COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA 26 (5) (2006) 521–532

PACKING NON-ZERO A-PATHS IN GROUP-LABELLED GRAPHS

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Received February 10, 2004

Let $G=(V,E)$ be an oriented graph whose edges are labelled by the elements of a group Γ and let $A\subseteq V$. An *A-path* is a path whose ends are both in A. The *weight* of a path P in G is the sum of the group values on forward oriented arcs minus the sum of the backward oriented arcs in P. (If Γ is not abelian, we sum the labels in their order along the path.) We are interested in the maximum number of vertex-disjoint A-paths each of non-zero weight. When $A=V$ this problem is equivalent to the maximum matching problem. The general case also includes Mader's $\mathcal{S}$-paths problem. We prove that for any positive integer k, either there are k vertex-disjoint A-paths each of non-zero weight, or there is a set of at most $2k-2$ vertices that meets each of the non-zero A-paths. This result is obtained as a consequence of an exact min-max theorem.

1. Introduction

Let Γ be a group, let $G=(V,E)$ be an oriented graph where each edge e of G is assigned a weight $\gamma_e\in\Gamma$, and let $A\subseteq V$. (We will use additive notation for groups, although they need not be abelian.) An *A-path* is a path (with at least one edge) in the underlying graph whose ends are both in A. Let e be an edge of G oriented with tail u and head v. We let $\gamma(e,u)=-\gamma_e$ and $\gamma(e,v)=\gamma_e$. Now, if $P=(v_0,e_1,v_1,e_2,v_2,\dots,e_k,v_k)$ is a path in G, then the *weight of P*, denoted $\gamma(P)$, is defined to be $\sum_{i=1}^{k}\gamma(e_i,v_i)$. Note that,

Mathematics Subject Classification (2000): 05C22

These results were obtained at a workshop on Structural Graph Theory at the PIMS Institute in Vancouver, Canada. This research was partially conducted during the period the first author served as a Clay Mathematics Institute Long-Term Prize Fellow.

0209–9683/106/$6.00

reversing the orientation on an edge e and replacing γ_e with $-\gamma_e$ does not change the weight of any path.

We are interested in the maximum number of vertex-disjoint A-paths each of non-zero weight. We prove the following result.

Theorem 1.1. *Let Γ be a group, let $G=(V,E)$ be an oriented graph with edge-labels from Γ, and let $A\subseteq V$. Then, for any integer k, either*

(1) there are k vertex-disjoint A-paths each of non-zero weight, or
(2) there is a set of at most $2k-2$ vertices that meets each non-zero A-path.

Let $\nu(G,A,\gamma)$ denote the the maximum number of vertex-disjoint A-paths each of non-zero weight. We prove Theorem 1.1 as a corollary to an exact min-max theorem for $\nu(G,A,\gamma)$. In fact we will give two different versions of the min-max theorem; the first provides a more intuitive upper-bound while the second is cleaner. Let $E(A,\gamma)$ denote the set of all edges $e\in E$ whose ends are both in A and that have $\gamma_e=0$; note that deleting such edges does not affect ν. Let $\mathrm{comp}(G)$ denote the set of components of G. Finally, let $X,A'\subseteq V$ such that $A–X\subseteq A'\subseteq V–X$ and let $G'=G–X–E(A',\gamma)$. Then

$$\begin{aligned}\nu(G,A,\gamma) &\leq |X| + \nu(G-X, A-X, \gamma)\\ &\leq |X| + \nu(G-X, A', \gamma)\\ &= |X| + \nu(G', A', \gamma)\\ &= |X| + \sum_{H\in\mathrm{comp}(G')} \nu(H, A'\cap V(H), \gamma)\\ &\leq |X| + \sum_{H\in\mathrm{comp}(G')} \left\lfloor \frac{|A'\cap V(H)|}{2} \right\rfloor .\end{aligned}$$

We will see that after an appropriate change of edge-weights we can find X and A' such that the above inequalities hold with equality.

Let $x\in V$ and let $\delta\in\Gamma$. For each edge e of G with tail u and head v we define

$$\gamma'_e = \begin{cases} \gamma_e+\delta, & \text{if } v=x\\ -\delta+\gamma_e, & \text{if } u=x\\ \gamma_e, & \text{otherwise.}\end{cases}$$

We say that γ' is obtained by *shifting* γ by δ at x. Note that, if $x\notin A$ then this shift does not change the weight of any A-path (even when Γ is non-abelian). The main theorem is:

Theorem 1.2. *Let Γ be a group, let $G=(V,E)$ be an oriented graph with edge labels $(\gamma_e : e\in E)$ from Γ, and let $A\subseteq X$. Then there exist edge-labels*

$(\gamma'_e : e \in E)$ obtained by shifting γ at vertices in $V - A$ and there exist sets $X, A' \subseteq V$ such that $A - X \subseteq A' \subseteq V - X$ and

$$\nu(G, A, \gamma) = |X| + \sum_{H \in \mathrm{comp}(G')} \left\lfloor \frac{|A' \cap V(H)|}{2} \right\rfloor,$$

where $G' = G - X - E(A', \gamma')$.

We now turn to an alternative min-max theorem. A set of edges $F \subseteq E$ is *A-balanced* if F contains no non-zero A-path and no non-zero circuit. We let $V(F)$ denote the set of all vertices in G that are incident with an edge in F. It is straightforward to prove that $F \subseteq E$ is A-balanced if and only if there exist edge-labels $(\gamma'_e : e \in E)$ obtained by shifting γ at vertices in $V - A$ such that $\gamma'_f = 0$ for all $f \in F$. With this in mind, the next result is an easy consequence of Theorem 1.2.

Corollary 1.3. *Let Γ be a group, let $G = (V, E)$ be an oriented graph with edge labels $(\gamma_e : e \in E)$ from Γ, and let $A \subseteq X$. Then*

$$\nu(G, A, \gamma) = \min \left(|X| + \sum_{H \in \mathrm{comp}(G - X - F)} \left\lfloor \frac{|(A \cup V(F)) \cap V(H)|}{2} \right\rfloor \right),$$

where the minimum is taken over all A-balanced sets $F \subseteq E$ and all sets $X \subseteq V$.

Note that, $\nu(G, V, \gamma)$ is the size of the largest matching in $G - E(V, \gamma)$. When $A = V$ it is easy to see that Theorem 1.2 is equivalent to the Tutte–Berge Formula (Theorem 3.1) for the size of a maximum matching. Our proof of Theorem 1.2 is modelled on an easy proof of the Tutte–Berge Formula that we give in Section 3. The referee found a remarkably short of Theorem 1.2 based on Schrijver's short proof [5] of Mader's $\mathcal{S}$-paths theorem. These short proofs rely on Gallai's A-path theorem [2], whereas our proof is self-contained.

2. Some special cases

In this section we mention some path-packing problems that can be modeled via non-zero A-paths. In each of the applications we are given an undirected graph $G = (V, E)$ and a set $A \subseteq V$. Then, we are interested in finding a maximum collection of "feasible" A-paths; where feasibility depends on the application. We then determine a group, a labelling of the edges, and an orientation of G so that the non-zero A-paths and feasible A-paths coincide. Unless explicitly defined, we assume that an arbitrary orientation of G has been prescribed.

A-paths. Here we consider any A-path to be feasible. We assign labels γ_e to edges $e \in E$ and let Γ be the free group generated by $\{\gamma_e : e \in E\}$. Thus, any non-trivial path has non-zero weight. Gallai [2] reduced this case to the maximum cardinality matching problem and proved the specialization of Theorem 1.1.

Odd A-paths. Here only the A-paths of odd length are feasible. We let $\Gamma = \mathbb{Z}_2$ and assign to each edge the label 1. Thus the non-zero paths are exactly those of odd length. The problem of finding a maximum collection of disjoint odd A-paths can be reduced to the maximum matching problem; see [1].

(The main result in [1] gives a structural characterization of signed-graphs with no odd-K_n minor. Signed-graphs can be considered as binary coextensions of graphic matroids. Theorem 1.1 allows us to extend those results to coextensions of graphic matroids over other finite fields.)

(S,T)-paths. Let (S,T) be a partition of A; an (S,T)-path is a path with one end in S and the other end in T. Let $\Gamma = \mathbb{Z}_2$. The edges with exactly one end in S are assigned a label of 1 and all other edges are labelled 0. Then, an A-path is an (S,T)-path if and only if it is non-zero. Now, $\nu(G,A,\gamma)$ is just the maximum number of vertex disjoint (S,T)-paths. It is an interesting exercise to deduce Menger's theorem from Theorem 1.2.

Composition of feasible sets. Suppose that we have groups Γ_1 and Γ_2 and two edge-labellings $(\alpha_e : e \in E)$ from Γ_1 and $(\beta_e : e \in E)$ from Γ_2. We can define $\Gamma = \Gamma_1 \times \Gamma_2$ and define new edge-labels $\gamma_e = (\alpha_e, \beta_e)$. Now, a path P is non-zero with respect to γ if and only if P is non-zero with respect to either α or β.

Mader's $\mathcal{S}$-paths. Let $\mathcal{S}$ be a partition of A and let $l = |S|$. A path is an $\mathcal{S}$-path if its ends are in different parts of $\mathcal{S}$. Thus, an A-path is an $\mathcal{S}$-path if and only if it is an $(S, A - S)$-path for some set $S \in \mathcal{S}$. Then, by composition, we can define a group $\Gamma = \mathbb{Z}_2^l$ and an edge-labelling γ from Γ such that the $\mathcal{S}$-paths are precisely the non-zero A-paths. (There is a more direct formulation in which $\Gamma = \mathbb{Z}_l$.) Mader [3] proved a min-max theorem for the maximum number of disjoint $\mathcal{S}$-paths; see Schrijver [5] for a shorter proof. Mader's Theorem is a direct specialization of Corollary 1.3.

(The problem of finding a maximum collection of vertex-disjoint $\mathcal{S}$-paths is equivalent to the problem of finding a maximum collection of internally vertex-disjoint A-paths. It is natural then to consider the problem of finding a maximum collection of internally vertex-disjoint non-zero A-paths. This contains the problem of finding a maximum collection of internally vertex-disjoint odd paths between a given pair of vertices; we suspect that this latter problem is NP-hard.)

Paths on surfaces. Suppose that $G=(V,E)$ is an oriented graph embedded on a surface S and that $A\subseteq V$ all lie on a common face F in the embedding, where F is a closed disk. We fix a basepoint x in F; then, we associate to each A-path P a simple closed curve $C(P)$ on S that is contained in $P\cup F$ and that has x as its basepoint. Now, we can designate an A-path P to be feasible, in different ways, according to the homotopy class of $C(P)$.

Example 1. P is feasible if $C(P)$ is non-contractible.
Example 2. P is feasible if $C(P)$ is non-separating.
Example 3. P is feasible if $C(P)$ is orientation reversing (that is, the neighbourhood of the curve $C(P)$ is a Möbius band).

Let $\Gamma=\pi(S,x)$ be the fundamental group of S with respect to the basepoint x; see Munkres [4]. Recall that the elements of Γ are the equivalence classes of (x,x)-curves on S with respect to homotopy; thus, the identity of Γ consists of the set of contractible (x,x)-curves. Readers familiar with topology will see that:

Lemma 2.1. *G can be assigned edge-labels $(\gamma_e : e\in E)$ from Γ such that, for any A-path P, $\gamma(P)$ is the homotopy class of $C(P)$.*

Thus, given the edge-labelling γ from Lemma 2.1, an A-path P is non-zero if and only if $C(P)$ is non-contractible. This gives us a formulation for the first example. In each of the other two examples, the homotopy classes corresponding to non-feasible A-paths determine a normal subgroup of Γ. Therefore, formulations for these examples can be obtained, via Lemma 2.1, by applying appropriate homomorphisms to Γ.

3. Matching

Let $G=(V,E)$ be a graph. The *matching number* of G, denoted $\nu(G)$, is the size of a maximum matching, and the *deficiency* of G is defined by $\operatorname{def}(G):=|V|-2\nu(G)$. We let $\operatorname{odd}(G)$ denote the number of components of G that have an odd number of vertices. Note that, for any $X\subseteq V$, we have

$$\operatorname{def}(G)\geq \operatorname{def}(G-X)-|X|\geq \operatorname{odd}(G-X)-|X|;$$

the following theorem shows that equality can be attained.

Theorem 3.1 (Tutte–Berge Formula). *For any graph G,*

$$\operatorname{def}(G)=\max(\operatorname{odd}(G-X)-|X| \,:\, X\subseteq V).$$

A set $S \subseteq V$ is *matchable* if there is a matching of G that covers every vertex in S (matchable sets need not have even cardinality). It is well-known that the matchable sets of G form the independent sets of a matroid on V; this is the *matching matroid* of G.

We require some elementary matroid theory. Let M be a matroid with ground set V and let $u,v \in V$. Then, u is a *coloop* of M if u is in every basis of M. The elements u and v are in *series* if neither u nor v are coloops, but there is no basis that avoids both u and v. It is easy to show that series pairs are transitive. That is, if u is in series with v and v is in series with w ($u \neq w$), then u is in series with w.

Lemma 3.2 (Gallai's Lemma). *If $G=(V,E)$ is a connected graph and $\nu(G-v)=\nu(G)$ for each vertex $v \in V$, then $\operatorname{def}(G)=1$ and $|V|$ is odd.*

Proof. The matching matroid M of G has no coloops since $\nu(G-v)=\nu(G)$ for each vertex $v \in V$. For each edge uv of G, we have $\nu(G-u-v)<\nu(G)$; that is, u is in series with v. Then, since G is connected, each pair of vertices is in series. Thus, no basis of M can avoid two or more vertices. Therefore, $\operatorname{def}(G)=1$ and, hence, $|V|$ is odd. ∎

Proof of the Tutte–Berge Formula. We have already seen that $\operatorname{def}(G) \geq \operatorname{odd}(G-X)-|X|$ for any set $X \subseteq V$, thus it suffices to prove that equality can be attained.

Choose $X \subseteq V$ maximal such that $\nu(G)=\nu(G-X)+|X|$. By our choice of X we have $\nu((G-X)-v)=\nu(G-X)$ for each $v \in V-X$. Then, applying Gallai's Lemma to each component H of $G-X$, we see that $\operatorname{def}(H)=1$ and $|V(H)|$ is odd. Thus, $\operatorname{def}(G-X)=\operatorname{odd}(G-X)$. Therefore, $\operatorname{def}(G) = |V|-2\nu(G) = |V|-2(\nu(G-X)+|X|) = (|V-X|-2\nu(G-X))-|X| = \operatorname{def}(G-X)-|X| = \operatorname{odd}(G-X)-|X|$; as required. ∎

4. A matroid from non-zero A-paths

Throughout this section we let Γ be a group, $G=(V,E)$ be an oriented graph with edge-labels $(\gamma_e : e \in E)$ from Γ, and $A \subseteq V$. We let $\operatorname{def}(G,A,\gamma) := |A|-2\nu(G,A,\gamma)$.

A *path* is a sequence $P=(v_0,e_1,v_1,e_2,v_2,\ldots,e_k,v_k)$ where $v_0,\ldots,v_k$ are distinct vertices of G and e_i has ends v_{i-1} and v_i for each $i \in \{1,\ldots,k\}$. Thus, P is ordered in that it has distinguished *start* (v_0) and *end* (v_k). However, P need not be "directed" in that an edge e_i of P may have v_{i-1} or v_i as its head. The path $(v_k,e_k,v_{k-1},\ldots,v_1,e_1,v_0)$ is denoted by $\bar{P}$. Also, for any $i,j \in \{1,\ldots,k\}$ with $i \leq j$, the path $(v_i,e_{i+1},v_{i+1},\ldots,e_j,v_j)$ is denoted by

$P[v_i, v_j]$. We allow paths consisting of a single vertex; we refer to such paths as *trivial*.

An A-*collection* is a set $\mathcal{P}$ of vertex disjoint paths such that:

1. each vertex in A is either the start or the end of a path in $\mathcal{P}$,
2. the start of each path $P \in \mathcal{P}$ is in A, and
3. if $P \in \mathcal{P}$ is non-trivial and has its end in A, then $\gamma(P) \neq 0$.

A path $P \in \mathcal{P}$ is *loose* if it is trivial or its end is not in A; thus each path in $\mathcal{P}$ is either an A-path or it is loose (not both). An A-collection is *optimal* if it contains $\nu(G, A, \gamma)$ A-paths; note that there are optimal A-collections.

Let $\Gamma' = \{\gamma(P) : P \text{ a path of } G\}$ (when Γ is finite we could just take $\Gamma' = \Gamma$). Now, let $S = \{(v, 0) : v \in A\} \cup \{(v, \gamma) : v \in V - A, \gamma \in \Gamma'\}$. We will define a matroid on the ground set S. Let $\mathcal{P}$ be an A-collection. We let $B(\mathcal{P})$ denote the set of pairs $(v, \gamma(P))$ where v is the end of a loose path $P \in \mathcal{P}$. Note that, $B(\mathcal{P}) \subseteq S$. Now let $\mathcal{B}$ denote the set of all $B(\mathcal{P})$ where $\mathcal{P}$ is an optimal A-collection.

Note that $|B| = \operatorname{def}(G, A, \gamma)$ for all $B \in \mathcal{B}$. Below we show that $\mathcal{B}$ is the collection of bases of a matroid on S. (In the special case that our original A-path problem was just matching, this matroid is isomorphic to the dual of the matching matroid.)

Lemma 4.1. *$\mathcal{B}$ is the set of bases of a matroid on S.*

Proof. As noted above, $\mathcal{B}$ is nonempty and all sets in $\mathcal{B}$ have the same cardinality. Suppose that $\mathcal{B}$ is not the collection of bases of a matroid. Thus, there exist $\mathcal{P}$, $\mathcal{P}'$, and (u, α) satisfying:

Claim 4.2. *$\mathcal{P}$ and $\mathcal{P}'$ are optimal A-collections and $(u, \alpha) \in B(\mathcal{P}) - B(\mathcal{P}')$ such that for each $(v, \beta) \in B(\mathcal{P}') - B(\mathcal{P})$ we have $(B(\mathcal{P}) - \{(u, \alpha)\}) \cup \{(v, \beta)\} \notin \mathcal{B}$.*

Now:

Claim 4.3. *we choose $\mathcal{P}$, $\mathcal{P}'$, and (u, α) satisfying 4.2 with $|E(\mathcal{P}) - E(\mathcal{P}')|$ as small as possible.*

We use the following claim repeatedly.

Claim 4.4. *There does not exist an optimal A-collection $\mathcal{P}''$ such that $B(\mathcal{P}) - B(\mathcal{P}'') = \{(u, \alpha)\}$ and $|E(\mathcal{P}'') - E(\mathcal{P}')| < |E(\mathcal{P}) - E(\mathcal{P}')|$.*

Subproof. Suppose that there does exist such an A–collection $\mathcal{P}''$. Since $|B(\mathcal{P}'')| = |B(\mathcal{P})|$ there is a unique element, say (u', α'), in $B(\mathcal{P}'') - B(\mathcal{P})$.

Moreover, by 4.2, $(u',\alpha')\notin B(\mathcal{P}')$. However, $|E(\mathcal{P}'')-E(\mathcal{P}')|<|E(\mathcal{P})-E(\mathcal{P}')|$. So, by 4.3, $\mathcal{P}''$, $\mathcal{P}'$, and (u',α') do not satisfy 4.2. That is, there exists an element $(v,\beta)\in B(\mathcal{P}')-B(\mathcal{P}'')$ such that $(B(\mathcal{P}')-\{(u',\alpha')\})\cup\{(v,\beta)\}\in\mathcal{B}$. However, $(B(\mathcal{P})-\{(u,\alpha)\})\cup\{(v,\beta)\} = (B(\mathcal{P}")-\{(u',\alpha')\})\cup\{(v,\beta)\}\in\mathcal{B}$, contradicting 4.2. ∎

Let $P=(v_0,e_1,v_1,\dots,e_k,v_k)$ be the path in $\mathcal{P}$ ending at u; thus, $u=v_k$. By possibly reversing the order, we may assume that there is a path $P'=(v'_0,e'_1,v'_1,\dots,e'_l,v'_l)$ in $\mathcal{P}'$ that starts at v_0. Suppose that P is not contained in P' and let e_a be the first edge of P not in P'. Now let $\mathcal{P}''$ be the A-collection obtained from $\mathcal{P}$ by replacing P with $P[v_0,v_{a-1}]$. Note that, $\mathcal{P}''$ is optimal. Moreover, $B(\mathcal{P})-B(\mathcal{P}'')=\{(u,\alpha)\}$ and $|E(\mathcal{P}'')-E(\mathcal{P}')|\le|E(\mathcal{P})-E(\mathcal{P}')|$; contradicting 4.4. Hence, P is contained in P'.

Suppose that P' is disjoint from each path in $\mathcal{P}$ other than the path P, and let $\mathcal{P}''$ be obtained from $\mathcal{P}$ by replacing P with P'. Since $\mathcal{P}$ is optimal, $\mathcal{P}''$ is also optimal and P' is loose. Note that, $(v'_l,\gamma(P'))\in B(\mathcal{P}')-B(\mathcal{P})$ and $(B(\mathcal{P})-\{(u,\alpha)\})\cup\{(v'_l,\gamma(P'))\}=B(\mathcal{P}'')\in\mathcal{B}$, contradicting 4.2. Therefore, there is some vertex that is both on P' and on a path in $\mathcal{P}$ other than P; let v'_i be the first such vertex on P' and let $Q=(u_0,f_1,u_1,\dots,f_m,u_m)$ be the path of $\mathcal{P}$ containing v'_i. Suppose that $u_j=v'_i$. We consider two cases.

Case 1: Q is a loose path.
Let P_1 be the A-path contained in $P'\cup Q$ and let P_2 be the path in $P'\cup Q$ that starts at u and ends at u_m. Since $\mathcal{P}$ is optimal, $\gamma(P_1)=0$. Thus, $\gamma(P'[v'_0,v'_i])=\gamma(Q[u_0,u_j])$ and, hence, $\gamma(P_2)=\gamma(Q)$. Now, let $\mathcal{P}''$ be the A-collection obtained from $\mathcal{P}$ by replacing P and Q with P_2 and the trivial path (u_0). Note that, $B(\mathcal{P})-B(\mathcal{P}'')=\{(u,\alpha)\}$. Moreover, since $\gamma(P_1)=0$, $P_1\neq P'$. Thus, there is an edge of $Q[u_0,u_j]$ that is not in $E(\mathcal{P}')$. So, $|E(\mathcal{P}'')-E(\mathcal{P}')|<|E(\mathcal{P})-E(\mathcal{P}')|$; contradicting 4.4.

Case 2: Q is an A-path.
Let P_1 and P_2 be the A-paths in $P'\cup Q$ that both start at u and that end with u_0 and u_m respectively. Note that $\gamma(P_1)+\gamma(Q)+\gamma(\bar{P}_2)=0$ and $\gamma(Q)\neq0$, so either $\gamma(P_1)\neq0$ or $\gamma(P_2)\neq0$. Moreover, either P' is loose (and hence different from P_1 and P_2) or $\gamma(P')\neq0$. Thus, either $\gamma(P_1)\neq0$ and $P_2\neq P'$ or $\gamma(P_2)\neq0$ and $P_1\neq P'$. By possibly swapping P_1 and P_2 and reversing Q, we may assume that $\gamma(P_2)\neq0$ and $P_1\neq P'$. Let $\mathcal{P}''$ be the A-collection obtained from $\mathcal{P}$ by replacing P and Q with P_2 and the trivial path (u_0). Note that, $B(\mathcal{P})-B(\mathcal{P}'')=\{(u,\alpha)\}$. Moreover, since $P_1\neq P'$ there is an edge of $Q[u_0,u_j]$ that is not in $E(\mathcal{P}'')\cup E(P')$. Thus, $|E(\mathcal{P}'')-E(\mathcal{P}')|<|E(\mathcal{P})-E(\mathcal{P}')|$; contradicting 4.4. This final contradiction completes the proof. ∎

5. Proofs of the main results

Let Γ be a group, $G=(V,E)$ be an oriented graph with edge-labels $(\gamma_e : e\in E)$ from Γ, and $A\subseteq V$. The triple (G,A,γ) is *critical* if

(i) G is connected,
(ii) $\nu(G-\{v\},A-\{v\},\gamma)=\nu(G,A,\gamma)$ for each $v\in V$,
(iii) for each $v\in V-A$ and for any edge-labelling γ' obtained from γ by shifting at v we have $\nu(G,A\cup\{v\},\gamma')>\nu(G,A,\gamma)$, and
(iv) $E(A,\gamma)=\emptyset$.

In Section 3 we defined "coloops" and "series pairs"; in this section we require the dual notions, "loops" and "parallel pairs". Let M be a matroid with ground set S and let $u,v\in S$. Then, u is a *loop* of M if u is not in any basis of M. The elements u and v are *parallel* if neither u nor v are loops, but there is no basis that contains both u and v. Parallel pairs are transitive; that is, if u is parallel with v and v is parallel with w $(u\neq w)$, then u is parallel with w.

Lemma 5.1. *Let Γ be a group, let $G=(V,E)$ be an oriented graph with edge labels $(\gamma_e : e\in E)$ from Γ, and let $A\subseteq X$. If (G,A,γ) is critical, then* $\operatorname{def}(G,A,\gamma)=1$ *and, hence, $|A|$ is odd.*

Proof. Suppose that (G,A,γ) is critical, and let $M=(S,\mathcal{B})$ be the matroid obtained from (G,A,γ) via Lemma 4.1. Let S' denote the set of all non-loop elements of M.

Claim 5.2. *Let e be and edge of G with tail u and head v, and let $(u,\alpha),(v,\beta)\in S'$. If $\alpha+\gamma_e-\beta\neq 0$, then (u,α) and (v,β) are parallel.*

Subproof. If (u,α) and (v,β) are not parallel, then there is a basis of M that contains them both. That is, there is an optimal A-collection $\mathcal{P}$ with $(u,\alpha),(v,\beta)\in B(\mathcal{P})$. Now, let P_u and P_v be the paths in $\mathcal{P}$ containing u and v respectively. Note that, $P=(P_u,e,\bar{P}_v)$ is an A-path with $\gamma(P)=\alpha+\gamma_e-\beta$. Then, since $\mathcal{P}$ is optimal, we have $\alpha+\gamma_e-\beta=0$, as required. ∎

Claim 5.3. *For each $v\in A$, we have $(v,0)\in S'$.*

Subproof. Since (G,A,γ) is critical, $\nu(G-v,A-v,\gamma)=\nu(G,A,\gamma)$. Thus, there exists a set $\mathcal{P}$ of $\nu(G,A,\gamma)$ non-zero A-paths each disjoint from v. Now, adding trivial A-paths to $\mathcal{P}$ we obtain an optimal A-collection $\mathcal{P}'$ with $(v,0)\in B(\mathcal{P}')$. Thus, $(v,0)\in S'$, as required. ∎

Claim 5.4. *For each $v\in V-A$, there exist two distinct elements $(v,\alpha),(v,\beta)\in S'$.*

Subproof. Consider any element $\delta \in \Gamma$, and let γ' be the edge labels obtained from γ by shifting at v by δ. Since (G,A,γ) is critical, $\nu(G,A\cup\{v\},\gamma') = \nu(G,A,\gamma)+1$. Let $\mathcal{P}$ be an optimal $A\cup\{v\}$-collection with respect to γ'. Since $\nu(G,A\cup\{v\},\gamma') > \nu(G,A,\gamma)$, v is the start or end of an $A\cup\{v\}$-path P in $\mathcal{P}$; by possibly reversing P we may assume that v is the end. Then, $\mathcal{P}$ is an optimal A-path collection in G and $\gamma'(P)=\gamma(P)+\delta\neq 0$. Now, $(v,\gamma(P))\in S'$ and $\gamma(P)\neq -\delta$. Since δ is any element of Γ, there must exist two distinct elements $(v,\alpha),(v,\beta)\in S'$. ∎

Claim 5.5. *Let e be an edge with tail u and head v. Then, there exist $(u,\alpha_u),(v,\alpha_v)\in S'$ that are parallel in M.*

Subproof. First suppose that $u,v\in A$. Let $\alpha_v=\alpha_u=0$. Since (G,A,γ) is critical, $0\neq\gamma_e=\alpha_u+\gamma_e-\alpha_v$. Then, by 5.2, (u,α_u) and (v,α_v) are parallel. Now we may assume that $u\notin A$ or $v\notin A$; by symmetry we may assume that $v\notin A$. Now, by 5.3 and 5.4, there exists $\alpha_u\in\Gamma$ such that $(u,\alpha_u)\in S'$, and, by 5.4, there exists $\alpha_v\in\Gamma$ such that $(v,\alpha_v)\in S'$ and $\alpha_v\neq\alpha_u+\gamma_e$. Then, by 5.2, (u,α_u) and (v,α_v) are parallel. ∎

For each $v\in V$, let $S'_v=\{(u,\alpha)\in S' : u=v\}$. Consider an optimal A-collection $\mathcal{P}$. Since there is at most one path in $\mathcal{P}$ that ends at v, $|B(\mathcal{P})\cap S'_v|\leq 1$. Thus, any two elements of S'_v are in parallel. Then, by 5.5 and since G is connected, each pair of elements in S' are parallel. Thus, if $\mathcal{P}$ is an optimal A-collection, then $|B(\mathcal{P})|=1$ and, hence, $\mathrm{def}(G,A,\gamma)=1$, as required. ∎

Proof of Theorem 1.2. Choose $X\subseteq V$ maximal such that $\nu(G-X,A-X,\gamma)=\nu(G,A,\gamma)-|X|$. Now among all sets $A'\subseteq V-X$ with $A-X\subseteq A'$ and edge-labellings γ' obtained from γ by shifting on the vertices in $A'-A$ such that $\nu(G-X,A',\gamma')=\nu(G-X,A-X,\gamma)$ we choose the pair (A',γ') with A' as large as possible. Now let $H_1,\dots,H_l$ be the components of $G-X-E(A',\gamma')$ and let A'_i denote $A'\cap V(H_i)$. Note that,

$$\nu(G,A,\gamma') = |X| + \sum_{i=1}^{l}\nu(H_i,A'_i,\gamma').$$

By our choice of X and A', it is easy to check that each of the triples (H_i,A'_i,γ') is critical. Then, by Lemma 5.1, $\nu(H_i,A'_i,\gamma')=\left\lfloor\frac{|A'_i|}{2}\right\rfloor$. So,

$$\nu(G,A,\gamma') = |X| + \sum_{i=1}^{l}\left\lfloor\frac{|A'_i|}{2}\right\rfloor,$$

as required. ∎

Proof of Theorem 1.1. By Theorem 1.2 there exist edge-labels $(\gamma'_e : e \in E)$ obtained by shifting γ at vertices in $V - A$ and there exist sets $X, A' \subseteq V$ such that $A - X \subseteq A' \subseteq V - X$ and

$$\nu(G,A,\gamma) = |X| + \sum_{i=1}^{l} \left\lfloor \frac{|V(H_i) \cap A'|}{2} \right\rfloor,$$

where $H_1, \ldots, H_l$ are the components of $G - X - E(A', \gamma')$. Let $i \in \{1, \ldots, l\}$ and let A'_i denote $V(H_i) \cap A'$. Now, let $X_i \subseteq A'_i$ with $|X_i| = |A'_i| - 1$, and let $X^* = X \cup X_1 \cup \cdots \cup X_l$. Note that, $\nu(H_i - X_i, A'_i - X_i, \gamma') = 0$ since $|A'_i - X_i| = 1$. Now,

$$\begin{aligned}
\nu(G - X^*, A - X^*, \gamma) &= \nu(G - X^*, A - X^*, \gamma') \\
&\le \nu(G - X^*, A' - X^*, \gamma') \\
&= \nu(G - X^* - E(A', \gamma'), A' - X^*, \gamma') \\
&\le \sum_{i=1}^{l} \nu(H_i - X_i, A'_i - X_i, \gamma') \\
&= 0.
\end{aligned}$$

Thus, X^* meets every non-zero A-path in G. Suppose that $\nu(G, A, \gamma) < k$. Then,

$$\begin{aligned}
2k - 2 &\ge 2\nu(G, A, \gamma) \\
&= 2|X| + \sum_{i=1}^{l} 2 \left\lfloor \frac{|A'_i|}{2} \right\rfloor \\
&\ge |X| + \sum_{i=1}^{l} (|A_i| - 1) \\
&= |X| + \sum_{i=1}^{l} |X_i| \\
&= |X^*|,
\end{aligned}$$

as required. ∎

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