(q)$ with no $M(K_n)$-minor. For all $n \geqslant 5$ and $q \geqslant 2$, the class $\mathcal{G}(n,q)$ contains all graphic matroids. The following result is a generalization of the Erdős–Pósa Theorem on edge-disjoint circuits [1].

Theorem 3.1. *(See Geelen, Gerards, and Whittle [6].) There exists an integer-valued function $\rho_1(n,q,m)$ such that for all positive integers n, q and m, if $M \in \mathcal{G}(n,q)$ has corank at least $\rho_1(n,q,m)$, then M has m disjoint circuits.*

The next result is a generalization of a theorem of Mader [10].

Theorem 3.2. *(See Geelen and Whittle [3].) There exists an integer-valued function $\beta(n,q)$ such that for any positive integers n and q, if $M \in \mathcal{G}^*(n,q)$ is simple, then $|E(M)| \leqslant \beta(n,q)r(M)$.*

Note that $\mathcal{A}(\theta,q) \subseteq \mathcal{G}((\theta+1)^2,q) \cap \mathcal{G}^*((\theta+1)^2,q)$, so we obtain the following easy corollary of Theorem 3.2.

Corollary 3.3. *There exists an integer-valued function $\alpha(\theta,q)$ such that for any positive integers θ and q, if $M \in \mathcal{A}(\theta,q)$ is simple, then $|E(M)| \leqslant \alpha(\theta,q)r(M)$.*

Combining Theorems 3.1 and 3.2 we obtain the following.

Corollary 3.4. *There exists an integer-valued function $\rho_2(n,q,t)$ such that for any positive integers n, q, and t, if e is an element of a cosimple matroid M in $\mathcal{G}(n,q)$ with corank at least $\rho_2(n,q,t)$, then $M \setminus e$ contains a circuit with length at most $r^*(M)/t$.*

Proof. Let $\rho_2(n,q,t) = \rho_1(n,q,2t\beta(n,q))$, and assume that $M \in \mathcal{G}(n,q)$ is a cosimple matroid with corank at least $\rho_2(n,q,t)$. By the dual of Theorem 3.2, $|E(M)| \leqslant \beta(n,q)r^*(M)$, and, by Theorem 3.1, M has a collection $2t\beta(n,q)$ disjoint circuits. The sum of the lengths of the two shortest of these circuits is at most $2|E(M)|/(2t\beta(n,q)) \leqslant 2\beta(n,q)r^*(M)/(2t\beta(n,q)) = r^*(M)/t$; one of these two circuits does not contain e. □

A *bundle* in a matroid M is a restriction N of M such that $\text{co}(N)$ is a matroid that either consists of only loops or has exactly one parallel class; the series classes of a bundle N are its *strands*. An *n-bundle* is a bundle with n strands. The main result of this section is as follows.

Theorem 3.5. *There exists an integer-valued function $\rho_3(n,q,\delta)$ such that for any positive integers n, q, and δ, if $M \in \mathcal{G}(n,q) \cap \mathcal{G}^*(n,q)$ has corank at least $\rho_3(n,q,\delta)$, then M has a δ-bundle.*

Before proving Theorem 3.5 we need some preliminary results.

Lemma 3.6. *Let $M \in \mathcal{G}^*(n,q)$ be a cosimple matroid, let C be a circuit of M, and let e an element of M. Then M has a restriction N containing C and e such that $\lambda_N(C \cup \{e\}) \leqslant 1$ and $r^*(N) \geqslant r^*(M)/(\beta(n,q)|C|)$.*

Proof. Let $t = |C| + 1 - r_M(C \cup e)$, let J be a basis of $M/(C \cup \{e\})$, and let $R = E(M) - (J \cup C \cup \{e\})$. Then, by Theorem 3.2, $\mathrm{si}(M/J)$ has at most $\beta(n,q)r(M/J) \leqslant \beta(n,q)|C|$ elements. So there exists a set Z in R with $r_{M/J}(Z) \leqslant 1$ and at least $|R|/(\beta(n,q)|C|) = (r^*(M) - t)/(\beta(n,q)|C|)$ elements. Hence the restriction N of M to $J \cup C \cup \{e\} \cup Z$ has corank at least $(|R|/(\beta(n,q)|C|)) + t \geqslant r^*(M)/(\beta(n,q)|C|)$. Moreover $\lambda_N(C \cup \{e\}) = r_N(C \cup \{e\}) + r_N(J \cup Z) - r_N(J \cup C \cup \{e\} \cup Z) = r(M) - r_M(J) + r_M(J \cup Z) - r(M) = r_{M/J}(Z) \leqslant 1$. □

Lemma 3.7. *There exists an integer-valued function $\rho_4(n,q,\rho)$ such that for any positive integers n, q, and ρ, if e is an element of a matroid $M \in \mathcal{G}(n,q) \cap \mathcal{G}^*(n,q)$ with corank at least $\rho_4(n,q,\rho)$, then there exist a circuit C in $M \setminus e$ and a restriction N of M containing $C \cup \{e\}$ such that $\lambda_N(C \cup \{e\}) \leqslant 1$ and $r^*(N) \geqslant \rho$.*

Proof. Let $\rho_4(n,q,\rho) = \rho_2(n,q,\rho\beta(n,q))$. Let M and e be as claimed; we may assume that M is cosimple. Then, by Corollary 3.4, $M \setminus e$ has a circuit C with length at most $r^*(M)/(\rho\beta(n,q))$. Hence, by Lemma 3.6, M has a restriction N containing $C \cup \{e\}$ such that $\lambda_N(C \cup \{e\}) \leqslant 1$ and $r^*(N) \geqslant r^*(M)/(\beta(n,q)|C|) \geqslant \rho$. □

The element e in the statement of the following result is only to facilitate induction.

Corollary 3.8. *There exists an integer-valued function $\rho_5(n,q,\delta)$ such that if e is an element of a matroid $M \in \mathcal{G}(n,q) \cap \mathcal{G}^*(n,q)$ with corank at least $\rho_5(n,q,\delta)$, then $M \setminus e$ contains a collection of disjoint circuits $C_0, \dots, C_\delta$ such that $\sqcap_M(C_0 \cup \dots \cup C_i, C_{i+1} \cup \dots \cup C_\delta \cup \{e\}) \leqslant 1$ for each $i \in \{0, \dots, \delta - 1\}$.*

Proof. Recursively we define $\rho_5(n,q,0) = 2$ and $\rho_5(n,q,\delta) = \rho_4(n,q,2 + \rho_5(n,q,\delta-1))$ for $\delta \geqslant 1$. Let M and e be as claimed. Then, by Lemma 3.7, there exist a circuit C_δ of $M \setminus e$ and a restriction N of M containing $C_\delta \cup \{e\}$ where $\lambda_N(C_\delta \cup \{e\}) \leqslant 1$ and $r^*(N) \geqslant 2 + \rho_5(n,q,\delta-1)$. Hence, N is the 1- or 2-sum of two matroids N_1 and N_2 where $C_\delta \cup \{e\} \subseteq E(N_1)$ and $E(N) - (C_\delta \cup \{e\}) \subseteq E(N_2)$. In case N is a 2-sum of N_1 and N_2, let f be the base-point of the 2-sum; otherwise, let f be any element of N_2. Clearly, $r^*(N_2) \geqslant r^*(N) - (|C \cup \{e\}| - r_M(C \cup \{e\})) \geqslant r^*(N) - 2 \geqslant \rho_5(n,q,\delta-1)$. Hence, by induction to δ, $N_2 \setminus f$ contains a collection of disjoint circuits $C_0, \dots, C_{\delta-1}$ such that $\sqcap_{N_2}(C_0 \cup \dots \cup C_i, C_{i+1} \cup \dots \cup C_{\delta-1} \cup \{f\}) \leqslant 1$ for $i = 0, \dots, \delta - 2$. Now $C_0, \dots, C_\delta$ are circuits in $M \setminus e$ with the required properties. □

From this we finally prove Theorem 3.5.

Proof of Theorem 3.5. Let $m = 2\delta - 1$ and $\rho_3(n,q,\delta) = \rho_5(n,q,(\delta-1)m)$. Take $M \in \mathcal{G}(n,q) \cap \mathcal{G}^*(n,q)$ with corank at least $\rho_3(n,q,\delta)$. So, by Lemma 3.8, M has a collection of disjoint circuits $C_0, C_1, \ldots, C_{(\delta-1)m}$ such that $\sqcap_M(C_1 \cup \cdots \cup C_i, C_{i+1} \cup \cdots \cup C_{(\delta-1)m}) \leqslant 1$ for $i = 0, 1, \ldots, (\delta-1)m - 1$.

We now break the proof into two cases. First suppose that for each $i \in \{0, 1, \ldots, (\delta-2)m\}$ we have $\sqcap_M(C_i, C_{i+m} \cup C_{i+m+1} \cup \cdots \cup C_{(\delta-1)m}) = 0$. Then, $\{C_0, C_m, \ldots, C_{(\delta-1)m}\}$ is a set of δ skew-circuits. So, in this case, M has a δ-bundle.

In the remaining case, there exists $i \in \{0, 1, \ldots, (\delta-2)m\}$ such that $\sqcap_M(C_i, C_{i+m} \cup \cdots \cup C_{(\delta-1)m}) = 1$. Therefore, for each $j \in \{i+1, \ldots, i+m-1\}$, we have $\sqcap_{M/C_i}(C_1 \cup \cdots \cup C_{i-1} \cup C_{i+1} \cup \cdots \cup C_j, C_{j+1} \cup \cdots \cup C_{(\delta-1)m}) = 0$. Thus $C_{i+1}, \ldots, C_{i+m}$ are skew in M/C_i. For each $k \in \{1, \ldots, m\}$ let $S_k \subseteq C_{i+k}$ be a circuit in M/C_i. Thus $S_1, \ldots, S_m$ are skew-circuits in M/C_i and $\sqcap_M(C_i, S_1 \cup \cdots \cup S_m) \leqslant 1$. Now, there exists $\epsilon \in \{0, 1\}$ and a δ-element subset $\mathcal{S}$ of $\{S_1, \ldots, S_m\}$ such that $\sqcap_M(C_i, S) = \epsilon$ for each $S \in \mathcal{S}$. It is now straightforward to check that $M|\mathcal{S}$ is a δ-bundle. □

4. Connectivity

For disjoint subsets A and B of $E(M)$ we let

$$\kappa_M(A, B) = \min\bigl(\lambda_M(X):\ A \subseteq X \subseteq E(M) - B\bigr).$$

Let S, T be disjoint subsets of $E(M)$. It is straightforward to show that $\sqcap_{M/J}(S, T) \leqslant \kappa_M(S, T)$ for any subset J of $E(M) - (S \cup T)$. Tutte [14] proved that there exists a J for which equality is attained.

Theorem 4.1 *(Tutte's Linking Theorem). If S and T are disjoint sets of elements in a matroid M, then there exists $J \subseteq E(M) - (X \cup Y)$ such that $\sqcap_{M/J}(S, T) = \kappa_M(S, T)$.*

Tutte's Linking Theorem is a generalization of Menger's Theorem. Indeed, let s and t be non-adjacent vertices in a connected graph G and let S and T be the sets of edges incident with s and t respectively. It is straightforward to show that the size of the smallest vertex cut separating s and t is $\kappa_{M(G)}(S, T) + 1$ and that there exist k internally vertex disjoint paths from s to t if and only if there exists $J \subseteq E(G) - (S \cup T)$ such that $\sqcap_{M(G/J)}(S, T) \geqslant k - 1$ (one such choice for J is the set of internal edges in the paths).

We will prove a slightly stronger version of Tutte's Linking Theorem. Let S and T be disjoint subsets of $E(M)$. A set $J \subseteq E(M) - (S \cup T)$ is called an *(S, T)-linking set* if

(i) J is independent, and
(ii) J is skew to S and to T;

the *capacity* of J is $\sqcap_{M/J}(S, T)$.

Theorem 4.2. *If S and T are disjoint sets of elements in a matroid M, then there exists an (S, T)-linking set of capacity $\kappa_M(S, T)$.*

Proof. By Theorem 4.1, there exists $J \subseteq E(M) - (S \cup T)$ such that $\sqcap_{M/J}(S, T) = \kappa_M(S, T)$. Among all such sets choose J as small as possible. It is routine to show that J is independent. Assume that J is not skew to both S and T. Then, up to symmetry, we may assume that there is an element $j \in J$ such that $j \in \mathrm{cl}_{M/(J-\{j\})}(S)$; set $J' = J - \{j\}$. Then $r_{M/J}(S \cup T) = r_{M/J'}(S \cup T) - 1$, $r_{M/J}(S) = r_{M/J'}(S) - 1$, and $r_{M/J}(T) \leqslant r_{M/J'}(T)$. Hence

$$\begin{aligned}\sqcap_{M/J'}(S,T) &= r_{M/J'}(S)+r_{M/J'}(T)-r_{M/J'}(S\cup T)\\ &\geqslant r_{M/J}(S)+r_{M/J}(T)-r_{M/J}(S\cup T)\\ &= \sqcap_{M/J}(S,T)\\ &= \kappa_M(S,T),\end{aligned}$$

contradicting the minimality of J. □

The series classes of $(M/(S\cup T))|J$ are called the *strands* of J. In the graphic case discussed above, a strand would be the set of internal edges of an (s,t)-path in G. The following results show that the strands of J behave somewhat like "(S,T)-paths."

Lemma 4.3. *If S and T are disjoint sets of elements in a matroid M and J be an (S,T)-linking set in M, then*

$$\sqcap_{M/J}(S,T)=\sqcap_M(S,T)+r^*\big(\big(M/(S\cup T)\big)|J\big).$$

Proof.

$$\begin{aligned}\sqcap_{M/J}(S,T) &= r_{M/J}(S)+r_{M/J}(T)-r_{M/J}(S\cup T)\\ &= r_M(S)+r_M(T)-r_M(S\cup T\cup J)+|J|\\ &= \sqcap_M(S,T)+|J|-\big(r_M(S\cup T\cup J)-r_M(S\cup T)\big)\\ &= \sqcap_M(S,T)+r^*\big(\big(M/(S\cup T)\big)|J\big),\end{aligned}$$

as required. □

Unfortunately there is one significant failure in extending from paths in graphs to linking sets in matroids; the number of strands of J may be considerably larger than $\sqcap_{M/J}(S,T)-\sqcap_M(S,T)$.

Lemma 4.4. *Let S and T be disjoint sets of elements in a matroid M, let J be an (S,T)-linking set, and let $\mathcal{P}$ be the set of strands of J. If $\mathcal{P}'\subseteq\mathcal{P}$ then*

$$|\mathcal{P}'|-r\big(\mathrm{co}\big(\big(M/(S\cup T)\big)|J\big)\big)\leqslant\sqcap_{M/\mathcal{P}'}(S,T)-\sqcap_M(S,T)\leqslant|\mathcal{P}'|.$$

Proof. Let $N=(M/(S\cup T))|J$, let $J'=\bigcup(P'\colon\ P'\in\mathcal{P}')$, and let J'' be the set of elements in $\mathrm{co}(N)$ obtained from J' in the cosimplification. By Lemma 4.3, $\sqcap_{M/\mathcal{P}'}(S,T)-\sqcap_M(S,T)=r^*(N|J')=r^*(\mathrm{co}(N)|J'')=|\mathcal{P}'|-r(\mathrm{co}(N)|J'')$. The result follows since $0\leqslant r(\mathrm{co}(N)|J'')\leqslant r(\mathrm{co}(N))$. □

Lemma 4.4 is very useful when $r(\mathrm{co}((M/(S\cup T))|J))$ is small. An (S,T)-linking set J is called *graphic* if $M/(S\cup T)|J$ is a bundle of $M/(S\cup T)$; if J is graphic, then $r(\mathrm{co}(M/(S\cup T))|J)\leqslant 1$. By Lemma 4.4, if J is a graphic (S,T)-linking set with δ strands, then J has capacity at least $\delta-1$. We now state the main result of this section; the function $\rho_3(n,q,\delta)$ is defined in Theorem 3.5.

Theorem 4.5. *For any positive integers n, q, $\delta\geqslant 3$, if $M\in\mathcal{G}(n,q)\cap\mathcal{G}^*(n,q)$ and $S,T\subseteq E(M)$ are skew with $\kappa_M(S,T)\geqslant\rho_3(n,q,\delta)$, then there exists a graphic (S,T)-linking set with δ strands.*

Proof. By Theorem 4.2, there exists an (S,T)-linking set J of capacity $\kappa_M(S,T)$. Let $N = (M/(S\cup T))|J$. By Lemma 4.3, $r^*(N) = \kappa_M(S,T)$. Therefore, by Theorem 3.5, there exists a δ-bundle N' in N. Thus $E(N')$ is a graphic (S,T)-linking set with δ-strands. □

We conclude this section with some easy results on connectivity.

Lemma 4.6. *Let T be a set of elements of a matroid M, and, for each $X \subseteq E(M) - T$, let $\psi(X) = \kappa_M(X,T)$. Then ψ is the rank function of a matroid on $E(M) - X$.*

Proof. It follows easily from the definition of ψ that:

(i) if $X \subseteq E(M) - T$, then $0 \leqslant \psi(X) \leqslant |X|$, and
(ii) if $X \subseteq Y \subseteq E(M) - T$, then $\psi(X) \leqslant \psi(Y)$.

Thus it only remains to prove that ψ is submodular. Consider $X_1, X_2 \subseteq E(M) - T$. By definition, for each $i \in \{1,2\}$, there exists a set A_i such that $X_i \subseteq A_i \subseteq E(M) - T$ and $\lambda_M(A_i) = \kappa_M(X_i,T) = \psi(X_i)$. Note that $X_1 \cap X_2 \subseteq A_1 \cap A_2$, so $\kappa_M(X_1 \cap X_2, T) \leqslant \lambda_M(A_1 \cap A_2)$. Similarly, $\kappa_M(X_1 \cup X_2, T) \leqslant \lambda_M(A_1 \cup A_2)$. Moreover, λ_M is submodular. Thus

$$\begin{aligned}\psi(X_1) + \psi(X_2) &= \lambda_M(A_1) + \lambda_M(A_2)\\ &\geqslant \lambda_M(A_1 \cap A_2) + \lambda_M(A_1 \cup A_2)\\ &\geqslant \kappa_M(A_1 \cap A_2, T) + \kappa_M(A_1 \cup A_2, T)\\ &= \psi(A_1 \cap A_2) + \psi(A_1 \cup A_2),\end{aligned}$$

as required. □

Lemma 4.7. *Let S and T be disjoint sets of elements of a matroid M. Then there exist sets $S_1 \subseteq S$ and $T_1 \subseteq T$ such that $|S_1| = |T_1| = \kappa_M(S_1,T_1) = \kappa_M(S,T)$.*

Proof. By Lemma 4.6, there exists $S_1 \subseteq S$ such that $|S_1| = \kappa_M(S_1,T) = \kappa_M(S,T)$ (indeed, take S_1 to be a maximal independent subset of the matroid defined in Lemma 4.6). Now, $\kappa_M(S_1,T) = \kappa_M(T,S_1)$, so, again by Lemma 4.6, there exists $T_1 \subseteq T$ such that $|T_1| = \kappa_M(T_1,S_1) = \kappa_M(T,S_1) = \kappa(S,T)$. □

Lemma 4.8. *Let S and T be disjoint sets of elements of a matroid M such that $|S| = |T| = \kappa_M(S,T)$. Then there exists $J \subseteq E(M) - (S\cup T)$ such that $J \cup S$ and $J \cup T$ are both bases of M.*

Proof. Firstly, since $|S| = |T| = \kappa_M(S,T)$, we easily see that S and T are independent. Let J be a maximal (S,T)-linking set of capacity $\kappa_M(S,T)$. Thus, $J \cup S$ and $J \cup T$ are independent and $\sqcap_{M/J}(S,T) = |S| = |T|$. So S and T span each other in M/J and, hence, $S \subseteq \mathrm{cl}_M(J\cup T)$ and $T \subseteq \mathrm{cl}_M(J \cup S)$. Now, if $S \cup J$ is not a basis, then there exists some $e \in E(M) - \mathrm{cl}_M(S\cup J)$. It is easy to see that $J \cup \{e\}$ is an (S,T)-linking set of capacity $\kappa_M(S,T)$; contradicting our choice of J. Thus $S \cup J$ and $T \cup J$ are bases of M, as required. □

Lemma 4.9. *Let S and T be disjoint subsets of a matroid M such that $S \cup T$ is independent and $\kappa_M(S,T) \geqslant \gamma$. Then, there exist sets $S' \subseteq S$ and $T' \subseteq T$ such that $\kappa_{M/((S\cup T)-(S'\cup T'))}(S',T') = |S'| = |T'| = \gamma$.*

Proof. Let $S' \subseteq S$ and $T' \subseteq T$ be minimal such that $\kappa_{M/((S\cup T)-(S'\cup T'))}(S',T') \geqslant \gamma$. Now suppose by way of contradiction that $|S'| > \gamma$. Let $M' = M/((S \cup T) - (S' \cup T'))$ and let J be an (S',T')-linking set in M' of capacity γ. Now, $\sqcap_{M'/J}(S',T') = \gamma < |S'|$. Thus, there exists $e \in S'$ such that $e \notin \mathrm{cl}_{M'/J}(T')$. So, $\sqcap_{M'/(J\cup\{e\})}(S'-\{e\},T') = \gamma$ and, hence, $\kappa_{M'/e}(S'-\{e\},T') \geqslant \gamma$. This contradicts our choice of S', so $|S'| = \gamma$. By symmetry, $|T'| = \gamma$. □

Lemma 4.10. *Let X, S, and T be disjoint sets of elements of a matroid M. If $\kappa_M(S,X) = \kappa_M(T,X) = |X|$, then $\kappa_M(S,T) \geqslant \frac{1}{2}|X|$.*

Proof. Let $A \subseteq E(M) - T$ with $S \subseteq A$ and $\lambda_M(A) = \kappa_M(S,T)$. By symmetry we may assume that $|X \cap A| \geqslant \frac{1}{2}|X|$. Now, $|X| = \kappa_M(X,T) \leqslant \kappa_M(X \cap A, T) + |X - A| \leqslant \lambda_M(A) + \frac{1}{2}|X| = \kappa_M(S,T) + \frac{1}{2}|X|$. Therefore $\kappa_M(S,T) \geqslant \frac{1}{2}|X|$, as required. □

Lemma 4.11. *Let X and Y be disjoint sets of elements of a matroid M with $\kappa_M(X,Y) \geqslant k$. If $E(M) - (X \cup Y) \neq \emptyset$ and $\kappa_{M/e}(X,Y) < k$ or $\kappa_{M\backslash e}(X,Y) < k$ for each $e \in E(M) - (X \cup Y)$, then $\lambda_M(X) = k$ and there exists an ordering $(x_1,\dots,x_l)$ of $E(M) - (X \cup Y)$ such that $\lambda_M(X \cup \{x_1,\dots,x_i\}) = k$ for each $i \in \{1,\dots,l\}$.*

Proof. Let $\mathcal{Z}$ be the family of sets Z with $X \subseteq Z \subseteq E(M) - Y$ such that $\lambda_M(Z) = k$. Note that, as $\kappa_M(X,Y) \geqslant k$, it follows by submodularity that $\mathcal{Z}$ is closed under union and intersection.

Choose a maximal collection $X_1,\dots,X_l$ in $\mathcal{Z}$ with $X \subsetneq X_1 \subsetneq \cdots \subsetneq X_l \subsetneq E(M) - Y$. Let $X_0 = X$ and $X_{l+1} = E(M) - Y$. Consider any $i \in \{1,\dots,l+1\}$ and any $e \in X_i - X_{i-1}$. Since $\kappa_{M/e}(X,Y) < k$ or $\kappa_{M\backslash e}(X,Y) < k$, there exists $A \subseteq E(M) - \{e\}$ such that A and $A \cup \{e\}$ are in $\mathcal{Z}$.

Let $A_1 = A \cup X_{i-1}$ and let $A_2 = A_1 \cap X_i$. If $i = 1$ then $A_1 = A$. If $i > 1$ then, since $\mathcal{Z}$ is closed under union, both of the sets A_1 and $A_1 \cup \{e\}$ are in $\mathcal{Z}$. In any case, $A_1, A_1 \cup \{e\} \in \mathcal{Z}$. Similarly, if $i = l+1$, then $A_2 = A_1$, and if $i \leqslant l$, then, since $\mathcal{Z}$ is closed under intersection, both of the sets A_2 and $A_2 \cup \{e\}$ are in $\mathcal{Z}$. In any case we have shown that $A_2, A_2 \cup \{e\} \in \mathcal{Z}$. However, note that $X_{i-1} \subseteq A_2 \subsetneq A_2 \cup \{e\} \subseteq X_i$. So, by our choice of $X_1,\dots,X_l$ we must have $X_{i-1} = A_2$ and $X_i = A_2 \cup \{e\}$. Thus, $|X_i - X_{i-1}| = 1$ for all i and, hence, we obtain the required ordering. □

Bundles play a significant role throughout this paper; the following lemma, in particular, is used frequently.

Lemma 4.12. *Let N be a bundle of M and let S be a strand of N. If S is not a series class of M, then $\lambda_{M\backslash S}(E(N) - S) < \lambda_M(E(N))$.*

Proof. Let $A = E(N) - S$ and $B = E(M) - E(N)$. By Lemma 2.1,

$$\begin{aligned}\lambda_M(B) &= \sqcap_M(A,B) + \lambda_{M/A}(B)\\ &= \lambda_{M\backslash S}\big(E(N) - S\big) + \lambda_{M/A}(S).\end{aligned}$$

Now suppose that $\lambda_{M\backslash S}(E(N) - S) = \lambda_M(E(N))$, and, hence, $\lambda_{M/A}(S) = 0$. Then, since $M|(A \cup S)$ is a bundle, S is a circuit in M/A. Thus, S is a series class of M/A and, hence, also of M. □

Note that if each strand of a bundle N of M is also a series class of M, then $\lambda_M(E(N)) \leqslant 1$. If, moreover, N is a set of skew-circuits, then $\lambda_M(E(N)) = 0$.

Note also that if N is an n-bundle in a matroid M with $\lambda_M(E(N)) = k$ where $k \leqslant n - 3$, then N has a collection $\mathcal{S}$ of k strands such that $N \setminus \mathcal{S}$ is an $(n-k)$-bundle whose strands are series-classes of $M \setminus \mathcal{S}$.

5. Extracting a grid

This section shows how to extract a grid from a particular structure. This is the only place in the proof where we are forced to explicitly identify a grid; in other cases we find cliques or cocliques using Theorem 3.2. The proofs in this section and in the next section rely heavily on the techniques of Johnson, Robertson, and Seymour [9].

The main result of this section is as follows.

Lemma 5.1. *For all positive integers θ and q there exist positive integers $n = n(\theta, q)$ and $m = m(\theta, q)$ such that, if $M \in \mathcal{U}(q) \cap \mathcal{U}^*(q)$ is a matroid and $(A_1, \dots, A_{n+1}, T_1, \dots, T_n)$ is a partition of $E(M)$ such that*:

(i) *$M|(A_1 \cup \cdots \cup A_{n+1})$ has m series classes,*
(ii) *for each series class S of $M|(A_1 \cup \cdots \cup A_{n+1})$ and each $i \in \{1, \dots, n+1\}$, $A_i \cap S \neq \emptyset$,*
(iii) *for each $j \in \{1, \dots, n\}$, T_j is spanned by both $A_1 \cup \cdots \cup A_j$ and $A_{j+1} \cup \cdots \cup A_{n+1}$, and*
(iv) *for each $j \in \{1, \dots, n\}$, $M|(A_1 \cup \cdots \cup A_j \cup T_j)$ is connected,*

then M has an $M(\mathrm{grid}_\theta)$-minor or $M^*(\mathrm{grid}_\theta)$-minor.

Before proving this lemma we need some preliminary results. The first of these allows us to recognize graphic matroids; this is essentially due to Seymour [13]. For a vertex v of a graph G we let $\delta_G(v)$ denote the set of edges of G that are incident with v.

Lemma 5.2. *Let $\hat{v}$ be a vertex of a connected graph $G = (V, E)$ and let M be a matroid on E such that*:

(i) *$r(M) = |V(G)| - 1$, and*
(ii) *$\delta_G(v)$ is a cocircuit of M for each $v \in V(G) - \{\hat{v}\}$.*

Then, for each spanning tree T of G, $E(T)$ is a basis of M. Moreover, if $\delta_G(\hat{v})$ is also a cocircuit of M, then $M = M(G)$.

Proof. The following claim is an immediate consequence of (ii).

5.2.1. *If C is a circuit of M and $v \neq \hat{v}$ is a vertex of G, then $|C \cap \delta_G(v)| \neq 1$.*

Let T be a spanning tree of G. By 5.2.1, $E(T)$ cannot contain a circuit of M. That is, $E(T)$ is independent in M. Then, by (i), $E(T)$ is a basis of M. Now suppose that $\delta_G(\hat{v})$ is also a cocircuit of M. It remains to prove that any circuit C of G is a circuit of M. Let $e \in C$ and let T be a spanning tree of G with $C - \{e\} \subseteq E(T)$. Now, $E(T)$ is a basis of M, so $E(T) \cup \{e\}$ contains a unique circuit C' of M. By an obvious extension of 5.2.1, the subgraph of G induced by C' has no vertices of degree 1, so $C = C'$. □

Lemma 5.3. *Let $G = (V, E)$ be a 2-connected planar map, let $\hat{F}$ be a face of G, and let M be a matroid on E such that*:

(i) $r(M) = |V(G)| - 1$,
(ii) *for each face F of G other than $\hat{F}$, $E(F)$ is a circuit of M, and*
(iii) *for each vertex v of G that is not on the boundary of $\hat{F}$, $\delta_G(v)$ is a cocircuit of M.*

Then $M = M(G)$.

Proof. By applying Lemma 5.2 to M^* and the plane dual G^* of G, it suffices to prove that $E(\hat{F})$ is a circuit of M. Moreover, we also see that if T is a spanning tree of G, then $E(T)$ is a basis of M. As each proper subset of $E(\hat{F})$ is contained in a spanning tree of G, all proper subsets of $E(\hat{F})$ are independent in M. Now, by way of contradiction, suppose that $E(\hat{F})$ is independent in M. Let $e \in E(\hat{F})$ and let T be a maximal tree in G such that $E(\hat{F}) - \{e\} \subseteq E(T)$ and $E(T) \cup \{e\}$ is independent in M. Since $r(M) = |V(G)| - 1$, T is not a spanning tree of G. Thus there exists an edge f of G with ends $u \in V(T)$ and $v \in V(G) - V(T)$. Now, v is not incident with $\hat{F}$ so $\delta_G(v)$ is a cocircuit of M and, hence, $(E(T) \cup \{e\}) \cup \{f\}$ is an independent set of M. This contradicts our choice of T. □

Lemma 5.4. *Let M be a matroid with $E(M) = \{a_{ij}\colon i, j \in \{1, \dots, n\}\} \cup \{e_{ij}\colon i \in \{1, \dots, n-1\}, j \in \{1, \dots, n\}\}$ such that*

(i) $\{a_{ij}\colon i, j \in \{1, \dots, n\}\}$ *is a basis of M,*
(ii) $\{a_{l1}, \dots, a_{ln}\}$ *is a series class of $(M/\{e_{1n}, \dots, e_{n-1,n}\})|\{a_{ij}\colon i, j \in \{1, \dots, n\}\}$ for each $l \in \{1, \dots, n\}$,*
(iii) $\{a_{ij}, a_{i+1,j}, e_{i,j-1}, e_{i,j}\}$ *is a circuit for each $i \in \{1, \dots, n-1\}$ and $j \in \{2, \dots, n\}$, and*
(iv) $\{a_{i1}, a_{i+1,1}, e_{i1}\}$ *is a circuit for each $i \in \{1, \dots, n-1\}$.*

Then M has an $M(\mathrm{grid}_{n-1})$-restriction.

Proof. Let G be a graph with $V(G) = \{x\} \cup \{v_{ij}\colon i \in \{1, \dots, n\}, j \in \{1, \dots, n\}\}$ and $E(G) = E(M)$ where e_{ij} is incident with v_{ij} and $v_{i+1,j}$ for each $i \in \{1, \dots, n-1\}$ and $j \in \{1, \dots, n\}$, a_{ij} is incident with $v_{i,j-1}$ and v_{ij} for each $i \in \{1, \dots, n\}$ and $j \in \{2, \dots, n\}$, and a_{i1} is incident with x and v_{i1} for each $i \in \{1, \dots, n\}$. It suffices to prove that $M = M(G)$, for which we shall use Lemma 5.3. Note that G is planar and, by (i), $r(M) = |V(G)| - 1$. Moreover, by (iii) and (iv), all but at most one face of G is a circuit in M. Thus, we need only show that $\delta_G(v_{i'j'})$ is a cocircuit of M for each $i' \in \{2, \dots, n-1\}$ and $j' \in \{1, \dots, n-1\}$.

Let $A = \{a_{ij}\colon i, j \in \{1, \dots, n\}\}$ and $P = \{e_{1n}, \dots, e_{n-1,n}\}$. By (ii), $\{a_{i'j'}, a_{i',j'+1}\}$ is a series pair of $M|(A \cup P)$. So the set $(A \cup P) - \{a_{i'j'}, a_{i',j'+1}\}$ does not span M. However, considering the small circuits we see that this set spans $E(M) - \delta_G(v_{i'j'})$. Thus, $\delta_G(v_{i'j'})$ contains a cocircuit C of M. Circuits and cocircuits cannot meet in a single element so, considering the small circuits of G incident with $v_{i'j'}$, we see that $C = \delta_G(v_{i'j'})$. □

Proof of Lemma 5.1. (Recall that the function α is defined in Corollary 3.3.) Let $t = 2\alpha(\theta, q)$, $d = \theta + 1$, $m = 1 + \sum_{i=0}^{d}(t-1)^i$, $n_1 = \max(t, d(q+1)^d)$, and $n = 2^{m(m-1)}n_1$. Now let M be a matroid satisfying the hypotheses of the theorem.

We may assume that M is cosimple. Let $A = A_1 \cup \cdots \cup A_{n+1}$ and let $S_1, \ldots, S_m$ be the series classes of $M|A$. Since each S_i and A_j have a nonempty intersection, $A - A_j$ is an independent set for each $j \in \{1, \ldots, n+1\}$. Consider some $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n+1\}$. Since $T_1, \ldots, T_{j-1} \subseteq \mathrm{cl}_M(A_1 \cup \cdots \cup A_{j-1})$ and $T_j, \ldots, T_n \subseteq \mathrm{cl}_M(A_{j+1} \cup \cdots \cup A_{n+1})$, we have $T_1 \cup \cdots \cup T_n \subseteq \mathrm{cl}_M(A - A_j)$. Hence the members of $S_i \cap A_j$ are in series in M. Then, since M is cosimple, $S_i \cap A_j$ consists of a single element, say a_{ij}.

For $I \subseteq \{1, \ldots, m\}$ and $J \subseteq \{1, \ldots, n+1\}$ we let $A(I, J)$ denote $\{a_{ij}\colon i \in I, j \in J\}$. For each $j \in \{1, \ldots, n\}$, let $B_j = A(\{1, \ldots, m\}, \{1, \ldots, j\})$ and let $M_j = M|(B_j \cup T_j)$. We may assume that T_j is minimal such that M_j is connected. Let $i \in \{1, \ldots, m\}$. Since the members of $A(\{i\}, \{1, \ldots, j\})$ are in series in $M|A$ and since $T_j \subseteq \mathrm{cl}_M(A(\{1, \ldots, m\}, \{j+1, \ldots, n+1\}))$ we see that the members of $A(\{i\}, \{1, \ldots, j\})$ are in series in $M|(A \cup T_j)$ and, hence, also in M_j. Since B_j is a basis of M_j, for each $e \in T_j$ there exists a unique circuit $C_e \subseteq B_j \cup \{e\}$. We let S_e be the set of all $i \in \{1, \ldots, m\}$ such that $C_e \cap A(\{i\}, \{1, \ldots, j\}) \neq \emptyset$. Thus, $C_e = A(S_e, \{1, \ldots, j\}) \cup \{e\}$. Now we define the hypergraph H_j with vertex set $\{1, \ldots, m\}$ and edge set $\{S_e\colon e \in T_j\}$. Since M_j is connected H_j is connected, and, by our choice of T_j, the hypergraph $H_j \setminus S_e$ is not connected for any $e \in T_j$. It follows that $|T_j| \leqslant m - 1$. Thus there exist $K \subseteq \{1, \ldots, n\}$ with $|K| = n_1$ and a hypergraph H on $\{1, \ldots, m\}$ such that $H_j = H$ for all $j \in K$.

5.4.1. *If there exists $S \in E(H)$ such that $|S| \geqslant t$, then M has an $M(\mathrm{grid}_\theta)$- or $M^*(\mathrm{grid}_\theta)$-minor.*

Subproof. Suppose that M has no $M(\mathrm{grid}_\theta)$- or $M^*(\mathrm{grid}_\theta)$-minor; thus, $M \in \mathcal{A}(\theta, q)$. Now, for each $j \in K$ there exists $e_j \in T_j$ such that $S_{e_j} = S$. Let $N = \mathrm{co}((M|(A \cup \{e_j\colon j \in K\}))/A(\{1, \ldots, m\} - S, \{1, \ldots, n+1\}))$. Now $|E(N)| = |S|(n_1 + 1) + n_1$ and $r(N) \geqslant |S|n_1$. So $r^*(N) \leqslant |S| + n_1$. Note that, $|S| \geqslant t = 2\alpha(\theta, q)$ and $n_1 \geqslant t = 2\alpha(\theta, q)$, so $\alpha(\theta, q)r^*(N) \leqslant \alpha(\theta, q)(|S| + n_1) \leqslant \frac{1}{2}n_1|S| + \frac{1}{2}|S|n_1 \leqslant |S|n_1 < |E(N)|$; contradicting Corollary 3.3. □

5.4.2. *If there exists $v \in V(H)$ that is in at least t hyperedges of H, then M has an $M(\mathrm{grid}_\theta)$- or $M^*(\mathrm{grid}_\theta)$-minor.*

Subproof. Let $S'_1, \ldots, S'_t$ be hyperedges of H containing v. Since H is minimally connected, none of the sets $S'_1, \ldots, S'_t$ is contained in the union of the rest. Thus there exist vertices $v_1, \ldots, v_t$ of H such that $v_i \in S'_j$ if and only if $i = j$. For each $j \in K$ and $i \in \{1, \ldots, t\}$ let e_{ij} be the element in T_j such that $S_{e_{ij}} = S'_i$. Now let $N_1 = (M|(A \cup \{e_{ij}\colon i \in \{1, \ldots, t\}, j \in K\}))/A(\{1, \ldots, m\} - \{v, v_1, \ldots, v_t\}, \{1, \ldots, n\})$. For each $j \in K$, let $M'_j = N_1|(A(\{v, v_1, \ldots, v_t\}, \{1, \ldots, j\}) \cup \{e_{1j}, \ldots, e_{tj}\})$. Thus, $A(\{v, v_1, \ldots, v_t\}, \{1, \ldots, j\})$ is a basis for M'_j and, for each $i \in \{1, \ldots, t\}$, $A(\{v, v_i\}, \{1, \ldots, j\}) \cup \{e_{ij}\}$ is a circuit of M'_j. Now, for elements $i, i' \in \{1, \ldots, t\}$ it is easy to check that $A(\{v_i, v_{i'}\}, \{1, \ldots, j\}) \cup \{e_{ij}, e_{i'j}\}$ is a circuit of M'_j.

Choose distinct elements $\{j_1, \ldots, j_t\}$ in K. Now let $N_2 = (N_1|(A(\{v_1, \ldots, v_t\}, \{1, \ldots, n\}) \cup \{e_{i,j_l}\colon\ i \in \{1, \ldots, t\}, l \in \{1, \ldots, t\}\}))/(A(\{v_1, \ldots, v_t\}, \{2, \ldots, n\}) \cup \{e_{1,j_1}, \ldots, e_{t,j_t}\})$. Now, $\{a_{v_1,1}, \ldots, a_{v_t,1}\}$ is a basis of N_2 and for each $l \in \{1, \ldots, t\}$ and $i \in \{1, \ldots, t\} - \{l\}$, the triple $\{a_{v_i,1}, a_{v_l,1}, e_{i,j_l}\}$ is a triangle of N_2. Thus $r(N_2) = t$ and $|E(\mathrm{si}(N_2))| \geqslant \frac{(t+1)t}{2} > \alpha(\theta, q)r(N_2)$, contradicting Corollary 3.3. □

Henceforth we may assume that each vertex of H is in at most t hyperedges and that each hyperedge has size at most t. It is now routine to show that H contains a long "induced" path. That is, H contains a sequence of vertices $(v_0, \ldots, v_d)$ and a sequence of hyperedges $(S'_1, \ldots, S'_d)$ such that $S'_i \cap \{v_0, \ldots, v_d\} = \{v_{i-1}, v_i\}$ for each $i \in \{1, \ldots, d\}$.

For each $j \in K$ and $i \in \{1, \ldots, d\}$ let e_{ij} be the element in T_j such that $S_{e_{ij}} = S'_i$. Now let $N_1 = (M|(A \cup \{e_{ij}\colon i \in \{1, \ldots, d\}, j \in K\}))/A(\{1, \ldots, m\} - \{v_0, \ldots, v_d\}, \{1, \ldots, n\})$ and let $N_2 = N_1/A(\{v_0, \ldots, v_d\}, \{2, \ldots, n\})$. Now, $\{e_{ij}\colon j \in K\} \subseteq \mathrm{cl}_{N_2}(a_{v_{i-1},1}, a_{v_i,1})$ for each $i \in \{1, \ldots, d\}$, so $r_{N_2}(\{e_{ij}\colon j \in K\}) \leqslant 2$. Since $|K| \geqslant d(q+1)^d$, it follows easily that there exists $K_1 \subseteq K$ such that $|K_1| = d$ and $r_{N_2}(\{e_{ij}\colon j \in K_1\}) = 1$ for each $i \in \{1, \ldots, d\}$. Now let $N_3 = N_1 \setminus \{e_{ij}\colon i \in \{1, \ldots, d\}, j \in K - K_1\}$. It is straightforward to show that N_3 satisfies the hypotheses of Lemma 5.4 and, hence, that N_3 has an $M(\mathrm{grid}_\theta)$-minor. □

6. Disentangling

In this section we obtain various results saying that given two highly connected sets X and Y and a very large bundle either we can route some of the connectivity between X and Y in a way that avoids many of the strands of the bundle or we will find a large grid.

Lemma 6.1. *There exist integer-valued functions $\delta_1(\delta, \gamma, \theta, q)$ and $\gamma_1(\gamma, \theta, q)$ such that for any positive integers δ, γ, θ, and $q \geqslant 2$, if $M \in \mathcal{A}(\theta, q)$, X and Y are disjoint subsets of $E(M)$ with $\kappa_M(X, Y) \geqslant \gamma_1(\gamma, \theta, q)$, and N is a $\delta_1(\delta, \gamma, \theta, q)$-bundle in $M \setminus (X \cup Y)$, then there exists a set $\mathcal{S}$ of strands of N such that $|\mathcal{S}| = \delta$ and $\kappa_{M/\mathcal{S}}(X, Y) \geqslant \gamma$.*

Proof. (Recall that α is defined in Corollary 3.3 and that m and n are defined in Lemma 5.1.) Let $\hat{\gamma}_1 = m(\theta, q) + \gamma + 1$ and $\gamma_1(\gamma, \theta, q) = \hat{\gamma}_1$. Let $n' = n(\theta, q)$, $n = 2(n' + 2\hat{\gamma}_1) + 1$, $l' = n^{\alpha(\theta,q)\hat{\gamma}_1}(\alpha(\theta, q)\hat{\gamma}_1 + 1)$, $\hat{\delta}_2 = l'(\hat{\gamma}_1 + 1)$, $\hat{\delta}_1 = \hat{\delta}_2 2^{\alpha(\theta,q)\hat{\gamma}_1}$, and $\delta_1(\delta, \gamma, \theta, q) = \hat{\delta}_1 + \delta$. Now, let M, N, X and Y be as given in the lemma.

By Lemma 4.7, we may assume that $|X| = |Y| = \kappa_M(X, Y) = \hat{\gamma}_1$. Let $C_1 \subseteq E(N)$ and $D_1 \subseteq E(M) - (E(N) \cup X \cup Y)$ be maximal sets such that $\kappa_{M \setminus D_1/C_1}(X, Y) \geqslant \hat{\gamma}_1$. Now let $M_1 = M \setminus D_1/C_1$ and $N_1 = N/C_1$. Note that N_1 is a bundle and we may assume that N_1 has at least $\hat{\delta}_1$ strands. Let $R = E(M_1) - (X \cup Y \cup E(N_1))$. By our choice of D_1 and C_1, if $e \in E(N_1)$, then $\kappa_{M_1/e}(X, Y) < \hat{\gamma}_1$, and if $e \in R$, then $\kappa_{M_1 \setminus e}(X, Y) < \hat{\gamma}_1$. By Lemma 4.11, $\lambda_{M_1}(X) = \hat{\gamma}_1$ and there exists an ordering $(x_1, \ldots, x_l)$ of $R \cup E(N_1)$ such that $\lambda_{M_1}(X \cup \{x_1, \ldots, x_i\}) = \hat{\gamma}_1$ for each $i \in \{1, \ldots, l\}$. Let $X_0 = X$ and $X_i = X \cup \{x_1, \ldots, x_i\}$ for each $i \in \{1, \ldots, l\}$.

6.1.1. *$X \cup R$ and $Y \cup R$ are bases of M_1 and for each $i \in \{1, \ldots, l\}$, $(X \cup R) \cap X_i$ spans X_i and $(Y \cup R) - X_i$ spans $E(M_1) - X_i$.*

Subproof. By Lemma 4.8, there exists $J \subseteq E(M_1) - (X \cup Y)$ such that $X \cup J$ and $Y \cup J$ are both bases of M_1. Evidently, J is an (X, Y)-linking set of capacity $\hat{\gamma}_1$. So, for each $e \in J$, $\kappa_{M_1/e}(X, Y) = \hat{\gamma}_1$. Thus, J is disjoint from $E(N_1)$. Similarly, if $f \in E(M_1) - (X \cup Y \cup J)$, then $\kappa_{M \setminus f}(X, Y) = \hat{\gamma}_1$. So $R \subseteq J$. It follows that $R = J$ and, hence, $X \cup R$ and $Y \cup R$ are bases of M_1.

Now consider $i \in \{1, \ldots, l\}$. Let $M' = M_1|(X \cup Y \cup R)$. Since R is an (X, Y)-linking set of M' of capacity $\hat{\gamma}_1$, we have $\lambda_{M'}(X_i \cap (X \cup R)) = \lambda_{M_1}(X_i)$. Now, since $r(M') = r(M_1)$, we have

$$r_{M'}\big(X_i \cap (X \cup R)\big) + r_{M'}\big((Y \cup R) - X_i\big)$$

$$= \lambda_{M'}\big(X_i \cap (X \cup R)\big) + r(M')$$
$$= \lambda_{M_1}(X_i) + r(M_1)$$
$$= r_{M_1}(X_i) + r_{M_1}\big(E(M_1) - X_i\big).$$

Then, since M' is a restriction of M_1, we have $r_{M'}(X_i \cap (X \cup R)) = r_{M_1}(X_i)$ and $r_{M'}((Y \cup R) - X_i) = r_{M_1}(E(M_1) - X_i)$, as required. $\square$

For each $e \in E(N_1)$, let F_e be the fundamental circuit of e with respect to the basis $X \cup R$ of M_1. Let $L_1 = (M_1/(X \cup Y))|R$; by Lemma 4.3, $r^*(L_1) \leqslant \hat{\gamma}_1$.

6.1.2. *For each $i \in \{1, \ldots, l\}$, if $x_i \in E(N_1)$ then $F_{x_i} \subseteq X_i$, $F_{x_i} \cap X \neq \emptyset$, and if S is a series class of L_1 such that $F_{x_i} \cap S \neq \emptyset$, then $S \cap X_i \subseteq F_{x_i}$.*

Subproof. Since $(X \cup R) \cap X_i$ spans X_i it follows that $F_{x_i} \subseteq X_i$. Now, if $F_{x_i} \cap X = \emptyset$, then x_i would be a loop in M_1/R and, hence, $\kappa_{M_1/x_i}(X, Y) = \hat{\gamma}_1$. However, this is not the case since $x_i \in E(N_1)$, so $F_{x_i} \cap X \neq \emptyset$. Now consider $M' = M_1|(X \cup Y \cup R \cup \{x_i\})$. If S is a series class of L_1 the elements of S are in series in $M' \setminus x_i$. Moreover, since $x_i \in \mathrm{cl}_{M_1}((Y \cup R) - X_i)$, the elements of $S \cap X_i$ are in series in M'. Therefore, if $S \cap F_{x_i} \neq \emptyset$, then $S \cap X_i \subseteq F_{x_i}$. $\square$

Now, for each $e \in E(N_1)$, let $\mathcal{S}_e$ be the set of series classes S of L_1 such that $S \cap F_e \neq \emptyset$, and, for each strand P of N_1, let $\mathcal{S}(P) = \bigcup(\mathcal{S}_e\colon\ e \in P)$. By Corollary 3.3 and the fact that $r^*(L_1) \leqslant \hat{\gamma}_1$, L_1 has at most $\alpha(\theta, q)\hat{\gamma}_1$ series classes. So there are at most $2^{\alpha(\theta,q)\hat{\gamma}_1}$ distinct sets among $(\mathcal{S}(P)\colon\ P$ a strand of $N_1)$. Thus, there exists a set $\mathcal{S}$ of series classes of L_1 and a set $\mathcal{P}$ of strands of N_1 such that $|\mathcal{P}| \geqslant \hat{\delta}_2$ and $\mathcal{S}(P) = \mathcal{S}$ for each $P \in \mathcal{P}$.

6.1.3. *If $\mathcal{P}' \subseteq \mathcal{P}$ is nonempty, then there exists $i \in \{1, \ldots, l\}$ such that X_i contains a strand in $\mathcal{P}'$ and X_i is disjoint from at least $|\mathcal{P}'| - \hat{\gamma}_1 - 1$ strands in $\mathcal{P}'$.*

Subproof. Choose i minimal such that X_i contains a strand, P_1 say, of $\mathcal{P}'$. Let, $\mathcal{P}''$ be the strands in $\mathcal{P}'$ that have a nonempty intersection with X_i. Note that, $P \cap X_i$ and $P - X_{i-1}$ are both nonempty for each $P \in \mathcal{P}''$. However, $|\mathcal{P}''| - 1 \leqslant \lambda_{N_1}(E(N_1) \cap X_{i-1}) \leqslant \lambda_{M_1}(X_{i-1}) = \hat{\gamma}_1$. Thus, X_i satisfies the claim. $\square$

By 6.1.3 and an easy inductive argument we have the following.

6.1.4. *There exists a subsequence $(Z_0, \ldots, Z_{l'})$ of $(X_0, \ldots, X_l)$ such that for each $i \in \{1, \ldots, l'\}$ the set $Z_i - Z_{i-1}$ contains a strand of $\mathcal{P}$.*

6.1.5. *There exists a subsequence $(W_0, \ldots, W_n)$ of $(Z_0, \ldots, Z_{l'})$ such that for each series class S of L_1 either*:

(1) $(W_i - W_{i-1}) \cap S \neq \emptyset$ *for all $i \in \{1, \ldots, n\}$, or*
(2) $(W_n - W_0) \cap S = \emptyset$.

Subproof. Let p be the number of series classes of L_1. By Corollary 3.3 and the fact that $r^*(L_1) \leqslant \hat{\gamma}_1$, $p \leqslant \alpha(\theta, q)\hat{\gamma}_1$. For each $i \in \{0, \ldots, p\}$, define $\beta_i = n^{p-i}(\alpha(\theta, q)\hat{\gamma}_1 + 1)$. Now choose $t \in \{0, \ldots, p\}$ maximal such that there exist $i, j \in \{1, \ldots, l'\}$, $j - i \geqslant \beta_t$, and such that $Z_j - Z_i$ is disjoint from at least t series classes of L_1.

If $t = p$, then $Z_j - Z_i \subseteq E(N_1)$. So, by 6.1.1, $Z_j - Z_i \subseteq \mathrm{cl}_M(Z_i)$. However $\lambda_{M_1}(Z_i) = \hat{\gamma}_1$, so $r(Z_j - Z_i) \leqslant \hat{\gamma}_1$. Therefore $|i - j| \leqslant |Z_j - Z_i| \leqslant \alpha(\theta, q)\hat{\gamma}_1$. This contradiction shows that $t < p$.

Now define $W_a = Z_{i+a\beta_{t+1}}$ for $a \in \{0, \ldots, n\}$. Evidently this satisfies the claim. □

For each $i \in \{1, \ldots, n\}$ let P_i be a strand of $\mathcal{P}$ where $P_i \subseteq W_i - W_{i-1}$. Recall that $\mathcal{S}(P_i) = \mathcal{S}$ for each $i \in \{1, \ldots, n\}$. Let $\mathcal{S}_1$ be the collection of sets $S \in \mathcal{S}$ such that $S \cap (W_n - W_0) \neq \emptyset$. Now let $X' = W_0$, $Y' = E(M_1) - W_n$, $C_2 = (R \cap (W_n - W_0)) - \bigcup \mathcal{S}_1$, $R' = (R \cap (W_n - W_0)) - C_2$ and $M_2 = M_1/C_2$. Note that R' is an (X', Y')-linking set in M_2 of capacity $\hat{\gamma}_1$; let $L' = (M_2/(X' \cup Y'))|R'$. For each $i \in \{1, \ldots, n\}$ let $W_i' = W_i - C_2$.

6.1.6. *If $i, j \in \{1, \ldots, n\}$ and $j > i$, then $W_i' - X'$ and $E(M_2) - W_j'$ are skew in M_2/X'.*

Subproof. Let $X'' = W_i'$, $Y'' = E(M_2) - W_j'$ and $R'' = R' \cap (W_j' - W_i')$. Then R'' is an (X'', Y'')-linking set of capacity $\hat{\gamma}_1$ in M_2; let $L'' = (M_2/(X'' \cup Y''))|R''$. By Lemma 4.3, $\sqcap_{M_2}(X', Y') + r^*(L') = \hat{\gamma}_1 = \sqcap_{M_2}(X'', Y'') + r^*(L'')$. However, by 6.1.5, $\mathrm{co}(L')$ and $\mathrm{co}(L'')$ are isomorphic. Thus, $r^*(L') = r^*(L'')$ and, hence, $\sqcap_{M_2}(X', Y') = \sqcap_{M_2}(X'', Y'')$. It follows that $X'' - X'$ and Y'' are skew in M_2/X'. □

6.1.7. *C_2 is skew to $P_1 \cup \cdots \cup P_n$ in M_1.*

Subproof. For each $e \in P_1 \cup \cdots \cup P_n$, the fundamental circuit F_e is disjoint from C_2. □

Thus, $M_2|(P_1 \cup \cdots \cup P_n)$ is an n-bundle.

6.1.8. *There exists $\mathcal{P}_2 \subseteq \{P_2, P_4, \ldots, P_{n-1}\}$ such that $|\mathcal{P}_2| \geqslant n'$, $\sqcap_{M_2}(\mathcal{P}_2, X' \cup Y') \leqslant 1$, and $\mathcal{P}_2$ is a set of skew-circuits in $M_2/(X' \cup Y')$.*

Subproof. Let $N_2 = M_2|(P_2 \cup P_4 \cup \cdots \cup P_{n-1})$. Note that $\sqcap_{M_2}(E(N_2), X' \cup Y') \leqslant \lambda_{M_2}(X') + \lambda_{M_2}(Y') = 2\hat{\gamma}_1$. Thus, by Lemma 4.12, there exists a set of strands $\mathcal{P}_2$ of N_2 such that $|\mathcal{P}_2| \geqslant n'$ and each member of $\mathcal{P}_2$ is a series class of $M_2|(X' \cup Y' \cup \mathcal{P}_2)$. Hence, $\sqcap_{M_2}(\mathcal{P}_2, X' \cup Y') \leqslant 1$ and $N_2|\mathcal{P}_2$ is a bundle in $M_2/(X' \cup Y')$. However, by 6.1.6, the sets in $\mathcal{P}_2$ are skew in $M_2/(X' \cup Y')$; thus $\mathcal{P}_2$ is a set of skew-circuits in $M_2/(X' \cup Y')$. □

6.1.9. *If $|\mathcal{S}_1| \leqslant \hat{\gamma}_1 - \gamma - 1$, then $\kappa_{M_1/\mathcal{P}_2}(X', Y') \geqslant \gamma$.*

Subproof. Suppose that $|\mathcal{S}_1| \leqslant \hat{\gamma}_1 - \gamma - 1$. By Lemma 4.3, $\sqcap_{M_2}(X', Y') \geqslant \gamma + 1$. However, $\sqcap_{M_2}(\mathcal{P}_2, X' \cup Y') \leqslant 1$, so $\sqcap_{M_2/\mathcal{P}_2}(X', Y') \geqslant \gamma$. Thus, $\kappa_{M_1/\mathcal{P}_2}(X', Y') \geqslant \gamma$. □

Thus, we may assume that $|\mathcal{S}_1| \geqslant \hat{\gamma}_1 - \gamma - 1 = m(\theta, q)$.

6.1.10. *For each $P \in \mathcal{P}_2$ and $e \in P$, $F_e - X'$ is a circuit in $M_2/(X' \cup Y')$.*

Subproof. Evidently, $F_e - X'$ is a circuit in M_2/X' and, by 6.1.6, $F_e - X'$ is skew to Y' in M_2/X'. Thus, $F_e - X'$ is a circuit in $M_2/(X' \cup Y')$. □

Let $(\sigma_1, \ldots, \sigma_{n'})$ be the subsequence of $(2, 4, \ldots, n-1)$ such that $\mathcal{P}_2 = \{P_{\sigma_1}, \ldots, P_{\sigma_{n'}}\}$. For $i \in \{1, \ldots, n'\}$, let $T_i = P_{\sigma_i}$ and $B_i = R' \cap (W'_{\sigma_i} - W'_{\sigma_i - 1})$. Let $\sigma_0 = 0$, for $i \in \{1, \ldots, n'\}$ let $A_i = R' \cap (W'_{\sigma_i - 1} - W'_{\sigma_{i-1}})$, and let $A_{n'+1} = R' - W'_{\sigma_{n'}}$. Now, let $A = A_1 \cup \cdots \cup A_{n'+1}$ and let M_3 be the restriction of $M_2/(X' \cup Y' \cup B_1 \cup \cdots \cup B_{n'})$ to $A \cup T_1 \cup \cdots \cup T_{n'}$. Now M_3 satisfies the hypotheses of Lemma 5.1. Indeed, for (i), (ii), and (iii) this is obvious, and (iv) follows as, for each $i \in \{1, \ldots, n'\}$ and $e \in T_i$, if S is a series class of $M_3|A$ and $S \cap F_e \neq \emptyset$, then $S \cap F_e = S \cap (A_1 \cup \cdots \cup A_i)$. Thus, by Lemma 5.1, we have a contradiction. □

Lemma 6.2. *There exist integer-valued functions $\delta_2(\delta, \gamma, \theta, q)$ and $\gamma_2(\gamma, \theta, q)$ such that for any positive integers δ, γ, θ, and $q \geqslant 2$, if $M \in \mathcal{A}(\theta, q)$, X and Y are disjoint subsets of $E(M)$ with $|X| = |Y| = \gamma_2(\gamma, \theta, q)$, and N is a $\delta_2(\delta, \gamma, \theta, q)$-bundle of $M \setminus (X \cup Y)$ such that $\kappa_{M/E(N)}(X, Y) = \gamma_2(\gamma, \theta, q)$, then there exist a δ-bundle N' in N and a flat F of M such that $X \cup Y \subseteq F \subseteq E(M) - E(N)$, $\kappa_{M|F}(X, Y) \geqslant \gamma$, and each strand of N' is a series class of $M|(E(N') \cup F)$.*

Proof. Let $\hat{\gamma} = \gamma + 2\alpha(\theta, q)$ and $\hat{\delta} = 2(\delta + \hat{\gamma} + 1)(\alpha(\theta, q)\hat{\gamma} + 1)^{2\alpha(\theta, q)}$. Now let $\delta_2(\delta, \gamma, \theta, q) = \hat{\delta}$ and $\gamma_2(\gamma, \theta, q) = \hat{\gamma}$, and let M, N, X and Y be as given in the statement of the lemma.

Let $t = r(\mathrm{co}(N))$; thus $t \in \{0, 1\}$. By Lemma 4.8, there exists $J \subseteq E(M) - (X \cup Y \cup E(N))$ such that $X \cup J$ and $Y \cup J$ are bases of $M/E(N)$. Let $M_1 = M|(X \cup Y \cup J \cup E(N))$ and let B be a basis for N. Note that $B \cup J \cup X$ is a basis for M_1, so $r^*(M_1) = r^*(N) + |Y| = \hat{\delta} - t + \hat{\gamma}$. By definition, $\hat{\delta} \geqslant \alpha(\theta, q)\hat{\gamma}$. Now, by Corollary 3.3,

$$\begin{aligned} \bigl|E\bigl(\mathrm{co}(M_1)\bigr)\bigr| &\leqslant \alpha(\theta, q) r^*(M_1) \\ &\leqslant \alpha(\theta, q)(\hat{\gamma} + \hat{\delta}) \\ &\leqslant \hat{\delta} + \alpha(\theta, q)\hat{\delta}. \end{aligned}$$

Now note that $B \cup J$ is an (X, Y)-linking set of capacity $\hat{\gamma}$ in M_1; let $L_1 = (M_1/(X \cup Y))|(B \cup J)$. Let $\mathcal{P}$ be the set of strands of N and for each strand $P \in \mathcal{P}$ let $\mathcal{S}_P$ be the set of series classes S of L_1 such that $P \cap S \neq \emptyset$. Thus, $|P \cap E(\mathrm{co}(M_1))| \geqslant |\mathcal{S}_P|$. Hence, $|E(\mathrm{co}(M_1))| \geqslant |\mathcal{P}| + \sum(|\mathcal{S}_P|\colon P \in \mathcal{P})$. Now, $|\mathcal{P}| = \hat{\delta}$ so

$$\sum\bigl(|\mathcal{S}_P|\colon\ P \in \mathcal{P}\bigr) \leqslant \alpha(\theta, q)|\mathcal{P}|.$$

Let $\mathcal{P}_1$ be the collection of sets $P \in \mathcal{P}$ such that $|\mathcal{S}_P| \leqslant 2\alpha(\theta, q)$. Now, $|\mathcal{P}|\alpha(\theta, q) \geqslant \sum(|\mathcal{S}_P|\colon P \in \mathcal{P}) > |\mathcal{P} - \mathcal{P}_1| 2\alpha(\theta, q)$ and, hence, $|\mathcal{P}_1| \geqslant \frac{1}{2}\hat{\delta}$. Now, by Lemma 4.3, $r^*(L_1) \leqslant \hat{\gamma}$, so, by Corollary 3.3, L_1 has at most $\alpha(\theta, q)\hat{\gamma}$ series classes. So there are at most $(\alpha(\theta, q)\hat{\gamma} + 1)^{2\alpha(\theta, q)}$ distinct sets among $(\mathcal{S}_P\colon P \in \mathcal{P}_1)$ and, hence, one of these sets is repeated at least $\delta + \hat{\gamma} + 1$ times. That is, there exist a set $\mathcal{S}$ of series classes of L_1 and $\mathcal{P}_2 \subseteq \mathcal{P}_1$ such that $|\mathcal{P}_2| = \delta + \hat{\gamma} + 1$ and $\mathcal{S}_P = \mathcal{S}$ for all $P \in \mathcal{P}_2$.

Let $Z = \bigcup \mathcal{P}_2$, $F' = X \cup Y \cup J \cup (B - Z)$, $B' = B \cap Z$, and $M_2 = M|(F' \cup Z)$. Now, Z intersects at most $2\alpha(\theta, q)$ series classes of L_1, so, by Lemma 4.4,

(i) $\kappa_{M_2 \setminus Z}(X, Y) \geqslant \hat{\gamma} - 2\alpha(\theta, q) = \gamma$.

Now, $\lambda_{M_2|(F' \cup B')}(B') \leqslant r^*(M_2|(F' \cup B')) = |Y| = \hat{\gamma}$. Moreover, since N is a bundle, $r_{M_2}(B') \geqslant r_{M_2}(Z) - t$, so

(ii) $\lambda_{M_2}(Z) \leqslant \hat{\gamma} + t$.

Thus, by Lemma 4.12, there exists $\mathcal{P}_3 \subseteq \mathcal{P}_2$ such that $|\mathcal{P}_3| \geqslant \delta$ and each strand of $N|\mathcal{P}_3$ is a series class of $M|(F' \cup (\bigcup \mathcal{P}_3))$. Therefore, the δ-bundle $N' = N|\mathcal{P}_3$ and the flat $F = \mathrm{cl}_M(F')$ satisfy the lemma. □

The following result is a strengthening of Lemma 6.1.

Lemma 6.3. *There exist integer-valued functions $\delta_3(\delta, \gamma, \theta, q)$ and $\gamma_3(\gamma, \theta, q)$ such that for any positive integers δ, γ, θ, and $q \geqslant 2$, if $M \in \mathcal{A}(\theta, q)$, X and Y are disjoint subsets of $E(M)$ with $|X| = |Y| = \kappa_M(X, Y) = \gamma_3(\gamma, \theta, q)$, and N is a $\delta_3(\delta, \gamma, \theta, q)$-bundle in $M \setminus (X \cup Y)$, then there exists a δ-bundle N' in N and a flat F of M containing X and Y such that $\kappa_{M|F}(X, Y) \geqslant \gamma$ and each strand of N' is a series class of $M|(F \cup E(N'))$.*

Proof. Let $\hat{\gamma}_2 = \gamma_2(\gamma, \theta, q)$, $\hat{\gamma}_1 = \gamma_1(\hat{\gamma}_2, \theta, q)$, and $\hat{\delta} = \delta_2(\delta + 2\hat{\gamma}_1 + 1, \gamma, \theta, q)$. Now, let $\delta_3(\delta, \gamma, \theta, q) = \delta_1(\hat{\delta}, \hat{\gamma}_2, \theta, q)$ and $\gamma_3(\gamma, \theta, q) = \hat{\gamma}_1$, and let M, X, Y and N be as stated above.

By Lemma 6.1, there exists a $\hat{\delta}$-bundle N_1 in N such that $\kappa_{M/E(N_1)}(X, Y) \geqslant \hat{\gamma}_2$. By Lemma 4.7, there exist sets $X_1 \subseteq X$ and $Y_1 \subseteq Y$ such that $|X_1| = |Y_1| = \kappa_{M/E(N_1)}(X_1, Y_1) = \hat{\gamma}_2$. Now, by Lemma 6.2, there exists a $(\delta + 2\hat{\gamma}_1 + 1)$-bundle N_2 in N_1 and a flat F_1 of M such that $X_1 \cup Y_1 \subseteq F_1$, $\kappa_{M|F_1}(X_1, Y_1) \geqslant \gamma$, and each strand of N_2 is a series class of $M|(F_1 \cup E(N_2))$. Let $F = \mathrm{cl}_M(F_1 \cup X \cup Y)$. Since $\sqcap_M(F_1, E(N_2)) \leqslant 1$, we have $\sqcap_M(F, E(N_2)) \leqslant 1 + |X \cup Y| \leqslant 1 + 2\hat{\gamma}_1$. So, by Lemma 4.12, there exists a δ-bundle N' of N_2 such that each strand of N' is a series class of $M|(F_1 \cup E(N'))$. Now, N' and F satisfy the lemma. □

The final strengthening establishes high connectivity avoiding several bundles at the same time.

Lemma 6.4. *There exist integer-valued functions $\delta_4(\delta, \gamma, l, \theta, q)$ and $\gamma_4(\gamma, l, \theta, q)$ such that for any positive integers δ, γ, l, θ, and $q \geqslant 2$, if $M \in \mathcal{A}(\theta, q)$, X and Y are disjoint subsets of $E(M)$ with $|X| = |Y| = \kappa_M(X, Y) = \gamma_4(\gamma, l, \theta, q)$, N is a restriction of M and $(Z_1, \ldots, Z_l)$ is a partition of $E(N)$ such that for each $i \in \{1, \ldots, l\}$ we have $N|Z_i$ is a $\delta_4(\delta, \gamma, l, \theta, q)$-bundle and each strand of $N|Z_i$ is a series class of N, then there exists a restriction N' of N and a flat F of M containing X and Y such that $\kappa_{M|F}(X, Y) \geqslant \gamma$ and, for each $i \in \{1, \ldots, l\}$, $N'|Z_i$ is a δ-bundle and each strand of $N'|Z_i$ is a series class of $M|(F \cup E(N'))$.*

Proof. The proof is by induction on l; the case $l = 1$ is proved in Lemma 6.3. Let $k \geqslant 2$ and suppose that the result holds for $l = k - 1$; now consider the case that $l = k$.

Let $\hat{\gamma}_3 = \gamma_4(\gamma, l - 1, \theta, q)$, $\hat{\gamma}_2 = \gamma_3(\hat{\gamma}_3, \theta, q)$, $\hat{\gamma}_1 = \gamma_4(\hat{\gamma}_2 + l, l - 1, \theta, q)$, and $\gamma_4(\gamma, l, \theta, q) = \hat{\gamma}_1$. Let $\hat{\delta}_2 = \delta + 2(\hat{\gamma}_1 + 1)$, $\hat{\delta}_1 = \delta_4(\hat{\delta}_2, \gamma, l - 1, \theta, q)$ and $\delta_4(\delta, \gamma, l, \theta, q) = \max(\delta_4(\hat{\delta}_1, \hat{\gamma}_2 + l, l - 1, \theta, q), \delta_3(\hat{\delta}_2, \hat{\gamma}_3, \theta, q))$. Now let M, N, $Z_1, \ldots, Z_l$, X, and Y be as given above.

By the induction hypothesis, there exist $Z'_1 \subseteq Z_1, \ldots, Z'_{l-1} \subseteq Z_{l-1}$ such that $N|Z'_1, \ldots, N|Z'_{l-1}$ are $\hat{\delta}_1$-bundles in M and $\kappa_{M/(Z'_1 \cup \cdots \cup Z'_{l-1})}(X, Y) \geqslant \hat{\gamma}_2$. Let $M_1 = M/(Z'_1 \cup \cdots \cup Z'_{l-1})$. By Lemma 4.7, there exist sets $X_1 \subseteq X$ and $Y_1 \subseteq Y$ such that $|X_1| = |Y_1| = \kappa_{M_1}(X_1, Y_1) = \hat{\gamma}_2$.

By Lemma 6.3, there exist $Z'_l \subseteq Z_l$ and a flat F_1 of M_1 such that $M_1|Z'_l$ is a $\hat{\delta}_2$-bundle, $\kappa_{M_1|F_1}(X_1, Y_1) \geqslant \hat{\gamma}_3$, and each strand of $M_1|Z'_l$ is a series class of $M_1|(F_1 \cup Z'_l)$. Let $M_2 = M|(X \cup Y \cup F_1 \cup Z'_1 \cup \cdots \cup Z'_l)$. Now

$$\lambda_{M_2}(Z'_l) \leqslant |X \cup Y| + \sqcap_{M_2}(Z'_l, F_1 \cup Z'_1 \cup \cdots \cup Z'_{l-1})$$

$$\leqslant 2\hat{\gamma}_1 + \sqcap_{M_1}\big(Z'_l, F_1\big) + \sqcap_{M_2}\big(Z'_l, Z'_1 \cup \cdots \cup Z'_{l-1}\big)$$
$$\leqslant 2\hat{\gamma}_1 + \sqcap_{M_1}\big(Z'_l, F_1\big) + \lambda_N(Z_l)$$
$$\leqslant 2\hat{\gamma}_1 + 2.$$

Therefore, there exists $Z''_l \subseteq Z'_l$ such that $M_2|Z''_l$ is a δ-bundle and each strand of $M_2|Z''_l$ is a series class of $M_2 \setminus (Z'_l - Z''_l)$.

Let $M_3 = M_2 \setminus (Z'_l - Z''_l)$. Note that $M_3|Z''_l = M_2|Z''_l$ is a δ-bundle, $\lambda_{M_3}(Z''_l) \leqslant r(\mathrm{co}(M_3|Z''_l))$, and $\kappa_{M_3}(X, Y) \geqslant \hat{\gamma}_3$. By Lemma 4.7, there exist sets $X_2 \subseteq X$ and $Y_2 \subseteq Y$ such that $|X_2| = |Y_2| = \kappa_{M_3}(X_2, Y_2) = \hat{\gamma}_3$. Now, by the induction hypothesis, there exists $Z \subseteq Z'_1 \cup \cdots \cup Z'_{l-1}$ and a flat F_2 of M_3 such that $X_2 \cup Y_2 \subseteq F_2$, $\kappa_{M_3|F_2}(X_2, Y_2) \geqslant \gamma$, and, for each $i \in \{1, \ldots, l-1\}$, $M_3|(Z \cap Z'_i)$ is a $\hat{\delta}_2$-bundle and each strand of $M_3|(Z \cap Z'_i)$ is a series class of $M_3|Z$.

Let $M_4 = M_3|(X \cup Y \cup F_2 \cup Z \cup Z''_l)$. Then, for each $i \in \{1, \ldots, l-1\}$,

$$\lambda_{M_4}\big(Z \cap Z'_i\big) \leqslant |X \cup Y| + \sqcap_{M_4}\big(Z \cap Z'_i, F_2 \cup \big(Z - Z'_i\big)\big)$$
$$= 2\hat{\gamma}_1 + 1.$$

Thus, there exists $Z''_i \subseteq Z'_i$ such that $M|Z''_i$ is a δ-bundle and each strand of $M|Z''_i$ is a series class of $M_4 \setminus (Z'_i - Z''_i)$. Let $M_5 = M_4|(X \cup Y \cup F_2 \cup Z''_1 \cup \cdots \cup Z''_l)$. Now it is easy to check that $N' = M_5|(Z''_1 \cup \cdots \cup Z''_l)$ and $F = \mathrm{cl}_M(X \cup Y \cup F_2)$ satisfy the lemma. □

7. An application of disentangling

In this section we prove the following corollary to Lemma 6.4.

Lemma 7.1. *There exists an integer-valued function $\gamma_5(\gamma, l, \theta, q)$ such that for positive integers γ, l, θ, and q, if I is an independent set of a matroid $M \in \mathcal{A}(\theta, q)$ and, for each $i \in \{1, \ldots, l\}$, (S_i, T_i) is a pair of disjoint subsets of I with $\kappa_{M/(I-(S_i \cup T_i))}(S_i, T_i) \geqslant \gamma_5(\gamma, l, \theta, q)$, then there exist disjoint subsets $J_1, \ldots, J_l$ of $E(M) - I$ such that for each $i \in \{1, \ldots, l\}$ J_i is an (S_i, T_i)-linking set in $M/(I - (S_i \cup T_i))$ with capacity γ and $\sqcap_{M/I}(J_i, (J_1 \cup \cdots \cup J_l) - J_i) \leqslant 1$.*

(Note that the sets $S_1 \cup T_1, \ldots, S_l \cup T_l$ above need not be disjoint.)

Proof. The proof is by induction on l; the case that $l = 1$ is trivial. For some integer $k \geqslant 2$, assume that the result holds when $l = k - 1$ and consider the case that $l = k$.

(Recall that the function ρ_3 was defined in Theorem 3.5 and the functions δ_4 and γ_4 were defined in Lemma 6.4.)

Let $\hat{\gamma}_3 = \rho_3((\theta + 1)^2, q, \gamma + l + 1)$, $\hat{\gamma}_2 = \gamma_4(\hat{\gamma}_3, l - 1, \theta, q)$, $\hat{\delta}_1 = \delta_4(\gamma + 1, \hat{\gamma}_3, l - 1, \theta, q)$, $\hat{\gamma}_1 = \rho_3((\theta + 1)^2, q, \hat{\delta}_1 + 2\hat{\gamma}_2 + 1)$, and $\gamma_5(\gamma, l, \theta, q) = \max(\gamma_5(\hat{\gamma}_1, l - 1, \theta, q), \hat{\gamma}_2)$. Now, let M, $(S_1, T_1), \ldots, (S_l, T_l)$, and I be as stated in the lemma.

By the induction hypothesis there exist disjoint subsets $J_1, \ldots, J_{l-1}$ of $E(M) - I$ such that for each $i \in \{1, \ldots, l-1\}$ J_i is an (S_i, T_i)-linking set in $M/(I - (S_i \cup T_i))$ with capacity $\hat{\gamma}_1$ and $\sqcap_{M/I}(J_i, (J_1 \cup \cdots \cup J_{l-1}) - J_i) \leqslant 1$. By Lemma 4.9, there exist sets $S \subseteq S_l$ and $T \subseteq T_l$ such that $|S| = |T| = \kappa_{M/(I-(S \cup T))}(S, T) = \hat{\gamma}_2$; let $M_1 = M/(I - (S \cup T))$. Let $i \in \{1, \ldots, l-1\}$. By Lemma 4.3, $r^*((M/I)|J_i) = \hat{\gamma}_1$, so $r^*(M_1|(S \cup T \cup J_i)) = \hat{\gamma}_1$. Thus, by Theorem 3.5, there exists a $(\hat{\delta}_1 + 2\hat{\gamma}_2 + 1)$-bundle N_i in $M_1|(S \cup T \cup J_i)$. Note that, $\sqcap_{M_1}(E(N_i), S \cup T \cup ((J_1 \cup \cdots \cup J_{l-1}) - J_i)) \leqslant |S| + |T| + \sqcap_{M/I}(J_i, (J_1 \cup \cdots \cup J_{l-1}) - J_i) \leqslant 2\hat{\gamma}_2 + 1$. Thus, there exists a $\hat{\delta}_1$-bundle N'_i in $N_i \setminus (S \cup T)$ such that $\sqcap_{M_1}(E(N'_i), S \cup T \cup ((J_1 \cup \cdots \cup J_{l-1}) - J_i)) \leqslant r(\mathrm{co}(N'_i))$.

Let $N = M_1|(E(N_1') \cup \cdots \cup E(N_{i-1}'))$. By Lemma 6.4, there exist a flat F of M_1 and a restriction N'' of N such that

(i) $S \cup T \subseteq F$,
(ii) $\kappa_{M_1|F}(S,T) \geqslant \hat{\gamma}_3$,
(iii) for each $i \in \{1,\ldots,l-1\}$, $N''|J_i$ is a $(\gamma+1)$-bundle, and
(iv) for each $i \in \{1,\ldots,l-1\}$, $\lambda_{M_1|(F \cup E(N''))}(J_i \cap E(N'')) \leqslant 1$.

For $i \in \{1,\ldots,l-1\}$, let $J_i' = E(N'') \cap J_i$. By Theorem 4.5, there exists a graphic (S,T)-linking set J_l in $M_1|F$ of capacity $\gamma + l$. Note that, $\sqcap_{M/I}(J_l, J_1' \cup \cdots \cup J_{l-1}') \leqslant l-1$. Thus, there exists a set $J_l' \subseteq J_l$ such that $M_1|J_l'$ is a $(\gamma+1)$-bundle and $\sqcap_{M/I}(J_l', J_1' \cup \cdots \cup J_{l-1}') \leqslant 1$. Thus, for each $i \in \{1,\ldots,l\}$, J_i' is an (S_i,T_i)-linking set in $M/(I-(S_i \cup T_i))$ of capacity $r(\mathrm{co}((M/I)|J_i')) \geqslant \gamma$. □

8. Finding highly-connected skew-circuits

In this section we find a large collection $\mathcal{C}$ of skew-circuits that are pairwise highly connected. Moreover, we would like to keep high connectivity between any given pair of circuits when all of the other circuits have been contracted. Eventually these circuits will be contracted to the vertices of a clique. In particular, we prove the following lemma.

Lemma 8.1. *There exists an integer-valued function $\omega_1(\delta,\gamma,\theta,q)$ such that for any positive integers δ, γ, θ, and $q \geqslant 2$, if M is a matroid in $\mathcal{A}(\theta,q)$ with branch-width at least $\omega_1(\delta,\gamma,\theta,q)$, then M or M^* contains a minor N such that N contains a collection $\{C_1,\ldots,C_\delta\}$ of skew-circuits where $\kappa_{N/((C_1\cup\cdots\cup C_\delta)-(C_i\cup C_j))}(C_i,C_j) \geqslant \gamma$ for each distinct pair $i,j \in \{1,\ldots,\delta\}$.*

The remainder of this section is devoted to proving Lemma 8.1; we need some preliminary results. For positive integers δ and γ, we define a *(δ,γ)-frame* in a matroid M to be a pair $(N,\mathcal{P})$ such that N is a minor of M, $\mathcal{P}$ is a set of series-classes of N, $|\mathcal{P}| \geqslant \delta$, and $|P| \geqslant \gamma$ for each $P \in \mathcal{P}$. The following result was proved in [7].

Lemma 8.2. *There exists an integer-valued function $\omega_2(\delta,\gamma,q)$ such that for any positive integers δ, γ and $q \geqslant 2$, if M is a matroid in $\mathcal{U}(q) \cap \mathcal{U}^*(q)$ with branch-width at least $\omega_2(\delta,\gamma,q)$, then M or M^* contains a (δ,γ)-frame.*

Let f be an integer-valued function defined on the set of positive integers. A matroid M is called *(m,f)-connected* if whenever (A,B) is a separation of order $\ell < m$, then either $|A| \leqslant f(\ell)$ or $|B| \leqslant f(\ell)$.

The following result was proved in [5].

Lemma 8.3. *Let $g(\ell) = \frac{6^{\ell-1}-1}{5}$ for all positive integers ℓ. If M is a minor-minimal matroid with branch-width k, then M is $(k+1,g)$-connected.*

Lemma 8.4. *There exists an integer-valued function $\omega_3(\delta,\gamma,\theta,q)$ such that for any positive integers $\delta \geqslant 3$, γ, θ, and $q \geqslant 2$, if M is a matroid in $\mathcal{A}(\theta,q)$ with branch-width at least $\omega_3(\delta,\gamma,\theta,q)$, then M or M^* contains a δ-bundle N and a set $A \subseteq E(M)-E(N)$ such that $|A| = \gamma$ and, for each strand S of N, $\kappa_M(S,A) = \gamma$.*

Proof. Let $\hat{\delta}_2 = q^{\rho_3((\theta+1)^2,q,\delta+1)}$ (recall that ρ_3 is defined in Theorem 3.5), $\hat{\delta}_1 = \hat{\delta}_2\binom{g(\gamma)}{\gamma}$, and $\omega_3(\delta,\gamma,\theta,q) = \max(\omega_2(\hat{\delta}_1, g(\gamma), q), g(\gamma))$. Now let M be a matroid in $\mathcal{A}(\theta,q)$ with branch-width at least $\omega_3(\delta,\gamma,\theta,q)$, and let M_1 be a minimal minor of M with branch-width $\omega_3(\delta,\gamma,\theta,q)$. By Lemma 8.2 and by possibly replacing M and M_1 by their duals, we may assume that there exists a $(\hat{\delta}_1, g(\gamma))$-frame $(N_1,\mathcal{P}_1)$ in M_1. Let $P_1 \in \mathcal{P}_1$ and let $A_1 \subseteq P_1$ with $|A_1| = g(\gamma)$. By Lemma 8.3, $\kappa_{M_1}(A_1, P) \geqslant \gamma$ for all $P \in \mathcal{P}_1 - \{P_1\}$. Thus, for each $P \in \mathcal{P}_1 - \{P_1\}$, there exists a subset A_P of A_1 such that $|A_P| = \kappa_M(P, A_P) = \gamma$. So there exist a subset $\mathcal{P}_2$ of $\mathcal{P}_1 - \{P_1\}$ of size at least $\hat{\delta}_2$ and a set $A \subseteq A_1$ such that $A_P = A$ for all $P \in \mathcal{P}_2$.

Now let N_2 be a restriction of M that contracts to N_1. Note that no two series classes of N_1 are in the same series class of N_2. Now let N_3 be a minimal restriction of N_2 that contains each of the sets in $\mathcal{P}_2$ and such that no two of these sets is in the same series class of N_3. Let $\mathcal{P}_3$ be the set of series classes of N_3 that contain the sets in $\mathcal{P}_2$. Note that $\kappa_M(A,P) = \gamma$ for each $P \in \mathcal{P}_3$.

Consider any element e of N_3 that is not contained in any of the sets in $\mathcal{P}_3$. By the definition of N_3, deleting e must merge two of the series classes in $\mathcal{P}_3$. Thus there is a triad containing e and two of the elements in $\bigcup \mathcal{P}_3$. Hence, any circuit containing e must contain an element of $\bigcup \mathcal{P}_3$. That is, $E(N_3) - (\bigcup \mathcal{P}_3)$ is an independent set of N_3.

Since N_3 has at least $\hat{\delta}_2$ series classes, $r^*(N_3) \geqslant \rho_3((\theta+1)^2, q, \delta+1)$. So, by Theorem 3.5, N_3 contains a $(\delta+1)$-bundle N_4. Since $r(\mathrm{co}(N_4)) \leqslant 1$, the union of any two series classes of N_4 contains a circuit. Moreover, any circuit of N_4 contains a set from $\mathcal{P}_3$. Thus there is at most one series class of N_4 that does not contain a set from $\mathcal{P}_3$. Hence, by deleting a single series class we can obtain a δ-bundle N in N_4 such that each series class of N contains a set from $\mathcal{P}_3$. Thus for each series class S of N we have $\kappa_M(S,A) \geqslant \gamma$ as required. □

Proof of Lemma 8.1. Let $\tilde{\delta}_\delta = 1$ and $\tilde{\gamma}_\delta = 2\gamma$. Now, for $i = \delta-1,\dots,0$, we inductively define $\tilde{\delta}_i = \delta_1(\tilde{\delta}_{i+1}+1, \tilde{\gamma}_{i+1}, \theta, q)$ and $\tilde{\gamma}_i = \gamma_1(\tilde{\gamma}_{i+1}, \theta, q)$ (where δ_1 and γ_1 are defined in Lemma 6.1). Let $\hat{\delta}_{\tilde{\delta}_0} = 1$ and $\hat{\gamma}_{\tilde{\delta}_0} = \tilde{\gamma}_0$. Now, for $i = \tilde{\delta}_0 - 1,\dots,0$, we inductively define $\hat{\delta}_i = \delta_1(\hat{\delta}_{i+1}+1, \hat{\gamma}_{i+1}, \theta, q)$ and $\hat{\gamma}_i = \gamma_1(\hat{\gamma}_{i+1}, \theta, q)$. Now let $\omega_1(\delta,\gamma,\theta,q) = \omega_3(\hat{\delta}_0, \hat{\gamma}_0, \theta, q)$ and let $M \in \mathcal{A}(\theta,q)$ be a matroid with branch-width at least $\omega_1(\delta,\gamma,\theta,q)$.

By Lemma 8.4 and by possibly replacing M with its dual, we may assume that M has a $\hat{\delta}_0$-bundle N and a set $A \subseteq E(M) - E(N)$ such that $|A| = \hat{\gamma}_0$ and, for each strand S of N, $\kappa_M(S,A) = \hat{\gamma}_0$.

8.4.1. *There exist*:

(i) *a sequence* $(P_1,\dots,P_{\tilde{\delta}_0})$ *of strands of* N;
(ii) *a sequence* $(\mathcal{S}_1,\dots,\mathcal{S}_{\tilde{\delta}_0})$ *of sets of strands of* N *where* $|\mathcal{S}_i| \geqslant \hat{\delta}_i + 1$ *and, for each* $i \in \{1,\dots,\tilde{\delta}_0\}$, $P_1,\dots,P_i \notin \mathcal{S}_i$ *and* $P_{i+1},\dots,P_{\tilde{\delta}_0} \in \mathcal{S}_i$; *and*
(iii) *a sequence* $(A_1,\dots,A_{\tilde{\delta}_0})$ *of subsets of* A *where* $A_{\tilde{\delta}_0} \subseteq \cdots \subseteq A_2 \subseteq A_1$ *and, for* $i \in \{1,\dots,\tilde{\delta}_0\}$, $|A_i| = \hat{\gamma}_i$,

such that, for each $i \in \{1,\dots,\tilde{\delta}_0\}$, $\kappa_{M/\mathcal{S}_i}(P_i, A_i) = |A_i|$.

Subproof. Let $\mathcal{S}_0$ be the set of strands of N and let $A_0 = A$; we construct the sequences inductively. Choose any $P_i \in \mathcal{S}_{i-1}$. Since $\kappa_M(P_i, A) = |A|$ and $A_{i-1} \subseteq A$, $\kappa_M(P_i, A_{i-1}) \geqslant \hat{\gamma}_{i-1}$. Therefore, by Lemma 6.1, there exists a subset $\mathcal{S}_i$ of $\mathcal{S}_{i-1} - \{P_i\}$ such that $|\mathcal{S}_i| = \hat{\delta}_i + 1$ and

$\kappa_{M/\mathcal{S}_i}(P_i, A_{i-1}) \geqslant \hat{\gamma}_i$. Then, by Lemma 4.7, there exists $A_i \subseteq A_{i-1}$ such that $\kappa_{M/\mathcal{S}_i}(P_i, A_i) = |A_i| = \hat{\gamma}_i$. □

Let $P \in \mathcal{S}_{\tilde{\delta}_0}$ and let $M_1 = M/P$. Now let $(C_1, \ldots, C_{\tilde{\delta}_0})$ denote $(P_{\tilde{\delta}_0}, \ldots, P_1)$, let $A'_0 = A_{\tilde{\delta}_0}$, and let $\mathcal{S}'_0 = \{1, \ldots, \tilde{\delta}_0\}$. Since N is a bundle, $C_1, \ldots, C_{\tilde{\delta}_0}$ are skew-circuits in M_1. Moreover:

8.4.2. $\kappa_{M_1}(C_1, A'_0) = |A'_0| = \hat{\delta}_0$ *and, for each* $i \in \{2, \ldots, \tilde{\delta}_0\}$, $\kappa_{M_1/(C_1\cup\cdots\cup C_{i-1})}(C_i, A'_0) = |A'_0|$.

8.4.3. *There exist*:

(i) *a subsequence* $(\sigma_1, \ldots, \sigma_\delta)$ *of* $(1, \ldots, \tilde{\delta}_0)$;
(ii) *a sequence* $(\mathcal{I}_1, \ldots, \mathcal{I}_\delta)$ *of sets where* $|\mathcal{I}_i| \geqslant \tilde{\delta}_i + 1$ *and, for each* $i \in \{1, \ldots, \delta\}$, $\sigma_{i+1}, \ldots, \sigma_\delta \in \mathcal{I}_i$ *and* $\mathcal{I}_i \subseteq \mathcal{I}_{i-1} \cap \{\sigma_i + 1, \ldots, \tilde{\delta}_0\}$; *and*
(iii) *a sequence* $(A''_1, \ldots, A''_\delta)$ *of sets where* $A'_\delta \subseteq \cdots \subseteq A'_1 \subseteq A'_0$ *and, for* $i \in \{1, \ldots, \delta\}$, $|A'_i| = \tilde{\gamma}_i$,

such that, for each $i \in \{1, \ldots, \delta\}$, $\kappa_{M_1/(\{C_{\sigma_1}, \ldots, C_{\sigma_{i-1}}\} \cup (\bigcup(C_j\colon j \in \mathcal{I}_i)))}(C_{\sigma_i}, A'_i) = |A'_i|$.

Subproof. The proof is very similar to that of 8.4.1; we construct the sequences inductively. Suppose that we have constructed the first $i-1$ terms in the sequences. Let σ_i be the smallest element in $\mathcal{I}_{i-1}$. If $i = 1$, let $M'_1 = M_1$ and, if $i \geqslant 2$, let $M'_i = M_1/(C_{\sigma_1} \cup \cdots \cup C_{\sigma_{i-1}})$. By 8.4.2, $\kappa_{M'_i}(C_{\sigma_i}, A'_0) = |A'_0|$. Since $A'_{i-1} \subseteq A'_0$, $\kappa_{M'_i}(C_{\sigma_i}, A'_{i-1}) = \hat{\gamma}_{i-1}$. Therefore, by Lemma 6.1, there exists a subset $\mathcal{I}_i$ of $\mathcal{I}_{i-1} - \{\sigma_i\}$ such that $|\mathcal{I}_i| = \tilde{\delta}_i + 1$ and $\kappa_{M'_i/(C_j\colon j \in \mathcal{I}_i)}(C_{\sigma_i}, A_{i-1}) \geqslant \tilde{\gamma}_i$. Then, by Lemma 4.6, there exists $A'_i \subseteq A'_{i-1}$ such that $\kappa_{M'_i/(C_j\colon j \in \mathcal{I}_i)}(C_{\sigma_i}, A'_i) = |A'_i| = \tilde{\gamma}_i$. □

Let $A' = A'_\delta$ and, for each $i \in \{1, \ldots, \delta\}$, let $C'_i = C_{\sigma_i}$. Thus, for each $i \in \{1, \ldots, \delta\}$, $\kappa_{M/((C'_1\cup\cdots\cup C'_\delta) - C'_i)}(C'_i, A') = |A'| = 2\gamma$. Therefore, by Lemma 4.10, for each distinct pair $i, j \in \{1, \ldots, \delta\}$, $\kappa_{M/((C'_1\cup\cdots\cup C'_\delta) - (C'_i \cup C'_j))}(C'_i, C'_j) \geqslant \gamma$. □

9. More disentangling

Using Lemma 8.1 we obtain many pairwise highly connected skew-circuits. We can then use Lemma 7.1 to disentangle the connectivities between these circuits. However, while we have disentangled the connecting sets from each other, the connecting sets for one pair of circuits may remain tangled with some of the other circuits; this is overcome by the following two lemmas. (We only use the following lemma with $\gamma = 2$.)

Lemma 9.1. *There exists an integer-valued function* $\omega_4(\delta, \gamma, \theta, q)$ *such that for any positive integers* δ, γ, θ, *and* q, *if* M *is a matroid in* $\mathcal{A}(\theta, q)$ *with branch-width at least* $\omega_4(\delta, \gamma, \theta, q)$, *then* M *or* M^* *contains a minor* N *such that* N *contains a collection* $\{C_1, \ldots, C_\delta\}$ *of skew-circuits and a collection* $\{J_{ij}\colon 1 \leqslant i < j \leqslant \delta\}$ *of disjoint subsets of* $E(N) - (C_1 \cup \cdots \cup C_\delta)$ *such that* $C_1 \cup \cdots \cup C_\delta$ *spans* N *and, for each* $i, j \in \{1, \ldots, \delta\}$ *with* $i < j$,

(i) $\kappa_{N|(C_i \cup C_j \cup J_{ij})}(C_i, C_j) \geqslant \gamma$, *and*
(ii) $r_N(C_i \cup C_j \cup J_{ij}) \leqslant r_N(C_i \cup C_j) + 1$.

Proof. (Recall that functions ρ_3, γ_5, and ω_1 are defined in Theorem 3.5, Lemma 7.1, and Lemma 8.1, respectively.) Let $\hat{\gamma}_2 = 3\alpha(\theta, q) + \rho_3((\theta+1)^2, q, \gamma + 3\alpha(\theta, q) + 2)$. For a function μ and positive integer n we let $\mu^{[n]}(n) = 0$ and, for each nonnegative integer $i < n$, we recursively define $\mu^{[i]}(n) = 1 + \mu(\mu^{[i+1]}(n))$. For any positive integer n we let $f(n) = (\max(n, \alpha(\theta, q)\hat{\gamma}_2))2^{\alpha(\theta,q)\hat{\gamma}_2}$. Let $h(n) = f^{[0]}(n)$, $g(n) = h^{[0]}(n)$, and $\hat{\delta}_1 = g(g(g(\delta)))$. Now, let $\hat{\gamma}_1 = \gamma_5(\hat{\gamma}_2, \binom{\hat{\delta}_1}{2}, \theta, q)$ and $\omega_4(\delta, \gamma, \theta, q) = \omega_1(\hat{\delta}_1, \hat{\gamma}_1, \theta, q)$.

By Lemma 8.1 and duality, we may assume that there is a minor N of M that contains a collection $\{C_1, \ldots, C_{\hat{\delta}_1}\}$ of skew-circuits where $\kappa_{N/((C_1 \cup \cdots \cup C_{\hat{\delta}_1}) - (C_i \cup C_j))}(C_i, C_j) \geqslant \hat{\gamma}_1$ for each distinct pair $i, j \in \{1, \ldots, \hat{\delta}_1\}$. Now, let $X = C_1 \cup \cdots \cup C_{\hat{\delta}_1}$ and let I be a maximum independent subset of X in N. Note that $\kappa_{N\setminus(X-I)/(I-(C_i\cup C_j))}(C_i \cap I, C_j \cap I) \geqslant \hat{\gamma}_1$ for each pair of distinct elements $i, j \in \{1, \ldots, \hat{\delta}_1\}$. Now, applying Lemma 7.1 to $N \setminus (X - I)$, there exist disjoint subsets $(J_{ij}\colon 1 \leqslant i \leqslant \hat{\delta}_1)$ of $E(N) - X$ such that J_{ij} is a (C_i, C_j)-linking set in $N/(X - (C_i \cup C_j))$ of capacity $\hat{\gamma}_2$ and $\lambda_{(N/X)|(J_{ij}\colon 1 \leqslant i < j \leqslant \hat{\delta}_1)}(J_{i'j'}) \leqslant 1$, for each $i', j' \in \{1, \ldots, \hat{\delta}_1\}$ where $i' < j'$. Let $N_{ij} = N|(X \cup J_{ij})$. For each $i \in \{1, \ldots, \hat{\delta}_1\}$, let e_i be the element in $C_i - I$.

The proof of the following claim is essentially the same as that of Lemma 6.2.

9.1.1. *Let $i, j \in \{1, \ldots, \hat{\delta}_1\}$ with $i < j$ and let n be a positive integer. If $\mathcal{C} \subseteq \{C_1, \ldots, C_{\hat{\delta}_1}\} - \{C_i, C_j\}$ and $|\mathcal{C}| \geqslant f(n)$, then there exists $\mathcal{C}_1 \subseteq \mathcal{C}$ such that $|\mathcal{C}_1| \geqslant n$ and, for any $\mathcal{C}' \subseteq \mathcal{C}_1$, $\lambda_{N_{ij}}(\mathcal{C}') \leqslant \alpha(\theta, q)$.*

Subproof. Assume that $\mathcal{C}$ satisfies the hypotheses above. Let $m = \max(n, \alpha(\theta, q)\hat{\gamma}_2)$. Let $X_1 = \cup(C\colon C \in \mathcal{C})$, and let $X_2 = X - (C_i \cup C_j \cup X_1)$. Note that J_{ij} is a $(C_i - \{e_i\}, C_j - \{e_j\})$-linking set of capacity $\hat{\gamma}_2$ in $N_{ij}/(X - (C_i \cup C_j)) \setminus \{e_i, e_j\}$. Therefore, by Lemma 4.9, there exists $Z_i \subseteq C_i - \{e_i\}$ and $Z_j \subseteq C_j - \{e_j\}$ such that $|Z_i| = |Z_j| = \hat{\gamma}_2$ and J_{ij} is a (Z_i, Z_j)-linking set of capacity $\hat{\gamma}_2$ in $(N_{ij}/(X - (C_i \cup C_j)) \setminus \{e_i, e_j\})/((C_i \cup C_j) - (Z_i \cup Z_j \cup \{e_i, e_j\}))$. Let $N' = N_{ij} \setminus \{e_i, e_j\}/((C_i \cup C_j) - (Z_i \cup Z_j \cup \{e_i, e_j\}))/X_2$, let $J' = J_{ij} \cup (X_1 \cap I)$, and let $L' = (N'/(Z_i \cup Z_j))|J'$.

Now, for each $C \in \mathcal{C}$, let $\mathcal{S}_C$ denote the set of series classes of L' that contain an element of C. Now, $r^*(L') = \hat{\gamma}_2$ so, by Corollary 3.3, L' has at most $\alpha(\theta, q)\hat{\gamma}_2$ series classes. Thus, there exists $\mathcal{C}_1 \subseteq \mathcal{C}$ and a set $\mathcal{S}$ of series classes of L' such that $|\mathcal{C}_1| = m$ and $\mathcal{S}_C = \mathcal{S}$ for all $C \in \mathcal{C}_1$. Let $N'' = N'/(\mathcal{C} - \mathcal{C}_1)$ and let $J'' = J' \cap E(N')$.

It is straightforward to check that $Z_i \cup J''$ and $Z_j \cup J''$ are bases of N''. Therefore, $r^*(N'') = |Z_j| + |\mathcal{C}_1| = \hat{\gamma}_2 + m$. So, by Corollary 3.3,

$$\begin{aligned} \bigl|E\bigl(\mathrm{co}(N'')\bigr)\bigr| &\leqslant \alpha(\theta, q) r^*\bigl(\mathrm{co}(N'')\bigr) \\ &= \alpha(\theta, q) r^*(N'') \\ &= \alpha(\theta, q)(\hat{\gamma}_2 + m) \\ &\leqslant m + \alpha(\theta, q)m \\ &= \bigl(\alpha(\theta, q) + 1\bigr)m. \end{aligned}$$

However, $|E(\mathrm{co}(N''))| \geqslant \sum(|\mathcal{S}_C| + 1\colon C \in \mathcal{C}_1) = (|\mathcal{S}| + 1)m$. Therefore, $|\mathcal{S}| \leqslant \alpha(\theta, q)$.

Let $\mathcal{C}' \subseteq \mathcal{C}_1$ and let $X' = \bigcup(C\colon C \in \mathcal{C}')$. Since the circuits in $\mathcal{C}_1$ are skew, $\lambda_{N_{ij}}(X') = \lambda_{N_{ij}/(X-X')}(X') + \sqcap_{N_{ij}}(X', X - X') = \lambda_{N_{ij}/(X-X')}(X')$. Moreover, $X' \cap I$ spans X', so $\lambda_{N_{ij}/(X-X')}(X') = \lambda_{N_{ij}/(X-X')\setminus(X'-I)}(X' \cap I)$. However, $N_{ij}/(X - X') \setminus (X' - I)$ is a minor of L'

and X' only intersects $|\mathcal{S}| \leqslant \alpha(\theta, q)$ series classes of L', so $\lambda_{N_{ij}/(X-X')\backslash(X'-I)}(X' \cap I) \leqslant \alpha(\theta, q)$. Thus, $\lambda_{N_{ij}}(X') \leqslant \alpha(\theta, q)$, as required. □

9.1.2. *Let $a \in \{1, \ldots, \hat{\delta}_1\}$ and let $(\sigma_1, \ldots, \sigma_m)$ be a sequence of distinct elements in $\{1, \ldots, \hat{\delta}_1\} - \{a\}$. If $m \geqslant h(n)$, then there exists a subsequence $(\hat{\sigma}_1, \ldots, \hat{\sigma}_n)$ of $(\sigma_1, \ldots, \sigma_m)$ such that for any $i \in \{1, \ldots, n\}$ and any subset $\mathcal{C}$ of $\{C_{\hat{\sigma}_{i+1}}, \ldots, C_{\hat{\sigma}_n}\}$ we have $\lambda_{N_{a,\hat{\sigma}_i}}(\mathcal{C}) \leqslant \alpha(\theta, q)$.*

Subproof. We construct the sequence inductively. Suppose that $m \geqslant h(n) = f^{[0]}(n)$, and, for some $l \geqslant 0$, we have a subsequence $(\hat{\sigma}_1, \ldots, \hat{\sigma}_l)$ of $(\sigma_1, \ldots, \sigma_m)$ where $\hat{\sigma}_l = \sigma_{l'}$ and we have a subsequence $(\sigma'_1, \ldots, \sigma'_{m'})$ of $(\sigma_{l'+1}, \ldots, \sigma_m)$ such that $m' \geqslant f^{[l]}(n)$ and, for any $i \in \{1, \ldots, l\}$ and any subset $\mathcal{C}$ of $\{C_{\hat{\sigma}_{i+1}}, \ldots, C_{\hat{\sigma}_l}\} \cup \{C_{\sigma'_1}, \ldots, C_{\sigma'_{m'}}\}$, we have $\lambda_{N_{a,\hat{\sigma}_i}}(\mathcal{C}) \leqslant \alpha(\theta, q)$.

Let $\hat{\sigma}_{l+1} = \sigma'_1$. By 9.1.1, there exists a subsequence $(\sigma''_1, \ldots, \sigma''_{m''})$ of $(\sigma'_2, \ldots, \sigma'_{m'})$ such that $m'' \geqslant f^{[l+1]}(n)$ and, for any subset $\mathcal{C}$ of $\{C_{\sigma''_1}, \ldots, C_{\sigma''_{m''}}\}$, we have $\lambda_{N_{a,\hat{\sigma}_{l+1}}}(\mathcal{C}) \leqslant \alpha(\theta, q)$. □

9.1.3. *Let $(\sigma_1, \ldots, \sigma_m)$ be a sequence of distinct elements in $\{1, \ldots, \hat{\delta}_1\}$. If $m \geqslant g(n)$, then there exists a subsequence $(\hat{\sigma}_1, \ldots, \hat{\sigma}_n)$ of $(\sigma_1, \ldots, \sigma_m)$ such that for any $i, j \in \{1, \ldots, n\}$ where $i < j$, and any subset $\mathcal{C}$ of $\{C_{\hat{\sigma}_{j+1}}, \ldots, C_{\hat{\sigma}_n}\}$ we have $\lambda_{N_{\hat{\sigma}_i,\hat{\sigma}_j}}(\mathcal{C}) \leqslant \alpha(\theta, q)$.*

Subproof. We construct the sequence inductively. Suppose that, for some $l \geqslant 0$, we have a subsequence $(\hat{\sigma}_1, \ldots, \hat{\sigma}_l)$ of $(\sigma_1, \ldots, \sigma_m)$ where $\hat{\sigma}_l = \sigma_{l'}$ and we have a subsequence $(\hat{\sigma}_{l+1}, \ldots, \hat{\sigma}_{l+m'})$ of $(\sigma_{l'+1}, \ldots, \sigma_m)$ such that $m' \geqslant h^{[l]}(n)$ and, for any $i, j \in \{1, \ldots, l + m'\}$ with $i \leqslant l$ and $j > i$ and any subset $\mathcal{C}$ of $\{C_{\hat{\sigma}_{j+1}}, \ldots, C_{\hat{\sigma}_{l+m'}}\}$, we have $\lambda_{N_{\hat{\sigma}_i,\hat{\sigma}_j}}(\mathcal{C}) \leqslant \alpha(\theta, q)$.

By 9.1.2, there exists a subsequence $(\sigma'_1, \ldots, \sigma'_{m''})$ of $(\hat{\sigma}_{l+2}, \ldots, \hat{\sigma}_{l+m'})$ such that $m'' \geqslant h^{[l+1]}(n)$ and, for any $j \in \{1, \ldots, m''\}$ and any subset $\mathcal{C}$ of $\{C_{\sigma'_{j+1}}, \ldots, C_{\sigma'_{m''}}\}$, we have $\lambda_{N_{\hat{\sigma}_{l+1},\sigma'_j}}(\mathcal{C}) \leqslant \alpha(\theta, q)$. Now, for $i \in \{1, \ldots, m''\}$ we redefine $\hat{\sigma}_{l+1+i}$ as σ'_i. □

9.1.4. *Let $(\sigma_1, \ldots, \sigma_m)$ be a sequence of distinct elements in $\{1, \ldots, \hat{\delta}_1\}$. If $m \geqslant g(n)$, then there exists a subsequence $(\hat{\sigma}_1, \ldots, \hat{\sigma}_n)$ of $(\sigma_1, \ldots, \sigma_m)$ such that for any $i, j \in \{1, \ldots, n\}$ where $i < j$, and any subset $\mathcal{C}$ of $\{C_{\hat{\sigma}_{i+1}}, \ldots, C_{\hat{\sigma}_{j-1}}\}$ we have $\lambda_{N_{\hat{\sigma}_i,\hat{\sigma}_j}}(\mathcal{C}) \leqslant \alpha(\theta, q)$.*

Subproof. We construct the sequence inductively. Suppose that, for some $l \geqslant 0$, we have a subsequence $(\hat{\sigma}_1, \ldots, \hat{\sigma}_l)$ of $(\sigma_1, \ldots, \sigma_m)$ where $\hat{\sigma}_l = \sigma_{l'}$ and we have a subsequence $(\hat{\sigma}_{l+1}, \ldots, \hat{\sigma}_{l+m'})$ of $(\sigma_{l'+1}, \ldots, \sigma_m)$ such that $m' \geqslant h^{[l]}(n)$ and, for any $i, j \in \{1, \ldots, l + m'\}$ with $i \leqslant l$ and $j > i$ and any subset $\mathcal{C}$ of $\{C_{\hat{\sigma}_{i+1}}, \ldots, C_{\hat{\sigma}_{j-1}}\}$, we have $\lambda_{N_{\hat{\sigma}_i,\hat{\sigma}_j}}(\mathcal{C}) \leqslant \alpha(\theta, q)$.

By 9.1.2, there exists a subsequence $(\sigma'_1, \ldots, \sigma'_{m''})$ of $(\hat{\sigma}_{l+m'}, \ldots, \hat{\sigma}_{l+2})$ such that $m'' \geqslant h^{[l+1]}(n)$ and, for any $j \in \{1, \ldots, m''\}$ and any subset $\mathcal{C}$ of $\{C_{\sigma'_{j+1}}, \ldots, C_{\sigma'_{m''}}\}$, we have $\lambda_{N_{\hat{\sigma}_{l+1},\sigma'_j}}(\mathcal{C}) \leqslant \alpha(\theta, q)$. Now, for $i \in \{1, \ldots, m''\}$ we redefine $\hat{\sigma}_{l+1+(m''+1-i)}$ as σ'_i. □

9.1.5. *There exists a subset $\{\sigma_1, \ldots, \sigma_\delta\}$ of $\{1, \ldots, \hat{\delta}_1\}$ such that, for each $i, j \in \{1, \ldots, \delta\}$ with $i < j$, $\lambda_{N_{\sigma_i,\sigma_j}}((C_{\sigma_1}, \ldots, C_{\sigma_\delta}) - (C_{\sigma_i} \cup C_{\sigma_j})) \leqslant 3\alpha(\theta, q)$.*

Subproof. Apply 9.1.3, reverse the order of the resulting subsequence, apply 9.1.3 again, and then apply 9.1.4. We obtain a subsequence $(\sigma_1, \ldots, \sigma_\delta)$ of $(1, \ldots, \hat{\delta}_1)$ such that, for each $i, j \in$

$\{1,\dots,\delta\}$ with $i<j$, we have $\lambda_{N_{\sigma_i,\sigma_j}}(C_{\sigma_1},\dots,C_{\sigma_{i-1}}) \leqslant \alpha(\theta,q)$, $\lambda_{N_{\sigma_i,\sigma_j}}(C_{\sigma_{i+1}},\dots,C_{\sigma_{j-1}}) \leqslant \alpha(\theta,q)$, and $\lambda_{N_{\sigma_i,\sigma_j}}(C_{\sigma_{j+1}},\dots,C_{\sigma_\delta}) \leqslant \alpha(\theta,q)$. □

Let $C'_i = C_{\sigma_i}$ for each $i \in \{1,\dots,\delta\}$ and $J'_{ij} = J_{\sigma_i\sigma_j}$ for each $i,j \in \{1,\dots,\delta\}$ with $i<j$. Let $X' = C'_1 \cup \cdots \cup C'_\delta$ and let $N_2 = N/((C_1 \cup \cdots \cup C_{\hat{\delta}_1}) - X')$. Note that, for each $i,j \in \{1,\dots,\delta\}$ with $i<j$, we have $\kappa_{N_2|(X'\cup J'_{ij})}(C'_i,C'_j) \geqslant \hat{\gamma}_2$ and $\lambda_{N_2|(X'\cup J'_{ij})}(X' - (C'_i \cup C'_j)) \leqslant 3\alpha(\theta,q)$; so $\kappa_{N_2|(C'_i\cup C'_j\cup J'_{ij})}(C'_i,C'_j) \geqslant \rho_3((\theta+1)^2, q, \gamma + 3\alpha(\theta,q)+2)$. Therefore, there exists a graphic (C'_i,C'_j)-linking set $J''_{ij} \subseteq J'_{ij}$ of capacity $\gamma + 3\alpha(\theta,q)+1$. Now, $(N_2/(C'_i \cup C'_j))|J''_{ij}$ is a bundle and $\sqcap_{N_2/(C'_i\cup C'_j)}(J''_{ij}, X' - (C'_i \cup C'_j)) \leqslant 3\alpha(\theta,q)$. Therefore, there exists a graphic (C'_i,C'_j)-linking set $\hat{J}_{ij} \subseteq J''_{ij}$ of capacity $\gamma+1$ such that $\sqcap_{N_2/(C'_i\cup C'_j)}(\hat{J}_{ij}, X' - (C'_i \cup C'_j)) \leqslant 1$. Let $N_3 = N_2|(X' \cup (\hat{J}_{ij}\colon 1 \leqslant i < j \leqslant \delta))$. By possibly contracting some elements of the sets $\hat{J}_{ij}$, we may assume that X' spans N_3. These contractions may reduce some of the connectivities, but since $\lambda_{(N/X)|(J_{i'j'}\colon 1\leqslant i'<j'\leqslant\hat{\delta}_1)}(J_{ij}) \leqslant 1$ for each $i<j$, the connectivities reduce by at most one. That is, $\kappa_{N_3|(C'_i\cup C'_j\cup\hat{J}_{ij})}(C'_i,C'_j) \geqslant \gamma$. Now, $r_{N_3}(C'_i \cup C'_j \cup \hat{J}_{ij}) - r_{N_3}(C'_i \cup C'_j) = \sqcap_{N_3/(C'_i\cup C'_j)}(\hat{J}_{ij}, X' - (C'_i \cup C'_j)) \leqslant 1$, as required. □

Lemma 9.2. *There exists an integer-valued function $\omega_5(\delta,\theta,q)$ such that for any positive integers δ, θ, and q, if M is a matroid in $\mathcal{A}(\theta,q)$ with branch-width at least $\omega_5(\delta,\theta,q)$, then M or M^* contains a minor N such that N contains a collection $\{C_1,\dots,C_\delta\}$ of skew-circuits and a collection $\{e_{ij}\colon 1 \leqslant i < j \leqslant \delta\}$ of elements such that $e_{ij} \in \mathrm{cl}_N(C_i \cup C_j) - (\mathrm{cl}_M(C_i) \cup \mathrm{cl}_M(C_j))$ for each $i,j \in \{1,\dots,\delta\}$ with $i<j$.*

Proof. For integers a and n with $1 \leqslant a < n$ we recursively define $f(a,n)$ as follows:

$$f(n-1,n) = n \quad \text{and} \quad f(a,n) = 2f(a+1,n) - a - 2 \quad \text{whenever } a < n-1.$$

Now, let $g(n) = f(0,n)$, $\hat{\delta}_1 = g(g(g(\delta)))$, $\omega_5(\delta,\theta,q) = \omega_4(\hat{\delta}_1, 2, \theta, q)$, and let M be a matroid in $\mathcal{A}(\theta,q)$ with branch-width at least $\omega_5(\delta,\theta,q)$.

By Lemma 9.1 and duality, we may assume the following.

9.2.1. *There exists a minor N of M such that N contains a collection $\{C_1,\dots,C_{\hat{\delta}_1}\}$ of skew-circuits and a collection $\{J_{ij}\colon 1 \leqslant i < j \leqslant \hat{\delta}_1\}$ of subsets of $E(N)$ such that $C_1 \cup \cdots \cup C_{\hat{\delta}_1}$ spans N and, for each $i,j \in \{1,\dots,\hat{\delta}_1\}$ with $i<j$,*

(i) $\kappa_{N|(C_i\cup C_j\cup J_{ij})}(C_i,C_j) \geqslant 2$, *and*
(ii) $r_N(C_i \cup C_j \cup J_{ij}) \leqslant r_N(C_i \cup C_j) + 1$.

Let $\hat{\delta}_2 = g(g(\delta))$.

9.2.2. *There exist a minor N_1 of N and a subsequence $(\sigma_1,\dots,\sigma_{\hat{\delta}_2})$ of $(1,\dots,\hat{\delta}_1)$ such that $(C_{\sigma_1},\dots,C_{\sigma_{\hat{\delta}_2}})$ are skew-circuits of N_1, $C_{\sigma_1} \cup \cdots \cup C_{\sigma_{\hat{\delta}_2}}$ spans N_1, and, for each $i,j \in \{1,\dots,\hat{\delta}_2\}$ with $i<j$,*

(i) $\kappa_{N_1|(C_{\sigma_i}\cup C_{\sigma_j}\cup J_{\sigma_i\sigma_j})}(C_{\sigma_i},C_{\sigma_j}) \geqslant 1 + r_{N_1}(C_{\sigma_i} \cup C_{\sigma_j} \cup J_{\sigma_i\sigma_j}) - r_{N_1}(C_{\sigma_i} \cup C_{\sigma_j})$,
(ii) $r_{N_1}(C_{\sigma_i} \cup C_{\sigma_j} \cup J_{\sigma_i\sigma_j}) \leqslant r_{N_1}(C_{\sigma_i} \cup C_{\sigma_j}) + 1$, *and*
(iii) $\lambda_{N_1|(C_{\sigma_1}\cup\cdots\cup C_{\sigma_{\hat{\delta}_2}}\cup J_{\sigma_i\sigma_j})}(C_{\sigma_k}) = 0$ *for all* $k \in \{j+1,\dots,\hat{\delta}_2\}$.

Subproof. We build the sequence inductively. Each step in the construction is associated with a pair (a,b) of positive integers; in the base case $a=b=1$, in the other cases we have $b>a$, and we index through the cases lexicographically. Let (a,b) be such a pair of integers, and let $m(a,b)=f(a-1,\hat{\delta}_2)-(b-a)$. Suppose that we have a minor N_1 of N and a subsequence $(\hat{\sigma}_1,\ldots,\hat{\sigma}_m)$ of $(1,\ldots,\hat{\delta}_1)$ with $m=m(a,b)$ such that $(C_{\hat{\sigma}_1},\ldots,C_{\hat{\sigma}_m})$ are skew-circuits of N_1, $C_{\hat{\sigma}_1}\cup\cdots\cup C_{\hat{\sigma}_m}$ spans N_1 and, for each $i,j\in\{1,\ldots,m\}$ with $i<j$,

(a) $\kappa_{N_1|(C_{\hat{\sigma}_i}\cup C_{\hat{\sigma}_j}\cup J_{\hat{\sigma}_i\hat{\sigma}_j})}(C_{\hat{\sigma}_i},C_{\hat{\sigma}_j})\geqslant 1+r_{N_1}(C_{\hat{\sigma}_i}\cup C_{\hat{\sigma}_j}\cup J_{\hat{\sigma}_i\hat{\sigma}_j})-r_{N_1}(C_{\hat{\sigma}_i}\cup C_{\hat{\sigma}_j})$,
(b) $r_{N_1}(C_{\hat{\sigma}_i}\cup C_{\hat{\sigma}_j}\cup J_{\hat{\sigma}_i\hat{\sigma}_j})\leqslant r_{N_1}(C_{\hat{\sigma}_i}\cup C_{\hat{\sigma}_j})+1$, and
(c) if either $i<a$ or both $i=a$ and $j\leqslant b$, then $\lambda_{N_1|(C_{\hat{\sigma}_1}\cup\cdots\cup C_{\hat{\sigma}_m}\cup J_{\hat{\sigma}_i\hat{\sigma}_j})}(C_{\hat{\sigma}_k})=0$ for all $k\in\{j+1,\ldots,m\}$.

(In the base case, when $a=b=1$, the condition (c) is vacuous.) If $a=\hat{\delta}_2-2$ and $b=\hat{\delta}_2-1$, then the claim is proved; suppose otherwise. If $b=m-1$, then let $a'=a+1$ and $b'=a'+1$. If $b<m-1$, then let $a'=a$ and $b'=b+1$. It is straightforward to check that $m(a',b')=m(a,b)-1$. If $\lambda_{N_1|(C_{\hat{\sigma}_1}\cup\cdots\cup C_{\hat{\sigma}_m}\cup J_{\hat{\sigma}_{a'}\hat{\sigma}_{b'}})}(C_{\hat{\sigma}_k})=0$ for all $k\in\{b'+1,\ldots,m\}$, then we are done with this step of the induction; we replace a by a', b by b', and N_1 by $N_1/C_{\hat{\sigma}_m}$. Thus, we may assume that there exists some $k\in\{b'+1,\ldots,m\}$ such that $\lambda_{N_1|(C_{\hat{\sigma}_1}\cup\cdots\cup C_{\hat{\sigma}_m}\cup J_{\hat{\sigma}_{a'}\hat{\sigma}_{b'}})}(C_{\hat{\sigma}_k})>0$; we choose the largest such k. Let C'_i denote $C_{\hat{\sigma}_i}$, let J'_{ij} denote $J_{\hat{\sigma}_i,\hat{\sigma}_j}\cap E(N_1)$, and let $Z=C'_1\cup\cdots\cup C'_m$. Since $(C'_1,\ldots,C'_m)$ are skew and $r_{N_1}(C'_{a'}\cup C'_{b'}\cup J'_{a'b'})\leqslant r_{N_1}(C'_{a'}\cup C'_{b'})+1$, we have $\lambda_{N_1|(C'_{a'}\cup C'_{b'}\cup J'_{a'b'})}(C'_k)=1$ and $r_{N_1}(C'_{a'}\cup C'_{b'}\cup J'_{a'b'})=r_{N_1}(C'_{a'}\cup C'_{b'})+1$.

Choose an element $e\in J'_{a'b'}$ that is not spanned by $C'_{a'}\cup C'_{b'}$. Now, let $I\subseteq C'_k\cup\{e\}$ be a maximal independent set of $N_1/(Z-C'_k)$ containing e. Let $N'=N_1/I$. Note that, I is skew to $Z-C'_k$; thus, the circuits $\{C'_1,\ldots,C'_m\}-\{C'_k\}$ are skew in N' and span N'. Now, since $J'_{a'b'}$ is spanned by $C'_{a'}\cup C'_{b'}\cup\{e\}$ in N_1, $J'_{a'b'}$ is spanned by $C'_{a'}\cup C'_{b'}$ in N'. So, for each $k'\in\{1,\ldots,m\}-\{a',b',k\}$, $\lambda_{N'|((Z-C'_k)\cup J'_{a'b'})}(C'_{k'})=0$. Moreover, it is easy to see that, for each $i,j\in\{1,\ldots,m\}-\{k\}$ with $i<j$,

$$\kappa_{N'|(C'_i\cup C'_j\cup J'_{ij})}\big(C'_i,C'_j\big)\geqslant 1+r_{N'}\big(C'_i\cup C'_j\cup J'_{ij}\big)-r_{N'}\big(C'_i\cup C'_j\big),\quad\text{and}$$
$$r_{N'}\big(C'_i\cup C'_j\cup J'_{ij}\big)\leqslant r_{N'}\big(C'_i\cup C'_j\big)+1.$$

Finally, consider elements $i,j,k'\in\{1,\ldots,m\}-\{k\}$ such that $k'\in\{j+1,\ldots,m\}$ and either $i<a$ or $i=a$ and $j\leqslant b$. By (c), $\lambda_{N_1|(Z\cup J'_{ij})}(C'_{k'})=0$. If J'_{ij} is in the closure of $Z-C'_k$ in N_1, then, since I is skew to $Z-C'_k$, $\lambda_{N'|((Z-C'_k)\cup J'_{ij})}(C'_{k'})=0$. So, suppose that J'_{ij} is not in the closure of $Z-C'_k$ in N_1, and, hence, that $\lambda_{N_1|(Z\cup J'_{ij})}(C'_k)=1$. Then, by (c), $k\leqslant j$; so $k<k'$. Hence, by our choice of k, $\lambda_{N_1|(Z\cup J'_{a'b'})}(C'_{k'})=0$. Therefore, $J'_{a'b'}$ is spanned by $Z-C'_{k'}$ and, hence, I is skew to $C'_{k'}$ in N_1. Then, $\lambda_{N'|((Z-C'_k)\cup J'_{ij})}(C'_{k'})=\lambda_{N_1|(Z\cup J'_{ij})}(C'_{k'})=0$; as required. Now, replace N_1 by N' and $(\hat{\sigma}_1,\ldots,\hat{\sigma}_m)$ by $(\hat{\sigma}_1,\ldots,\hat{\sigma}_{k-1},\hat{\sigma}_{k+1},\ldots,\hat{\sigma}_m)$; we have shown that (a), (b), and (c) are satisfied by this choice. □

In the proof of the above claim we considered pairs $i,j\in\{1,\ldots,m\}$ with $i<j$ and "cleaned" the stretch after j. Repeating the same proof we can also clean the stretch preceding i and again repeating the proof we can clean the stretch between i and j. Thus, we obtain the following.

9.2.3. *There exist a minor N_2 of N_1 and a subsequence $(\tau_1, \ldots, \tau_\delta)$ of $(\sigma_1, \ldots, \sigma_{\hat{\delta}_2})$ such that $(C_{\tau_1}, \ldots, C_{\tau_\delta})$ are skew-circuits of N_2, $C_{\tau_1} \cup \cdots \cup C_{\tau_\delta}$ spans N_2, and, for each $i, j \in \{1, \ldots, \delta\}$ with $i < j$,*

(i) $\kappa_{N_2|(C_{\tau_i} \cup C_{\tau_j} \cup J_{\tau_i \tau_j})}(C_{\tau_i}, C_{\tau_j}) \geqslant 1 + r_{N_2}(C_{\tau_i} \cup C_{\tau_j} \cup J_{\tau_i \tau_j}) - r_{N_2}(C_{\tau_i} \cup C_{\tau_j})$,
(ii) $r_{N_2}(C_{\tau_i} \cup C_{\tau_j} \cup J_{\tau_i \tau_j}) \leqslant r_{N_2}(C_{\tau_i} \cup C_{\tau_j}) + 1$, *and*
(iii) $\lambda_{N_2|(C_{\tau_1} \cup \cdots \cup C_{\tau_\delta} \cup J_{\tau_i \tau_j})}(C_{\tau_k}) = 0$ *for all* $k \in \{1, \ldots, \delta\} - \{i, j\}$.

For each $i \in \{1, \ldots, \delta\}$ let C'_i denote C_{τ_i}, for each $i, j \in \{1, \ldots, \delta\}$ with $i < j$, let J'_{ij} denote $J_{ij} \cap E(N_2)$, and let $Z = C'_1, \ldots, C'_\delta$.

Consider $i, j \in \{1, \ldots, \delta\}$ with $i < j$. By (iii), $\lambda_{N_2|(Z \cup J'_{ij})}(C'_k) = 0$ for all $k \in \{1, \ldots, \delta\} - \{i, j\}$. It follows that $\lambda_{N_2|(Z \cup J'_{ij})}(C'_i \cup C'_j \cup J'_{ij}) = 0$. Then, since $(C'_1, \ldots, C'_\delta)$ are skew and span N_2, J'_{ij} is spanned by $C'_i \cup C'_j$ in N_2. Moreover, $\kappa_{N_2|(C'_i \cup C'_j \cup J'_{ij})}(C'_i, C'_j) \geqslant 1$, so there exists $e_{ij} \in J'_{ij}$ that is spanned by neither C'_i nor C'_j in N_2. This completes the proof. □

10. Finding cliques

Lemma 9.2 provides us with a lot of structure. In this section we will show that this structure implies the existence of either a large clique or the dual of a large clique as a minor. Recall that the function β was defined in Theorem 3.2.

Lemma 10.1. *Let m, q, and n be positive integers with $m \geqslant 2(\beta(n, q) + 1)$. Now, let $M \in \mathcal{U}^*(q)$ be a matroid with $E(M) = \{a_{ij}\colon i, j \in \{1, \ldots, m\}, i \neq j\} \cup \{e_{ij}\colon 1 \leqslant i < j \leqslant m\}$ such that $(\{a_{ij}\colon j \in \{1, \ldots, m\} - \{i\}\}\colon i \in \{1, \ldots, m\})$ is a collection of skew-circuits and, for each $i, j \in \{1, \ldots, m\}$ with $i < j$, $\{a_{ij}, e_{ij}, a_{ji}\}$ is a circuit. Then, M has an $M^*(K_n)$-minor.*

Proof. Let C_i denote the circuit $\{a_{ij}\colon j \in \{1, \ldots, m\} - \{i\}\}$ of M and let $J = \{e_{ij}\colon 1 \leqslant i < j \leqslant m\}$. Let N be the matroid obtained from M/J by simplifying the parallel pairs $\{a_{ij}, a_{ji}\}$. Thus, in N we have $a_{ij} = a_{ji}$. We claim that N is cosimple; consider an element a_{ij} of N. Since a_{ij} is in the circuit C_i in M, there is a circuit C'_i of N with $a_{ij} \in C'_i \subseteq C_i$. Similarly, there is a circuit C'_j of N with $a_{ij} \in C'_j \subseteq C_j$. Since $C'_i \cap C'_j = \{a_{ij}\}$, a_{ij} is not in series with any other element in N; thus, N is cosimple as claimed.

Note that, $|E(N)| = \frac{m(m-1)}{2} > \beta(n, q)m$. Now, each circuit C_i of M has $m - 1$ elements, so $r(M) = m(m-2)$. Moreover, $|J| = \binom{m}{2}$. Hence, $r^*(N) = |E(N)| - r(N) \leqslant |E(N)| - r(M) + |J| = 2\binom{m}{2} - m(m-2) = m$. Therefore, applying Theorem 3.2 to N^*, we see that M has an $M^*(K_n)$-minor. □

Lemma 10.2. *Let m, q, and n be positive integers with $m \geqslant 2\beta(n, q)$. Now, let $M \in \mathcal{U}(q)$ be a matroid with $E(M) = \{a_1, \ldots, a_m\} \cup \{e_{ij}\colon 1 \leqslant i < j \leqslant m\}$ such that $\{a_1, \ldots, a_m\}$ is an independent set of M and, for each $i, j \in \{1, \ldots, m\}$ with $i < j$, $\{a_i, e_{ij}, a_j\}$ is a circuit. Then, M has an $M(K_n)$-minor.*

Proof. Note that $r(M) = m$ and $E(M) = m + \binom{m}{2} = \frac{(m+1)m}{2} > \beta(n, q)r(M)$. Thus, by Theorem 3.2, M has an $M(K_n)$-minor. □

Now, the last step.

Lemma 10.3. *For any positive integers n and q there exists a positive integer m such that if $M \in \mathcal{U}(q) \cap \mathcal{U}^*(q)$ is a matroid containing skew-circuits $(C_1, \dots, C_m)$ and elements $\{e_{ij}\colon 1 \leqslant i < j \leqslant m\}$ where $e_{ij} \in \mathrm{cl}_M(C_i \cup C_j) - (\mathrm{cl}_M(C_i) \cup \mathrm{cl}_M(C_j))$, then M has an $M(K_n)$- or an $M^*(K_n)$-minor.*

Proof. Let $f(k) = \beta(n,q)k^2$, and let f^j denote the composition of f with itself j times; that is, $f^1(k) = f(k)$ and, for $j > 1$, $f^j(k) = f(f^{j-1}(k))$. Now, let $t_1 = 2(\beta(n,q)+1)$, $t_2 = 2\beta(n,q)$, and $t = t_1 + t_2$; let $m = t + f^t\left(\frac{t(t-1)}{2}\right)$; and let M, $(C_1, \dots, C_m)$, and $\{e_{ij}\colon 1 \leqslant i < j \leqslant m\}$ be as stated above.

10.3.1. *If $i \in \{1, \dots, m\}$ and $S \subseteq \{i+1, \dots, m\}$ where $|S| \geqslant f(k)$, then there exists $S' \subseteq S$ with $|S'| = k$ and there exists $X \subseteq C_i$ such that either*

(i) *$|C_i - X| = 2$ and, for each $j \in S'$, $e_{ij} \notin \mathrm{cl}_{M/X}(C_j)$, or*
(ii) *there is a bijection $\pi : (C_i - X) \to S'$ such that, for each $a \in C_i - X$, $a \in \mathrm{cl}_{M/X}(C_{\pi(a)} \cup \{e_{i\pi(a)}\})$.*

Subproof. Choose $X \subseteq C_i$ maximal such that for each $j \in S$ $e_{ij} \notin \mathrm{cl}_{M/X}(C_j)$ and let $N = M/X$. We may assume that $|C_i - X| > 2$. By our choice of X, for each element $a \in C_i - X$ there exists an element $\pi(a)$ of S such that $e_{i\pi(a)} \in \mathrm{cl}_{N/a}(C_{\pi(a)})$. Since the circuits C_i and $C_{\pi(a)}$ are skew, $a \in \mathrm{cl}_N(C_{\pi(a)} \cup \{e_{i\pi(a)}\})$. Let $S' = \{\pi(a)\colon a \in C_i - X\}$. Since $|C_i - X| > 2$, $\mathrm{cl}_N(C_{\pi(a)} \cup \{e_{i\pi(a)}\}) \cap (C_i - X) = \{a\}$. Thus, π is a bijection from $C_i - X$ to S'. Therefore, we may assume that $|S'| < k$ and, hence, $|C_i - X| < k$.

Let $J = \bigcup(C_j\colon j \in S)$ and let $A = \{e_{ij}\colon j \in S\}$. Note that $A \subseteq \mathrm{cl}_{N/J}(C_i - X)$. Thus, $r_{N/J}(A) \leqslant r_{N/J}(C_i - X) < k$. Then, by Theorem 3.2, $|E(\mathrm{si}((N/J)|A))| < \beta(n,q)k$. However, $|A| \geqslant f(k) = \beta(n,q)k^2 > k|E(\mathrm{si}((N/J)|A))|$. Therefore, there exists $A' \subseteq A$ with $|A'| = k$ such that $r_{N/J}(A') = 1$. Let $S'' = \{j\colon j \in S, e_{ij} \in A'\}$ and let $J' = \bigcup(C_j\colon j \in S'')$. Since $C_i - X$ is skew to J in N, $\sqcap_N(C_i - X, J' \cup A') \leqslant r_{N/J}(A') = 1$. Now choose a set X' with $X \subseteq X' \subseteq C_i$ that is maximal such that $e_{ij} \notin \mathrm{cl}_{M/X'}(C_j)$ for each $j \in S''$. Since $\sqcap_N(C_i - X, J' \cup A') \leqslant 1$, $C_i - X'$ will be a parallel pair in N/X'. □

By repeatedly applying the above claim for $i = 1, \dots, t$ we get the following.

10.3.2. *There exist $S \subseteq \{t+1, \dots, m\}$ with $|S| = \frac{t(t-1)}{2}$ and $X \subseteq C_1 \cup \dots \cup C_t$ such that for each $i \in \{1, \dots, t\}$ either*

(i) *$|C_i - X| = 2$ and, for each $j \in S$, $e_{ij} \notin \mathrm{cl}_{M/X}(C_j)$, or*
(ii) *there is a bijection $\pi : (C_i - X) \to S$ such that, for each $a \in C_i - X$, $a \in \mathrm{cl}_{M/X}(C_{\pi(a)} \cup \{e_{i\pi(a)}\})$.*

Let S and X be as given by 10.3.2. Now, for $i \in \{1, \dots, t\}$, let $C_i' = C_i - X$ and let $N = M/X$. Note that either $|C_i'| = 2$ or $|C_i'| = |S|$. We break the proof into two cases; among $(C_1', \dots, C_t')$ either there are t_1 circuits of size $|S|$ or there are t_2 circuits of size two.

Case 1. *At least t_1 of the circuits $(C_1', \dots, C_t')$ have size $|S|$.*

By possibly reordering we may assume that $|C_i'| = |S|$ for each $i \in \{1, \dots, t_1\}$.

Consider some $k \in S$ and $i \in \{1, \dots, t_1\}$. By construction, $C'_i \cap \mathrm{cl}_N(C_k \cup \{e_{ik}\})$ contains exactly one element, say a^i_k. Moreover, the elements $(a^i_j\colon j \in S)$ are distinct.

Since there are at least as many elements in S as there are pairs of elements in $\{1, \dots, t_1\}$, we can choose a sequence $(k_{ij}\colon 1 \leqslant i < j \leqslant t_1)$ of distinct elements in S. For each $i, j \in \{1, \dots, t_1\}$ with $i < j$, we let $a_{ij} = a^i_{k_{ij}}$ and $a_{ji} = a^j_{k_{ij}}$.

Consider some $i, j \in \{1, \dots, t_1\}$ with $i < j$, and let $k = k_{ij}$. Obviously $N|(C_k \cup \{e_{ik}, a_{ij}\})$ is connected, and, hence, $N|(C_k \cup \{e_{ik}, e_{jk}, a_{ij}, a_{ji}\})$ is connected. Therefore, there exists a circuit C of N such that $\{a_{ij}, a_{ji}\} \subseteq C \subseteq C_k \cup \{e_{ik}, e_{jk}, a_{ij}, a_{ji}\}$. We can then contract elements of C to make a triangle through the pair $\{a_{ij}, a_{ji}\}$; moreover, we can do this for each pair. Note that the circuits $(C'_1, \dots, C'_{t_1})$ remain skew after these contractions. Then, by Lemma 10.1, M has an $M^*(K_n)$-minor.

Case 2. *At least t_1 of the circuits $(C'_1, \dots, C'_t)$ have size two.*

This is essentially the same as the first case, but we use Lemma 10.2 in place of Lemma 10.1. □

Finally, Theorem 2.2 is an immediate corollary of Lemmas 9.2 and 10.3.

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References

[1] P. Erdős, L. Pósa, On the maximal number of disjoint circuits of a graph, Publ. Math. Debrecen 9 (1962) 3–12.
[2] J.F. Geelen, G.P. Whittle, Branch-width and Rota's Conjecture, J. Combin. Theory Ser. B 86 (2002) 315–330.
[3] J.F. Geelen, G.P. Whittle, Cliques in dense $GF(q)$-representable matroids, J. Combin. Theory Ser. B 87 (2003) 264–269.
[4] J.F. Geelen, A.M.H. Gerards, G.P. Whittle, Branch-width and well-quasi-ordering in matroids and graphs, J. Combin. Theory Ser. B 84 (2002) 270–290.
[5] J.F. Geelen, A.M.H. Gerards, N. Robertson, G.P. Whittle, On the excluded-minors for the matroids of branch-width k, J. Combin. Theory Ser. B 88 (2003) 261–265.
[6] J.F. Geelen, A.M.H. Gerards, G.P. Whittle, Disjoint cocircuits in matroids with large rank, J. Combin. Theory Ser. B 87 (2003) 270–279.
[7] J.F. Geelen, A.M.H. Gerards, N. Robertson, G.P. Whittle, Obstructions to branch-decomposition of matroids, J. Combin. Theory Ser. B 96 (2006) 560–570.
[8] P. Hliněný, Branch-width, parse trees, and monadic second-order logic for matroids, J. Combin. Theory Ser. B 96 (2006) 325–351.
[9] T. Johnson, N. Robertson, P.D. Seymour, Connectivity in binary matroids, handwritten manuscript.
[10] W. Mader, Homomorphieeigenschaften und mittlere Kantendichte von Graphen, Math. Ann. 174 (1967) 265–268.
[11] J.G. Oxley, Matroid Theory, Oxford Univ. Press, New York, 1992.
[12] N. Robertson, P.D. Seymour, Graph minors V: Excluding a planar graph, J. Combin. Theory Ser. B 41 (1986) 92–114.
[13] P.D. Seymour, Recognizing graphic matroids, Combinatorica 1 (1981) 75–78.
[14] W.T. Tutte, Menger's theorem for matroids, J. Res. Natl. Bureau of Standards—B. Math. Math. Phys. 69B (1965) 49–53.