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AN ALGEBRAIC MATCHING ALGORITHM

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Tutte introduced a V by V skew-symmetric matrix $T=(t_{ij})$, called the *Tutte matrix*, associated with a simple graph $G=(V,E)$. He associates an indeterminate z_e with each $e\in E$, then defines $t_{ij}=\pm z_e$ when $ij=e\in E$, and $t_{ij}=0$ otherwise. The rank of the Tutte matrix is exactly twice the size of a maximum matching of G. Using linear algebra and ideas from the Gallai–Edmonds decomposition, we describe a very simple yet efficient algorithm that replaces the indeterminates with constants without losing rank. Hence, by computing the rank of the resulting matrix, we can efficiently compute the size of a maximum matching of a graph.

1. Introduction

Let $G=(V,E)$ be a simple graph, and let $(z_e:e\in E)$ be algebraically independent commuting indeterminates. We define a V by V skew-symmetric matrix $T=(t_{ij})$, called the *Tutte matrix* of G, such that $t_{ij}=\pm z_e$ if $ij=e\in E$, and $t_{ij}=0$ otherwise. Tutte observed that T is nonsingular (that is, its determinant is not identically zero) if and only if G admits a perfect matching. In fact, the rank of T is exactly twice the size of a maximum cardinality matchable set in G. (A subset X of V is called *matchable* if $G[X]$, the subgraph induced by X, admits a perfect matching.) By applying elementary linear algebra to the Tutte matrix, Tutte proved his famous matching theorem [12].

It is not immediately clear how to obtain an efficient matching algorithm from the Tutte matrix. The determinant of T is a polynomial that may have exponentially many terms, so it cannot be computed efficiently. We

Mathematics Subject Classification (1991): 05C70 circumvent this problem by substituting constants in place of the indeterminates. In doing so the rank of T does not increase. This idea was originally proposed by Lovász [7] who gave an efficient randomized algorithm for finding the size of the largest matching of a graph. Here, using ideas from the Gallai–Edmonds structure theorem and from linear algebra, we describe an efficient deterministic algorithm for choosing the constants so that the rank of T does not decrease.

Let T be the Tutte matrix of G. An *evaluation* of T is a matrix $T' = (t'_{ij})$ obtained by replacing the indeterminates in T by integers in $\{1,\ldots,n\}$, where $n = |V|$. For an edge ij of G, we construct a matrix $T'(a;ij)$ from T' by replacing t'_{ij} and t'_{ji} by a and $-a$ respectively. $T'[X]$ denotes the principal submatrix of T' whose rows and columns are indexed by the set X, and $T' \setminus X$ denotes $T'[V \setminus X]$. Let $D(T')$ be the set of all $x \in V$ such that $\operatorname{rank} T' \setminus x = \operatorname{rank} T'$. We refer to the elements of $D(T')$ as the *deficient* elements of T'. For evaluations T'_1 and T'_2 of T, we write

$\boldsymbol{T'_1 \preceq T'_2}$ if either $\operatorname{rank} T'_2 > \operatorname{rank} T'_1$ or $\operatorname{rank} T'_2 = \operatorname{rank} T'_1$ and $D(T'_1) \subseteq D(T'_2)$.
$\boldsymbol{T'_1 \approx T'_2}$ if $\operatorname{rank} T'_1 = \operatorname{rank} T'_2$ and $D(T'_1) = D(T'_2)$. And
$\boldsymbol{T'_1 \prec T'_2}$ if $T'_1 \preceq T'_2$ but $T'_1 \not\approx T'_2$.

Obviously, for any evaluation T' of T, we have $T' \preceq T$. Our main theorem is the following.

Theorem 1.1. *Let T be the Tutte matrix of a simple graph $G = (V,E)$, and let T' be an evaluation of T. Then either $T' \approx T$ or there exists $ij \in E$ and $a \in \{1,\ldots,n\}$ such that $T'(a;ij) \succ T'$.*

An obvious consequence of the theorem is that we can efficiently determine the size of the largest matching in G. Indeed, take any evaluation T' of T and apply Theorem 1.1. We will either find $a \in \{1,\ldots,n\}$ and $ij \in E$ such that $T'(a;ij) \succ T'$ (in which case we replace T' by $T'(a;ij)$ and then repeat the above procedure), or we conclude that $T' \approx T$ (in which case the largest matchable set has size $\operatorname{rank} T'$). Hence, in at most n^2 iterations we will know the size of the largest matchable set. Our algorithm is not computationally competitive with Edmonds' augmenting path algorithm [5] for the matching problem. However, there are a number of important combinatorial problems that can be formulated in terms of matrices of indeterminates (for example, path-matching [3], exact matching [10], and linear matroid parity [7]). The results in this paper provide some hope of finding similar solutions to these more general problems.

One drawback of the algorithm is that it does not actually identify a maximum matching. Suppose we have found an evaluation T' of T such that $T' \approx T$, how then do we find a maximum cardinality matching? The most obvious solution is to run the algorithm $|E|$ times (throwing out edges if doing so does not decrease the rank of T). With more sophisticated methods from linear algebra, a maximum matching can be extracted from T' in $O(n^3)$. This is a moot point, since a maximum matching can be found in $O(n^{2.5})$ using augmenting path algorithms [9].

Consider a naive implementation of our algorithm. We need to apply Theorem 1.1 $O(n^2)$ times. Each time we apply Theorem 1.1, we may have to compute $D(T'(a;ij))$ for each $ij \in E$ and each $a \in \{1,\ldots,n\}$. Computing $D(T'(a;ij))$ requires $O(n)$ matrix inversions. So, in total, we require $O(n^6)$ matrix inversions. There are a number of ways to improve this bound. For example, Cheriyan [2] shows how to compute $D(T'(a;ij))$ in approximately the same time as it takes to perform one matrix inversion. Furthermore, following the proof of Theorem 1.1, we do not need to check all edges; we need only check $O(n)$ edges. However, the algorithm remains computationally unattractive.

2. Skew-symmetric matrices

A V by V matrix $T' = (t'_{ij})$ is *skew-symmetric* if $-T'$ is equal to T'^T (that is, the transpose of T'). Suppose that T' is skew-symmetric. Define $\mathcal{F}(T') = \{X \subseteq V : \operatorname{rank} T'[X] = |X|\}$. Note that,

$$\det(T'[X]) = \det(T'[X]^T) = \det(-T'[X]) = (-1)^{|X|} \det(T'[X]).$$

Thus, if $|X|$ is odd then $T'[X]$ is singular. Hence, $|X|$ is even for all $X \in \mathcal{F}(T')$. The set-system $(V, \mathcal{F}(T'))$ is an example of an "even delta-matroid" (see Bouchet [1]). Then, by a result obtained independently by Duchamp [4] and Wenzel [13], $\mathcal{F}(T')$ satisfies the following axiom:

Simultaneous exchange axiom. Given $X,Y \in \mathcal{F}(T')$ and $x \in X \Delta Y$, there exists $y \in X \Delta Y$ such that $X \Delta \{x,y\}, Y \Delta \{x,y\} \in \mathcal{F}(T')$.

We denote by $\mathcal{F}^*(T')$ the set of maximum cardinality members of $\mathcal{F}(T')$.

Lemma 2.1. *Let T' be a V by V skew-symmetric matrix, and let X be a subset of V. Then $X \in \mathcal{F}^*(T')$ if and only if X indexes a maximal set of linearly independent columns of T'.*

Proof. It is clear that, for $X \in \mathcal{F}(T')$, the columns of T' indexed by X are linearly independent. Therefore, it suffices to prove, for a maximal set X of linearly independent columns of T', that $T'[X]$ is nonsingular.

Since T' is skew-symmetric, X also indexes a maximal set of linearly independent rows of T'. Note that deleting dependent rows does not affect column dependencies. Thus X is a maximal set indexing linearly independent columns of $T'[X,V]$. Therefore, $T'[X]$ is nonsingular. ∎

Note that the previous lemma implies that the rank of T' is equal to the size of a largest nonsingular principal submatrix of T'. In particular, the rank of a skew-symmetric matrix is always even. The previous result also implies that $\mathcal{F}^*(T')$ is the family of bases of a representable matroid. We use some elementary matroid theory in the proof of the following lemma; see Oxley [11] for a good introduction to matroid theory.

Lemma 2.2. *Let T' be a V by V skew-symmetric matrix. Then there exists a partition $\mathcal{D}(T')$ of $D(T')$, such that, for distinct elements i,j in $D(T')$, i,j are in the same part of the partition if and only if* $\operatorname{rank} T' \backslash \{i,j\} = \operatorname{rank} T' - 2$.

Proof. Given $x,y \in D(T')$, $\operatorname{rank} T' \setminus \{x,y\}$ is either $\operatorname{rank} T'$ or $\operatorname{rank} T' - 2$. Let M be the matroid $(V, \mathcal{F}^*(T'))$. The coloops of M are the elements of $V \setminus D(T')$. For $x,y \in D(T')$, $\{x,y\}$ is coindependent in M if and only if $\operatorname{rank} T' \setminus \{x,y\} = \operatorname{rank} T'$. Let $(D_1, \ldots, D_k)$ be the series-classes in M. Then $\mathcal{D} = (D_1, \ldots, D_k)$ is a partition of $D(T')$, and, for distinct $i,j \in D(T')$, i,j are in a common part of the partition if and only if $\{i,j\}$ is codependent in M. ∎

A structure theorem

Let $A(T')$ be the set of all $v \in V \setminus D(T')$ such that $D(T' \setminus v) = D(T')$. Then define $C(T') = V \setminus (A(T') \cup D(T'))$, and define $\operatorname{odd}(T')$ to be the number of sets in $\mathcal{D}(T')$ having odd cardinality.

Theorem 2.2. *Let $T' = (t'_{ij})$ be a V by V skew-symmetric matrix satisfying*

(a) *For each $i \in D(T')$ and $j \in C(T')$, $t'_{ij} = 0$, and*
(b) *For each $i,j \in D(T')$ in different parts of $\mathcal{D}(T'[D(T')])$, $t'_{ij} = 0$.*

Then T' satisfies the following conditions,

(i) *$T'[C(T')]$ is nonsingular,*
(ii) *For each $X \in \mathcal{D}(T'[D(T')])$ and $x \in X$, $T'[X - x]$ is nonsingular,*
(iii) $\operatorname{rank} T' = |V| - (\operatorname{odd}(T'[D(T')]) - A(T'))$, *and*
(iv) *For each $i \in A(T')$ there exists $j \in D(T')$ such that $t'_{ij} \neq 0$.*

We require the following lemma.

Lemma 2.3. *For $A' \subseteq A(T')$, we have*

(i) $\operatorname{rank} T' \setminus A' = \operatorname{rank} T' - 2|A'|$, *and*
(ii) *$D(T' \setminus A') = D(T')$.*

Proof of Lemma 2.3. We prove the lemma by induction on $|A'|$. By definition, the result holds for $|A'| \le 1$. Choose any $a \in A'$; then we may suppose that the result holds for $A' - a$. In particular $\operatorname{rank} T' \setminus (A' - a) = \operatorname{rank} T' - 2|A' - a|$ and $D(T' \setminus (A' - a)) = D(T')$. Since $a \in A(T')$, a is not a member of $D(T')$, and hence it is also not in $D(T' \setminus (A' - a))$. Therefore, $\operatorname{rank} T' \setminus A' < \operatorname{rank} T' \setminus (A' - a)$. However, since skew-symmetric matrices have even rank, $\operatorname{rank} T' \setminus A' = \operatorname{rank} T' \setminus (A' - a) - 2 = \operatorname{rank} T' - 2|A'|$. Hence, (i) is satisfied.

It is straightforward that $D(T' \setminus (A' - a)) \subseteq D(T' \setminus A')$. Suppose that $D(T' \setminus A') \ne D(T')$. Then there exists $y \in D(T' \setminus A') \setminus D(T')$. Choose sets $X \in \mathcal{F}^*(T')$ and $Y \in \mathcal{F}^*(T' \setminus A')$ such that $y \notin Y$. Since $a \notin D(T')$, $a \in X$. Now $a \in X \setminus Y$, so, by the simultaneous exchange axiom, there exists $b \in X \Delta Y$ such that $X \Delta \{a, b\}$ and $Y \Delta \{a, b\}$ are both contained in $\mathcal{F}(T')$. If $b \in Y \setminus X$, then $X \Delta \{a, b\} \in \mathcal{F}^*(T')$, contradicting that $a \notin D(T')$. Hence, $b \in X \setminus Y$. Note that $X \setminus \{a, b\} \in \mathcal{F}^*(T' \setminus a)$, so $b \in D(T' \setminus a) = D(T')$. Therefore, $b \ne y$. Now $Y \cup \{a, b\} \in \mathcal{F}^*(T' \setminus (A' - a))$, and $y \notin Y \cup \{a, b\}$. Hence, $y \in D(T' \setminus (A' - a))$. This contradiction completes the proof. ∎

Proof of Theorem 2.2. For simplicity, we denote $D(T')$, $A(T')$, and $C(T')$ by D, A and C respectively. Let A' be the set of all $i \in V \setminus D$ for which there exists $j \in D$ such that $t'_{ij} \ne 0$. Then define $C' = V \setminus (A' \cup D)$. By (a), $A' \subseteq A$. Suppose that $\mathcal{D}(T'[D]) = (X_1, \ldots, X_k)$. Then $T'[V \setminus A']$ is a block diagonal matrix with blocks $T'[X_1], \ldots, T'[X_k], T'[C']$. By Lemma 2.3, $D(T'[X_i]) = X_i$ for $i = 1, \ldots, k$, and $D(T'[C'])$ is the empty set. Hence $T'[C']$ is nonsingular. However, for $a \in C' \setminus C$, $D(T' \setminus (A' \cup a)) = D$, so $D(T'[C' - a])$ is empty. This is a contradiction, since $|C' - a|$ is odd so $T'[C' - a]$ is singular. Hence $C' = C$ and $A' = A$. This proves (i) and (iv). By the definition of $\mathcal{D}(T')$, each $Y \in \mathcal{F}^*(T'[X_i])$ has size $|X_i| - 1$. Hence $\operatorname{rank} T'[X_i] = |X_i| - 1$, which proves (ii). However $\operatorname{rank} T'[X_i]$ is even, so $|X_i|$ is odd for $i = 1, \ldots k$. Hence $\operatorname{rank} T' \setminus A(T') = |V \setminus A(T')| - \operatorname{odd}(T'[D(T')])$. Then, by Lemma 2.3,

$$\begin{aligned}
\operatorname{rank} T' &= \operatorname{rank} T' \setminus A(T') + 2\,|A(T')| \\
&= |V \setminus A(T')| - \operatorname{odd}(T'[D(T')]) + 2\,|A(T')| \\
&= |V| - (\operatorname{odd}(T'[D(T')]) - A(T')),
\end{aligned}$$

proving (iii). ∎

Lemma 2.4. *Let T' be a V by V skew-symmetric matrix, and suppose $x \in C(T')$. Then, $D(T') \subset D(T' \setminus \{x\})$. Furthermore, for $v \in D(T' \setminus \{x\}) \setminus D(T')$ and for any $X \in \mathcal{F}^*(T')$, $X \setminus \{v, x\} \in \mathcal{F}(T')$.*

Proof. Since $x \in C(T')$ it is straightforward that $D(T') \subseteq D(T' \setminus \{x\})$. Since neither x nor v is deficient, $x, v \in X$. Choose $Y \in \mathcal{F}^*(T' \setminus \{x\})$ such that $v \notin Y$. Then $v \in X \Delta Y$. So, by the simultaneous exchange axiom, there exists $y \in X \Delta Y$ such that $X \Delta \{v, y\}, Y \Delta \{v, y\} \in \mathcal{F}(T')$. Suppose that $y \in Y \setminus X$. Then $X \Delta \{v, y\} \in \mathcal{F}^*(T')$. However, this contradicts that $v \notin D(T')$. Therefore, $y \in X \setminus Y$, so $Y \Delta \{v, y\} \in \mathcal{F}^*(T')$. However, $x \notin D(T')$, and therefore $y = x$. Then $X \Delta \{v, y\} = X \setminus \{v, x\} \in \mathcal{F}(T')$, as required. ∎

Lemma 2.5. *Let T' be a V by V skew-symmetric matrix such that for each $i \in D(T')$ and $j \in C(T')$, $t'_{ij} = 0$. Furthermore, suppose that $i, j \in D(T')$ such that i and j are in different parts of $\mathcal{D}(T'[D(T')])$. Then either i and j are in different parts of $\mathcal{D}(T')$, or there exists $a \in A(T')$ such that i and j are in different parts of $\mathcal{D}(T' \setminus a)$.*

Proof. Note that $T' \setminus A(T')$ is block diagonal with blocks $T'[D(T')]$ and $T'[C(T')]$. Then, by Lemma 2.3, $\mathcal{D}(T' \setminus A(T')) = \mathcal{D}(T'[D(T')])$. Therefore, i and j are in different parts of $\mathcal{D}(T' \setminus A(T'))$. Let A' be a minimal subset of $A(T')$ such that i and j are in different parts of $\mathcal{D}(T' \setminus A')$. We may assume that $|A'| \geq 2$. Choose $a \in A'$. By the definitions, there exists $X \in \mathcal{F}^*(T')$ such that $j \notin X$, and $Y \in \mathcal{F}^*(T' \setminus A')$ such that $i, j \notin X$. Now $a \in X \setminus Y$. So, by the simultaneous exchange axiom, there exists $b \in X \Delta Y$ such that $X \Delta \{a, b\}, Y \Delta \{a, b\} \in \mathcal{F}(T')$. Since $a \notin D(T')$, $b \in X \setminus Y$. Note that $X \setminus \{a, b\} \in \mathcal{F}^*(T' \setminus a)$. Thus we may assume that $i \in X \setminus \{a, b\}$, since otherwise i and j are in different parts of $\mathcal{D}(T' \setminus a)$. Therefore, $b \neq i$. Now note that $Y \cup \{a, b\} \in \mathcal{F}^*(T' \setminus (A' - a))$. Therefore, i and j are in different parts of $\mathcal{D}(T' \setminus (A' - a))$. However, this contradicts the minimality of A'. ∎

Pfaffians and matrix perturbations

Pfaffians are a powerful tool for studying skew-symmetric matrices. We now review some basic results about Pfaffians; see Godsil [6] for a more detailed overview.

Let $T' = (t'_{ij})$ be a V by V skew-symmetric matrix, where $V = \{1, \ldots, n\}$. Let $G(T')$ denote the graph (V, E') where $E' = \{ij : t'_{ij} \neq 0\}$, and let $\mathcal{M}_{T'}$ denote the set of perfect matchings of $G(T')$. A pair of edges $u_1 v_1$, $u_2 v_2$ of $G(T')$, where $u_1 < v_1$ and $u_2 < v_2$, is said to *cross* if $u_1 < u_2 < v_1 < v_2$ or $u_2 < u_1 < v_2 < v_1$. The *sign* of a perfect matching M of $G(T')$, denoted σ_M, is $(-1)^k$ where k is the number of pairs of crossing edges in M. The *Pfaffian* of T', denoted $\mathrm{Pf}(T')$, is defined as follows:

$$\mathrm{Pf}(T') = \sum_{M \in \mathcal{M}_{T'}} \sigma_M \prod_{\substack{uv \in M \\ u<v}} t'_{uv}. \tag{1}$$

Pfaffians satisfy the identity $\det(T') = \mathrm{Pf}(T')^2$; and are often more convenient to work with than determinants. Like determinants, Pfaffians can be calculated by "row expansion" [6]:

$$\mathrm{Pf}(T') = \sum_{k=2}^{n} (-1)^{k+1} t'_{1k} \, \mathrm{Pf}(T' \setminus \{1,k\}). \tag{2}$$

Consider $T'_a = T'(a, ij)$ for $i, j \in V$ and an indeterminate a. Note that, by (1), $\mathrm{Pf}(\tilde{T}'_a[X])$ is linear in a. Furthermore, by (2),

$$\mathrm{Pf}(T'_a[X]) = \pm \, \mathrm{Pf}(T'[X \setminus \{i,j\}])a + \mathrm{Pf}(T'_0) \tag{3}$$

Lemma 2.6. *Let $ij \in E$. Then there exists $a \in \{1, \ldots, n\} \setminus \{t'_{ij}\}$ such that $T'(a;ij) \succeq T'$.*

Proof. Let T'_a denote $T'(a, ij)$, where a is yet to be determined. For each $x \in D(T')$, there exists $X \in \mathcal{F}^*_{T'}$ with $x \notin X$. There is at most one choice for a that makes $T'_a[X]$ singular. For any other choice of a, we have $\operatorname{rank} T'_a \geq \operatorname{rank} T'$, and if $\operatorname{rank} T'_a = \operatorname{rank} T'$ then $x \in D(T'_a)$. Note that if x is either i or j, then $T'_a[X] = T'[X]$ which is nonsingular. Hence, for each $x \in D(T') \setminus \{i,j\}$, there is at most one forbidden choice for a. So the number of forbidden choices is at most $|D(T) \setminus \{i,j\}| \leq n-2$, but there are $n-1$ choices available for a. ∎

Lemma 2.7. *Let $ij \in E$, and let $X \subseteq V \setminus \{i,j\}$ such that $X \in \mathcal{F}(T')$ but $X \cup \{i,j\} \notin \mathcal{F}(T')$. Then, for any $a' \neq t'_{ij}$, $X \cup \{i,j\} \in \mathcal{F}(T'(a';ij))$.*

Proof. By (3), $\mathrm{Pf}(T'_a[X \cup \{i,j\}])$ is linear in a and not identically zero. So there is at most one choice for a that makes $\mathrm{Pf}(T'_a[X \cup \{i,j\}])$ zero, this choice is $a = t_{ij}$. ∎

Building good evaluations

We conclude this section by proving Theorem 1.1. We require some preliminary results.

Theorem 2.3. *Let $G=(V,E)$ be a graph, and let T' be an evaluation of its Tutte matrix. If ij is an edge with $i\in D(T')$ and $j\in C(T')$, then there exists $a\in\{1,\dots,n\}$ such that $T'(a,ij)\succ T'$.*

Proof. Take $X\in\mathcal{F}^*(T')$ such that $i\notin X$, and $y\in D(T'\setminus j)\setminus D(T')$. By Lemma 2.4, $X\setminus\{j,y\}\in\mathcal{F}(T')$. Now since $y\notin D(T')$, $X\setminus\{y\}\cup\{i\}\notin\mathcal{F}(T')$. By Lemma 2.6, there exists $a\in\{1,\dots,n\}\setminus\{t'_{ij}\}$ such that $T'(a;ij)\succeq T'$. Then, by Lemma 2.7, $X\setminus\{y\}\cup\{i\}\in\mathcal{F}(T'(a;ij))$. Hence, either $\operatorname{rank}T'(a;ij)>\operatorname{rank}T')$ or $y\in D(T'(a;ij))$. So $T(a;ij)\succ T'$. ∎

Theorem 2.4. *Let $G=(V,E)$ be a graph, and let T' be an evaluation of its Tutte matrix, such that, for all $x\in D(T')$ and all $y\in C(T')$, $xy\notin E$. If $i,j\in D(T')$ such that $ij\in E$ and i and j are in different parts of $\mathcal{D}(T'[D(T')])$, then there exists $a\in\{1,\dots,n\}$ such that $T'(a;ij)\succ T'$.*

Proof. Suppose that i and j are in different parts of $\mathcal{D}(T')$. Then there exists $X\in\mathcal{F}^*(T')$ such that $i,j\notin X$. Take any $a\in\{1,\dots,n\}\setminus\{t'_{ij}\}$. Then, by Lemma 2.7, $X\cup\{i,j\}\in\mathcal{F}(T'(a;ij))$. Hence $T'(a;ij)\succ T'$, as required. Therefore, by Lemma 2.5, we may assume that there exists $x\in A(T')$ such that i and j are in different parts of $\mathcal{D}(T'\setminus x)$. Take $X\in\mathcal{F}^*(T'\setminus x)$ such that $i,j\notin X$. Now since $x\notin D(T')$, $X\cup\{i,j\}\notin\mathcal{F}(T')$. By Lemma 2.6, there exists $a\in\{1,\dots,n\}\setminus\{t'_{ij}\}$ such that $T'(a;ij)\succeq T'$. Then, by Lemma 2.7, $X\cup\{i,j\}\in\mathcal{F}(T'(a;ij))$. Hence, either $\operatorname{rank}T'(a;ij)>\operatorname{rank}T'$ or $x\in D(T'(a;ij))$. So $T(a;ij)\succ T'$. ∎

Proof of Theorem 1.1. Suppose that there does not exist $ij\in E$ and $a\in\{1,\dots,n\}$ such that $T'(a;ij)\succ T'$. Then, by Theorems 2.3 and 2.4, T' satisfies conditions (a) and (b) of Theorem 2.2. Then, by Theorem 2.2, $\operatorname{rank}T'=|V'|-(\operatorname{odd}(T'[D(T')])-|A(T')|)$. Therefore it is routine to see that every matchable set of G misses at least $\operatorname{odd}(T'[D(T')])-|A(T')|$ elements of $D(T')$. It follows that $\operatorname{rank}T'=\operatorname{rank}T$, and that each set in $\mathcal{F}^*(T)$ avoids only elements of $D(T')$. Hence $T'\approx T$ as required. ∎

3. The Gallai–Edmonds structure theorem

As previously mentioned, our proof of Theorem 1.1 is motivated by the Gallai–Edmonds structure theorem. In this section we give a proof of this theorem, using Theorem 2.2. For a more detailed discussion of the structure theorem, and for a comprehensive introduction to matching theory, see Lovász and Plummer [8].

We require the following definitions. Let $G=(V,E)$ be a simple graph. We denote by $D(G)$ the set of vertices in G that are avoided by some maximum cardinality matchable set. Let $A(G)$ be the set of vertices in $V\setminus D(G)$ that have a neighbour in $D(G)$. Then define $C(G) := V\setminus(A(G)\cup D(G))$. We denote by $\mathrm{odd}(G)$ the number of connected components in G having an odd number of vertices. If $V-v$ is a matchable set of G for every $v\in V$ then we call G *hypomatchable.*

Theorem 3.5 (Gallai–Edmonds Structure Theorem). *For a graph $G=(V,E)$, we have*

(i) *$C(G)$ is a matchable set of G,*
(ii) *every connected component of $G[D(G)]$ is hypomatchable,*
(iii) *the size of the largest matchable set of G is $|V|-(\mathrm{odd}(G[D(G)])-|A(G)|)$.*

Proof. Let T be the Tutte matrix of G. Note that $D(G)=D(T)$. Furthermore, it is clear that there is no edge ij of G such that i and j are in different parts of $\mathcal{D}(T[D(T)])$. Suppose that there exists an edge ij of G such that $i\in D(T)$ and $j\in C(T)$. Let X be a maximum cardinality matchable set of G that does not contain i, and let $y\in D(T\setminus j)\setminus D(T)$. Then, by Lemma 2.4, $X\setminus\{j,y\}$ is a matchable set. But, as ij is an edge, $X\setminus\{y\}\cup\{i\}$ is a maximum cardinality matchable set. This contradicts that $y\notin D(T)$. Therefore there are no edges between $D(T)$ and $C(T)$ in G. Hence we can apply Theorem 2.2 to T. By Theorem 2.2 (iv), $A(G)=A(T)$ and hence $C(G)=C(T)$. So (i), (ii) and (iii) are immediate consequences of their counterparts in Theorem 2.2. ∎

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