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Inequivalent representations of matroids over prime fields

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ABSTRACT

It is proved that for each prime field $GF(p)$, there is an integer n_p such that a 4-connected matroid has at most n_p inequivalent representations over $GF(p)$. We also prove a stronger theorem that obtains the same conclusion for matroids satisfying a connectivity condition, intermediate between 3-connectivity and 4-connectivity that we term "k-coherence".

We obtain a variety of other results on inequivalent representations including the following curious one. For a prime power q, let $\mathcal{R}(q)$ denote the set of matroids representable over all fields with at least q elements. Then there are infinitely many Mersenne primes if and only if, for each prime power q, there is an integer m_q such that a 3-connected member of $\mathcal{R}(q)$ has at most m_q inequivalent $GF(7)$-representations.

The theorems on inequivalent representations of matroids are consequences of structural results that do not rely on representability. The bulk of this paper is devoted to proving such results.

Chapter 1. Introduction

In this paper we prove the following theorem.

Theorem 1.1. *Let p be a prime number. Then there is an integer $\gamma(p)$ such that a 4-connected matroid has at most $\gamma(p)$ inequivalent $GF(p)$-representations.*

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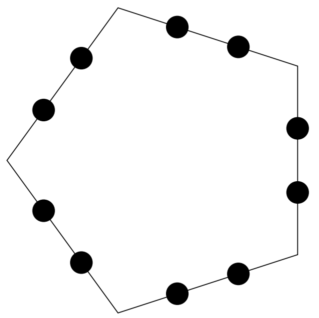

Fig. 1.1. A rank-5 free swirl.

We also prove a somewhat stronger theorem that obtains the same conclusion for a weaker notion of connectivity. Before discussing this, and other results in this paper, we provide some background.

It is easily seen that if a matroid M is binary, then M is uniquely representable over any field for which it is representable. It is also straightforward to show that ternary matroids are uniquely representable over $GF(3)$. In [15], Kahn proved that 3-connected quaternary matroids are uniquely representable over $GF(4)$. In that paper he made the conjecture that for any finite field $GF(q)$, there is an integer $\mu(q)$ such that a 3-connected matroid has at most $\mu(q)$ inequivalent $GF(q)$-representations. In [25] Oxley, Vertigan and Whittle proved that Kahn's Conjecture holds for $GF(5)$, but, unfortunately, examples in that paper show that the conjecture fails for all fields larger than $GF(5)$.

One may hope to recover the situation by increasing the connectivity. What about 4-connected matroids? For non-prime fields, the situation is somewhat dire. It is shown in [10] that there are 4-connected matroids with an arbitrary number of inequivalent representations over any non-prime field with at least 9 elements. Indeed, for $n \geqslant m$, there is a vertically $(m+1)$-connected matroid that has at least 2^{n-1} inequivalent representations over any finite field of non-prime order $q \geqslant m^m$.

As Theorem 1.1 shows, the situation is much better for prime fields. Before turning to a more detailed discussion of the contents of this paper we mention a significant application of the results of this paper. Seymour [28] showed that in the worst case it requires exponentially many rank evaluations to prove that a matroid is binary and this negative result extends easily to other fields [14]. In contrast to this, it is proved in [14, Theorem 1.1] that, for any prime p, an n-element matroid can be proved to be not representable over $GF(p)$ using only $O(n^2)$ rank evaluations. Results from this paper form an essential ingredient in the proof of this result.

We also obtain other consequences that we believe are interesting in their own right. Here is one. Recall that a *Mersenne prime* is one that has the form $2^n - 1$ for some integer n. A very well-known conjecture is that the number of Mersenne primes is infinite. For a prime power q, let $\mathcal{R}(q)$ denote the set of matroids representable over all fields with at least q elements.

Theorem 1.2. *There are infinitely many Mersenne primes if and only if, for each prime power q, there is an integer m_q such that a 3-connected member of $\mathcal{R}(q)$ has at most m_q inequivalent $GF(7)$-representations.*

The results that we have highlighted so far are all to do with representations of matroids over fields. It is striking that it is not until the short last chapter of this paper that fields play a role. Until then all of our results are purely structural and are valid independently of any assumptions about representability. The potential for a matroid to have inequivalent representations is due to certain structural features that are inherent in the matroid. These are the features that we focus on. The applications to fields are obtained as consequences of our structural results.

We now outline our stronger theorem on inequivalent representations. Let M be a matroid with a basis $B = \{b_1, b_2, \dots, b_n\}$ such that, for all $i \in \{1, 2, \dots, n-1\}$, there are elements $P_i = \{p_i, q_i\}$ placed freely on the line spanned by $\{b_i, b_{i+1}\}$ and a pair of points $P_n = \{p_n, q_n\}$ placed freely on the line spanned by $\{b_n, b_1\}$. Then the matroid $M \backslash B$ is the rank-n *free swirl*, denoted Δ_n. Fig. 1.1 illustrates Δ_5. For prime fields, the class of free swirls provide the counterexamples to Kahn's Conjecture given

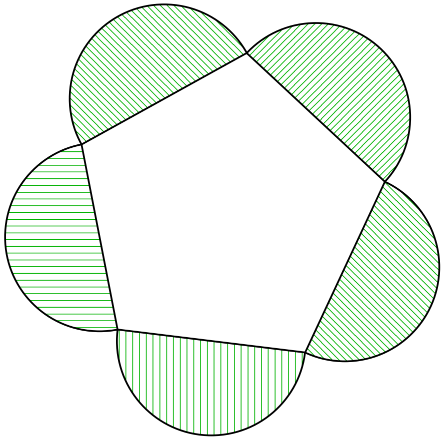

Fig. 1.2. A swirl-like flower.

in [25]. It is natural to ask if all such counterexamples are, in some sense, related to free swirls. In this paper we provide what is, essentially, a positive answer to that question.

Note that a set X of elements of Δ_n is non-trivially 3-separating if and only if it is a union of members of $(P_1, P_2, \dots, P_n)$ that are consecutive in the cyclic order. More generally, a 3-connected matroid M has a *swirl-like flower* with n petals if there is a partition $\mathbf{P} = (P_1, P_2, \dots, P_n)$ of $E(M)$ into exactly 3-separating sets called the *petals* of $\mathbf{P}$ such that a union of petals is 3-separating if and only if it is consecutive in the cyclic order. Fig. 1.2 illustrates a swirl-like flower with five petals. It is possible for petals to be degenerate in a way that we explain later. If $n \geqslant 4$, then the *order* of a swirl-like flower is the number of non-degenerate petals it has. To control inequivalent representations we control swirl-like flowers. Let $k \geqslant 5$ be an integer. Then a matroid is *k-coherent* if it is 3-connected and has no swirl-like flowers of order k. The main theorem of this paper is really the following.

Theorem 1.3. *Let $k \geqslant 5$ be an integer and p be a prime number. Then there is a function $\gamma(k, p)$ such that a k-coherent matroid has at most $\gamma(k, p)$ inequivalent representations over $GF(p)$.*

As 4-connected matroids are k-coherent for any $k \geqslant 5$, Theorem 1.1 is an immediate corollary of Theorem 1.3. While Theorem 1.3 is somewhat more technical than Theorem 1.1, it is considerably stronger. We also observe that it is, in general, easiest to prove a theorem for the weakest version of connectivity for which the theorem is true. This is because weaker notions of connectivity are usually easier to keep in minors and therefore facilitate inductive arguments. We know of no way to obtain a bound on the number of inequivalent representations of 4-connected matroids other than as a consequence of a stronger theorem using a weaker connectivity notion.

We now consider the structure of this paper. It was always clear that this was going to be a long paper, although it never occurred to us that it would be *this* long. Our original intention was to partition it into a sequence of papers, but the interconnectivity of the material was so high that this strategy seemed increasingly artificial and we eventually abandoned it. In the discussion that follows we use loosely a number of terms that are defined formally later in the paper.

Chapter 2 contains known material, mainly on connectivity, that is fundamental to this paper. Flowers are structures in matroids that arise when 3-separations cross. Flowers come in several different types and there is a natural equivalence and partial order on the flowers in a matroid. Flowers were introduced and studied in [20] and further studied in [21]. It seems that just about every known elementary fact on flowers is needed at some stage in this paper, and some facts are needed many times. Chapter 3 is primarily a survey of basic properties of flowers.

Let $k \geqslant 5$ be an integer. As noted above, a k-coherent matroid is one that is 3-connected and has no swirl-like flower of order at least k. This is our basic notion of connectivity. Chapter 4 describes properties of this connectivity notion. If M is a k-coherent matroid and T is a triangle of M, then one would typically expect there to be an element $t \in T$ such that $M\backslash t$ is k-coherent. Unfortunately this is not always the case and triangles that do not have this property are called *k-wild*. The structure of M relative to a k-wild triangle is described. If f is an element of the k-coherent matroid M and $M\backslash f$ and M/f are both 3-connected, then one might hope that at least one of these matroids is k-coherent. Again this is not always the case. If neither $M\backslash f$ nor M/f is k-coherent, then we say that f is *feral*. The structure of M relative to a feral element is described. Feral elements and k-wild triangles arise repeatedly in proofs in later chapters of the paper. We prove a wheels and whirls type theorem for k-coherent matroids.

The underlying cause of inequivalent representations in matroids is that an element may have *freedom*. A *k-skeleton* is a k-coherent matroid whose elements have, in some sense, maximum freedom. It is easily seen, and shown in Chapter 12, that the number of inequivalent representations of a k-coherent matroid over a finite field is bounded above by the maximum of the number of inequivalent representations of its k-skeleton minors. In Chapter 5 k-skeletons and their basic properties are described. Again we find that certain structures arise that are counterexamples to expected behaviour; we call these structures *bogan couples* and *gangs of three*. In a way that is entirely analogous to the theorems for k-wild triangles and feral elements, we give theorems that describe the local structure of a matroid relative to a bogan couple or a gang of three.

Chapter 6 gives a chain theorem for k-skeletons. It is shown that if M is a k-skeleton, then unless M is trivially small, M has a k-skeleton minor N such that $|E(M) - E(N)| \leqslant 4$. Viewed from a bottom up perspective it gives us a way of building all k-skeletons in a class. A 4-element jump may seem somewhat daunting, but it is shown that the structure when 3- and 4-element jumps are required is quite specific. Indeed reasonably special structure arises even in the 2-element case. Apart from a handful of lemmas the results of Chapter 6 are not needed in the rest of the paper. We expect that these results will be used to obtain an explicit bound on the number of inequivalent representations of 4-connected matroids over $GF(7)$.

Chapter 7 marks a sharp change in the techniques of this paper. Up to this stage we have focussed on exact structure. From now on techniques are extremal. The rank-q *free spike* Λ_q is defined in Chapter 2.4. Let $\mathcal{E}(q)$ denote the class of matroids with no $U_{2,q+2}$-, $U_{q,q+2}$- or Λ_q-minor. The goal is to eventually show that there are only a finite number of k-skeletons in $\mathcal{E}(q)$. This gives immediate corollaries for matroids representable over fields as, if q is prime, the matroids representable over $GF(q)$ are a subclass of $\mathcal{E}(q)$.

A *path of* 3-*separations* in a matroid is an ordered partition of the ground set that induces a nested sequence of 3-separations in M. In Chapter 7 it is shown that a k-skeleton with a sufficiently long path of 3-separations cannot be in $\mathcal{E}(q)$. This begins the process of taming the structure of a sufficiently large k-skeleton in $\mathcal{E}(q)$. In Chapter 8 this process is continued. We show that a sufficiently large k-skeleton must contain, as a minor, a large 4-connected matroid all elements of which are neither fixed nor cofixed. This refining process is continued and it is shown that a sufficiently large 4-connected matroid with the above property has a large 4-connected minor whose ground set contains many clonal pairs.

The refining process is further continued in Chapter 9 where we consider unavoidable minors of 4-connected matroids whose ground set contains many clonal pairs. In fact it suffices to focus on 3-connected matroids whose ground set has a partition into clonal pairs. It is shown that a sufficiently large such matroid M in $\mathcal{E}(q)$ must have a large free-swirl minor. Moreover, we can also guarantee that all of the clonal pairs in this minor are clonal pairs in M.

We now have a guaranteed large free-swirl minor. If we delete a clonal pair from this minor we obtain a matroid with a natural partition that gives a certain type of path of 2-separations. This is a minor of a matroid in which all of the 2-separations are bridged. Moreover the minor has a partition into clonal pairs that remain clonal pairs in the large matroid. In Chapter 10 it is shown that if the path of 2-separations is sufficiently long, then unless the 2-separations are bridged in a very specific way, the bridging matroid cannot be in $\mathcal{E}(q)$.

Chapter 11 returns attention to our large free-swirl minor. Such a matroid displays a path of 3-separations. These 3-separations are bridged by our larger matroid. Here we obtain the final win by showing that, if the swirl is sufficiently large and the clonal pairs of the free swirl remain clonal in the matroid M that bridges the separations, then M cannot be in $\mathcal{E}(q)$. In other words there are a finite number of k-skeletons in $\mathcal{E}(q)$.

Having achieved the climax of the paper—which has no doubt been reached in a fever pitch of excitement—the reader can relax and enjoy the denouement which consists of the applications to fields given in the final chapter. Alternatively one could cheat and begin by reading the last chapter first. This strategy is recommended.

Chapter 2. Background material

It is assumed that the reader is familiar with matroid theory as set forth in Oxley [19]. Terminology and notation generally follows [19]. This chapter attempts to cover some of the additional terminology and known results that we will need. It is largely an accumulation of material scattered in research papers, although results covered in [19] that are of particular significance here are restated. Much of it will be needed throughout although some of it is not needed until later in the paper; for example blocking sequences are not used until Chapter 9. It just seemed more natural to include blocking sequences in a chapter primarily concerned with connectivity techniques. On the other hand some introductory material is delayed; for example basic facts on freedom in matroids are not introduced until Chapter 5.

We note that duality plays an integral role throughout this paper. Almost all of our concepts and results are either self-dual or have dual formulations that need to be grasped. This applies, for example, to something as elementary as the closure operator of a matroid where we need to understand the behaviour of the dual operator, the *coclosure* operator. We often neglect to state dual versions of results and in any unexplained context the phrase "by Lemma x" should always be taken to mean "by Lemma x or its dual".

We note that, as with all matroid structure theory, in one way or another it is all about connectivity.

Let M be a matroid on ground set E with rank function r. The *connectivity function* λ_M of M is defined, for all subsets A of E, by $\lambda_M(A) = r(A) + r(E - A) - r(M)$. If the matroid is clear from the context then $\lambda_M(A)$ will be denoted by $\lambda(A)$. We extend the notation to a partition $(A, E - A)$ of E, by defining $\lambda_M(A, E - A) = \lambda_M(A)$.

The set A or the partition $(A, E - A)$ is *k-separating* if $\lambda(A) < k$. The partition $(A, E - A)$ is a *k-separation* if A is k-separating and $|A|, |E - A| \geqslant k$. Fussing about the distinction between 3-separating partitions and 3-separations leads to constipated prose, but the distinction is at times important, so we appear to be stuck with it. The matroid M is *k-connected* if it has no $(k-1)$-separations. A k-separating set A, or k-separation $(A, E - A)$ is *exact* if $\lambda(A) = k - 1$. A matroid is *connected* if it is 2-connected.

It is immediate from the definition that the connectivity function of a matroid is *symmetric*, that is, $\lambda_M(X) = \lambda_M(E - X)$ for all subsets X of E. Moreover, one readily checks that if r^* is the rank function of the dual M^* of the matroid M, then $\lambda_M(X) = r(X) + r^*(X) - |X|$ for all subsets X of E. This establishes the next lemma.

Lemma 2.1. *For any matroid M we have $\lambda_M = \lambda_{M^*}$.*

We freely use Lemma 2.1 without mention throughout the paper. Another elementary fact about λ is that it is monotone under minors.

Lemma 2.2. *Let N be a minor of the matroid M. Then $\lambda_N(A) \leqslant \lambda_M(A)$ for any subset A of $E(N)$.*

An easy rank calculation proves that λ is *submodular*, that is $\lambda(X) + \lambda(Y) \geqslant \lambda(X \cup Y) + \lambda(X \cap Y)$ for all $X, Y \subseteq E$. The submodularity of λ is frequently used to establish that certain sets have bounded connectivity. The next lemma is an instance of this.

Lemma 2.3. *Let M be a matroid and let X and Y be subsets of $E(M)$ such that $\lambda(X) = \lambda(Y) = 2$. If $\lambda(X \cup Y) \geqslant 2$, then $\lambda(X \cap Y) \leqslant 2$. In particular, if M is 3-connected and $\lambda(X) = \lambda(Y) = 2$, the following hold.*

(i) *If* $|X \cap Y| \geqslant 2$, *then* $X \cup Y$ *is* 3*-separating.*
(ii) *If* $|E(M) - (X \cup Y)| \geqslant 2$, *then* $X \cap Y$ *is* 3*-separating.*

We make frequent use of Lemma 2.3 and often write *by uncrossing* to mean "by an application of Lemma 2.3".

Keeping connectivity A key role that connectivity plays in matroid structure theory is to eliminate degeneracies caused by low-order separations. This is precisely the role played by connectivity when sufficient connectivity enables us to bound the number of inequivalent representations of a matroid. To make inductive arguments possible it is necessary to have theorems that enable us to remove elements keeping a given type of connectivity. It seems that Tutte was the first to appreciate the need for such results. Indeed a number of the results proved by Tutte have become basic tools. The following is the most elementary.

Lemma 2.4. *Let* M *be a connected matroid. Then, for any element* e *of* M*, either* $M\backslash e$ *or* M/e *is connected.*

Let $n \geqslant 2$ be an integer. Recall that the *wheel* W_n is the graph consisting of a cycle of length n together with another vertex v that is incident with all of the vertices in the cycle. The *rim* edges of W_n are the edges in the cycle. The remaining edges are the *spoke* edges. If we lapse and say that a matroid is a wheel we mean that it is the cycle matroid of a wheel. The rim edges of W_n form a circuit-hyperplane of $M(W_n)$. The *whirl* W^n is obtained from $M(W_n)$ by declaring this circuit-hyperplane to be a basis; see [19, Chapter 8.4]. The next theorem of Tutte [30] is fundamental.

Theorem 2.5 *(Tutte's Wheels and Whirls Theorem). Let* M *be a* 3*-connected matroid. Then, unless* M *is a whirl or the cycle matroid of a wheel, there is an element* e *of* M *such that either* $M\backslash e$ *or* M/e *is* 3*-connected.*

A set X of a matroid M is a *parallel set* if every 2-element subset of X is a circuit. A *parallel class* of M is a maximal parallel set. Dually, X is a *series set* of M if every 2-element subset of M is a cocircuit. A *series class* of M is a maximal series set. A matroid M is 3*-connected up to parallel pairs* if whenever (X, Y) is a 2-separation of M, then either X or Y is a parallel pair and is 3*-connected up to series pairs* if whenever (X, Y) is a 2-separation of M, then either X or Y is a series pair. Bixby [3] proved the next very useful result.

Theorem 2.6 *(Bixby's Lemma). Let* e *be an element of the* 3*-connected matroid* M*. Then either* $M\backslash e$ *is* 3*-connected up to series pairs or* M/e *is* 3*-connected up to parallel pairs.*

To say that a matroid M has the matroid N *as a minor* means that we may delete or contract elements from M to obtain a matroid equal to N. To say that M has an N*-minor* means that M has a matroid isomorphic to N as a minor. It is often the case that we would like to keep connectivity and keep a minor. Typically isomorphism needs to be invoked. The version of Seymour's Splitter Theorem [27] below is not the strongest possible. See Oxley [19, Chapter 11] for a more detailed discussion of the Splitter Theorem and its consequences.

Theorem 2.7 *(Seymour's Splitter Theorem). Let* N *be a* 3*-connected matroid that is not a wheel or a whirl. If* N *is a proper minor of the* 3*-connected matroid* M*, then there is an element* e *of* M *such that either* $M\backslash e$ *or* M/e *is* 3*-connected with an* N*-minor.*

Tutte's Linking Theorem Let M be a matroid and let X and Y be disjoint subsets of $E(M)$. We let $\kappa_M(X, Y) = \min(\lambda_M(A)\colon\ X \subseteq A \subseteq E(M) - Y)$. If the matroid M is clear we abbreviate κ_M to κ. Intuitively $\kappa(X, Y)$ measures the connectivity between X and Y provided by the rest of the matroid.

If N is a minor of M and $X, Y \subseteq E(N)$, then $\kappa_N(X, Y) \leqslant \kappa_M(X, Y)$. The next theorem provides a good characterisation for $\kappa_M(X, Y)$. This theorem is, in fact, a generalisation of Menger's Theorem.

Theorem 2.8 *(Tutte's Linking Theorem). (See [29].) Let M be a matroid and let X and Y be disjoint subsets of $E(M)$. Then there exists a minor N on $X \cup Y$ such that $\lambda_N(X) = \kappa_M(X, Y)$.*

The following lemma shows that if we apply Tutte's Linking Theorem when $\lambda_M(X) = \kappa_M(X, Y)$, the resulting minor N satisfies $M|X = N|X$.

Lemma 2.9. *Let N be a minor of a matroid M and let X be a subset of $E(N)$. If $\lambda_M(X) = \lambda_N(X)$, then $M|X = N|X$.*

Local connectivity For subsets X and Y in M, the *local connectivity* between X and Y, denoted $\sqcap_M(X, Y)$, or $\sqcap(X, Y)$ if the matroid is clear from the context, is defined by $\sqcap_M(X, Y) = r(X) + r(Y) - r(X \cup Y)$. Evidently $\sqcap_M(X, Y) = \lambda_{M|(X\cup Y)}(X, Y)$. We denote $\sqcap_{M^*}(X, Y)$ by $\sqcap^*_M(X, Y)$. The next lemma follows from an easy rank calculation.

Lemma 2.10. *Let x be an element of the matroid M and let A and B be disjoint subsets of $E(M) - \{x\}$. Then*

(i) $\sqcap_{M/x}(A, B) = \sqcap_M(A, B)$ *if either* $x \notin \mathrm{cl}(A \cup B)$ *or* $x \in \mathrm{cl}(A)$, *but* $x \notin \mathrm{cl}(B)$;
(ii) $\sqcap_{M/x}(A, B) = \sqcap_M(A, B) + 1$ *if* $x \in \mathrm{cl}(A \cup B)$ *but* $x \notin \mathrm{cl}(A)$ *and* $x \notin \mathrm{cl}(B)$; *and*
(iii) $\sqcap_{M/x}(A, B) = \sqcap_M(A, B) - 1$ *if* $x \in \mathrm{cl}(A)$ *and* $x \in \mathrm{cl}(B)$.

Let $\{A, B, C\}$ be a partition of the ground set E of a matroid M. Then A and B are *skew* in M if $r(A \cup B) = r(A) + r(B)$ and are *coskew* in M if they are skew in M^*. Equivalently, A and B are skew if $\sqcap_M(A, B) = 0$ and are coskew if $\sqcap^*_M(A, B) = 0$.

Recall that sets X and Y in a matroid M form a *modular pair* if $r(X) + r(Y) = r(X \cup Y) + r(X \cap Y)$. An easy argument from duality proves

Lemma 2.11. *Let M be a matroid and let $\{A, B, C\}$ be a partition of $E(M)$. Then the following are equivalent.*

(i) *A and B are coskew in M.*
(ii) $r_{M/C}(A) + r_{M/C}(B) = r_{M/C}(A \cup B)$.
(iii) $r(A \cup C) + r(B \cup C) = r(M) + r(C)$.
(iv) *$E - A$ and $E - B$ form a modular pair in M.*

The next lemma is proved in [20].

Lemma 2.12. *For disjoint subsets X and Y of M,*

$$\lambda(X \cup Y) = \lambda(X) + \lambda(Y) - \sqcap(X, Y) - \sqcap^*(X, Y).$$

We will say that A and B are *fully skew* if they are both skew and coskew. An immediate corollary of Lemma 2.12 and the submodularity of the connectivity function is

Corollary 2.13. *A and B are fully skew if and only if $\lambda(A \cup B) = \lambda(A) + \lambda(B)$.*

1. Structure related to connectivity

Sequential and equivalent 3-separations Let A be a set in a matroid M. The *coclosure* $\mathrm{cl}^*(A)$ of A is the closure of A in M^*. If $\mathrm{cl}^*(A) = A$, then A is *coclosed* in M. Let $(A, \{x\}, B)$ be a partition of $E(M)$. Then $x \in \mathrm{cl}(A)$ if and only if x is a loop of M/A. So by duality, $x \in \mathrm{cl}^*(A)$ if and only if x is a coloop of $M\backslash A = M|(A \cup \{x\})$. In other words, we have

Lemma 2.14. *If $(A, \{x\}, B)$ is a partition of $E(M)$, then $x \in \mathrm{cl}^*(A)$ if and only if $x \notin \mathrm{cl}(B)$.*

Throughout this paper we freely use properties of coclosure that are obtained by dualising standard properties of closure. For example, $x \in \mathrm{cl}^*_{M\backslash a}(Z)$ if and only if $x \in \mathrm{cl}^*_M(Z \cup \{a\})$.

If A is both closed and coclosed in M, then A is *fully closed*. The *full closure* of a set A, denoted $\mathrm{fcl}_M(A)$, or $\mathrm{fcl}(A)$ if M is clear from the context, is the intersection of all the fully-closed sets containing A. Evidently the full closure is a closure operator in that it satisfies the properties that $\mathrm{fcl}(A) \supseteq A$ and $\mathrm{fcl}(\mathrm{fcl}(A)) = \mathrm{fcl}(A)$ for all subsets A of $E(M)$. The set A is *cohesive* if $E(M) - A$ is fully closed.

We use the notation $x \in \mathrm{cl}^{(*)}(A)$ as a shorthand way of saying that either $x \in \mathrm{cl}(A)$ or $x \in \mathrm{cl}^*(A)$. Note that $x \in \mathrm{cl}^{(*)}(A)$ if and only if $\lambda(A \cup \{x\}) \leqslant \lambda(X)$.

Lemma 2.15. *Let A be a set of elements of the matroid M. Then the following are equivalent*:

(i) $B = \mathrm{fcl}(A)$.
(ii) *B is a maximal set containing A, for which there is an ordering $(a_1, \dots, a_n)$ of $B - A$ such that $a_i \in \mathrm{cl}^{(*)}(A \cup \{a_1, \dots, a_{i-1}\})$ for all $i \in \{1, \dots, n\}$.*
(iii) *B is a maximal set containing A for which there is an ordering $(a_1, \dots, a_n)$ of $B - A$ such that $\lambda(A \cup \{a_1, \dots, a_i\}) \leqslant \lambda(A \cup \{a_1, \dots, a_{i-1}\})$ for all $i \in \{1, \dots, n\}$.*

Via the full closure operator we can obtain an equivalence on 3-separating sets of a 3-connected matroid M as follows. Say $\lambda(A) = \lambda(B) = 2$. Then A is *equivalent* to B, denoted $A \cong B$, if $\mathrm{fcl}(A) = \mathrm{fcl}(B)$. Say that (A_1, A_2) and (B_1, B_2) are exactly 3-separating partitions in M. Then (A_1, A_2) is *equivalent* to (B_1, B_2), denoted $(A_1, A_2) \cong (B_1, B_2)$ if, for some ordering (C_1, C_2) of $\{B_1, B_2\}$, we have $A_1 \cong C_1$ and $A_2 \cong C_2$.

Let X be a 3-separating set of the 3-connected matroid M. Then X is *sequential* if it has an ordering $(x_1, \dots, x_n)$ such that $\{x_1, \dots, x_i\}$ is 3-separating for all $i \in \{1, \dots, n\}$. By the symmetry of the connectivity function, X is sequential if and only if $\mathrm{fcl}(E(M) - X) = E(M)$. Under the equivalence defined earlier one can regard sequential 3-separating partitions as being equivalent to trivial 3-separating partitions. If X is not sequential, then we say that X is a *non-sequential* 3-separating set. A 3-separation (X, Y) of M is *non-sequential* if neither X nor Y is sequential; otherwise (X, Y) is *sequential*. Thus (X, Y) is non-sequential if $\mathrm{fcl}(X) \neq E(M)$ and $\mathrm{fcl}(Y) \neq E(M)$.

The next lemma gives a test for equivalence of 3-separations.

Lemma 2.16. *Let (A_1, A_2) and (B_1, B_2) be 3-separating partitions of the 3-connected matroid M.*

(i) *If (A_1, A_2) is sequential, then $(A_1, A_2) \cong (B_1, B_2)$ if and only if there is an ordering (C_1, C_2) of $\{B_1, B_2\}$ such that $\mathrm{fcl}(A_1) = \mathrm{fcl}(C_1)$ and $\mathrm{fcl}(A_2) = \mathrm{fcl}(C_2)$.*
(ii) *If (A_1, A_2) is non-sequential, then $(A_1, A_2) \cong (B_1, B_2)$ if and only if there is an ordering (C_1, C_2) of $\{B_1, B_2\}$ such that $\mathrm{fcl}(A_1) = \mathrm{fcl}(C_1)$.*

Guts, coguts, blocking, coblocking Let $(A, \{x\}, B)$ be a partition of the ground set of the matroid M. Assume that $\lambda(A) = \lambda(A \cup \{x\}) = k$. Then either $x \in \mathrm{cl}(A)$, in which case we say that x is in the *guts* of $(A, B \cup \{x\})$, or $x \in \mathrm{cl}^*(A)$, in which case we say that x is in the *coguts* of $(A, B \cup \{x\})$. Note that, as $\lambda(A) = \lambda(A \cup \{x\})$, it is also the case that if x is in the guts of $(A, B \cup \{x\})$, then $x \in \mathrm{cl}(B)$ and if x is in the coguts of $(A, B \cup \{x\})$, then $x \in \mathrm{cl}^{(*)}(B)$.

Let N be a minor of M and (A, B) be a partition of $E(N)$ such that $\lambda_N(A) = k$. Then (A, B) is *induced* in M if there is a partition (A', B') of $E(M)$ with $A \subseteq A'$ and $B \subseteq B'$ such that $\lambda_M(A') = k$. If (A, B) is not induced in M, then (A, B) is *bridged* by M.

The case when N is obtained by removing a single element from M is of particular interest and has its own terminology. Assume that $N = M\backslash x$ and that (A, B) is bridged by M. Then we say that (A, B) is *blocked* by x. At times we will say that x *blocks* A, or A is *blocked by* x to mean that (A, B) is blocked by x. On the other hand, if $N = M/x$ and (A, B) is bridged by M, then we say that (A, B) is *coblocked* by x. As with blocking we frequently say that x *coblocks* A or A is *coblocked by* x to meant that (A, B) is coblocked by x.

The next lemma follows from easy rank calculations. Recall that A is k-separating if $\lambda(A) < k$.

Lemma 2.17. *Let M be a matroid and let $(A, \{x\}, B)$ be a partition of $E(M)$ where $\lambda(A) = \lambda(A \cup \{x\}) = k$. Then the following are equivalent.*

(i) *x is in the guts of $(A, B \cup \{x\})$.*
(ii) *$x \in \mathrm{cl}(A)$ and $x \in \mathrm{cl}(B)$.*
(iii) *$x \notin \mathrm{cl}^*(A)$ and $x \notin \mathrm{cl}^*(B)$.*
(iv) *If x is not a loop of M, then (A, B) is a k-separation of M/x that is coblocked by x.*

Dualising we obtain:

Lemma 2.18. *Let M be a matroid and let $(A, \{x\}, B)$ be a partition of $E(M)$ where $\lambda(A) = \lambda(A \cup \{x\}) = k$. Then the following are equivalent.*

(i) *x is in the coguts of $(A, B \cup \{x\})$.*
(ii) *$x \in \mathrm{cl}^*(A)$ and $x \in \mathrm{cl}^*(B)$.*
(iii) *$x \notin \mathrm{cl}(A)$ and $x \notin \mathrm{cl}(B)$.*
(iv) *If x is not a coloop of M, then (A, B) is a k-separation of $M\backslash x$ that is blocked by x.*

The fact that to coblock the k-separation (A, B) of M/x we must have $x \in \mathrm{cl}_M(A)$ and $x \in \mathrm{cl}_M(B)$ is used many times in this paper. We similarly use the dual observation on blocking, although rather than observe that $x \in \mathrm{cl}^*_M(A)$ and $x \in \mathrm{cl}^*_M(B)$ we more typically make the equivalent observation that $x \notin \mathrm{cl}_M(A)$ and $x \notin \mathrm{cl}_M(B)$; no doubt this is because we are, after all, more habituated to the closure operator.

Exposed 3-separations Let N be a 3-connected minor of the 3-connected matroid M. A 3-separation (A, B) of N is *exposed* in N if every 3-separating partition of N that is equivalent to (A, B) is bridged in M. Again the case where N is obtained by deleting or contracting a single element is of most interest. If (A, B) is exposed in N and $N = M\backslash x$ or M/x for some element x, then we say that (A, B) is *exposed by x*. The next lemma is immediate.

Lemma 2.19. *Let M be a 3-connected matroid with an element x such that $M\backslash x$ is 3-connected. If the 3-separation (A, B) is exposed by x, then (A, B) is non-sequential.*

It is shown in [24] that, if M is a 3-connected matroid that is not a wheel or a whirl, then there is an element x in $E(M)$ such that either $M\backslash x$ or M/x is 3-connected and does not expose any 3-separations. This result has an important application in this paper and we state it formally as Theorem 4.3 immediately prior to its use.

If we are looking from a different perspective we alter the terminology. Let x be an element of the 3-connected matroid M such that $M\backslash x$ is 3-connected. Let (A, B) be a 3-separation of $M\backslash x$. Then we say that (A, B) is *well blocked* by x if every 3-separating partition of $M\backslash x$ that is equivalent to (A, B) is blocked by x. In other words, (A, B) is well blocked by x if and only if (A, B) is exposed by x.

Split sets A set B of a matroid M is *split* if some partition (B', B'') of B into nonempty subsets has the property that B' and B'' are fully skew. An element $b \in B$ is *isolated* in B if $B - \{b\}$ and $\{b\}$ are fully skew.

Lemma 2.20. *Let M be a 3-connected matroid.*

(i) *Let (X, Y) be a 3-separation of M. Then neither X nor Y is split.*
(ii) *Let (X, Y) be an exact 4-separation of M. Then X is split if and only if there is an element $x \in X$ that is isolated in X, and this holds if and only if $\lambda(X - \{x\}) = 2$ for some element $x \in X$.*

We can regard split 4-separating sets as being in some sense degenerate. We omit the easy proof of the next lemma.

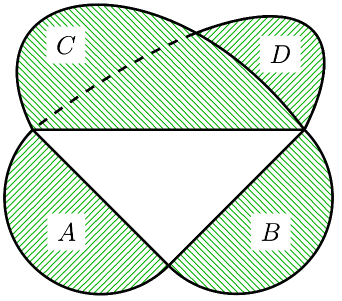

Fig. 2.1. A schematic diagram.

Lemma 2.21. *Let M be a 3-connected matroid and B be a set of elements of M such that $\lambda(B) = 3$. If $b \in B$, then the following are equivalent.*

(i) *b is isolated in B.*
(ii) *$b \notin \mathrm{cl}(B - \{b\})$ and $b \notin \mathrm{cl}^*(B - \{b\})$.*
(iii) *$b \in \mathrm{cl}(E(M) - B)$ and $b \in \mathrm{cl}^*(E(M) - B)$.*

Lemma 2.22. *Let z be an element of the 3-connected matroid M such that M/z is 3-connected. Let (R, B) be a 3-separation of M/z that is coblocked by z. Assume that R is split in M with isolated element x. Then, in M/z, the 3-separation (R, B) is equivalent to $(R - \{x\}, R \cup \{x\})$ and x is in the guts of (R, B).*

Proof. By Lemma 2.21, $x \in \mathrm{cl}_M(E(M) - R)$, that is $x \in \mathrm{cl}_M(B \cup \{z\})$. Hence $x \in \mathrm{cl}_{M/z}(B)$, so that x is in the guts of (R, B). $\square$

As a consequence of Lemma 2.22, we obtain the useful fact that exposed 3-separations in M/z correspond to certain unsplit 4-separations in M and $M\backslash z$.

Corollary 2.23. *Let M be a 3-connected matroid, let z be an element of M for which M/z is 3-connected, and let (R, B) is a 3-separation of M/z that is exposed by z. Then the following hold.*

(i) *$(R, B \cup \{z\})$ is an unsplit 4-separation of M.*
(ii) *If $M\backslash z$ is 3-connected, then (R, B) is an unsplit 4-separation of $M\backslash z$.*

Proof. Assume that R is split in either M or $M\backslash z$. Let x be an isolated element. By Lemma 2.22, $x \in \mathrm{cl}_{M/z}(B)$ so that, in M/z, (R, B) is equivalent to $(R - \{z\}, B \cup \{z\})$. But $R - \{z\}$ is not coblocked by z, that is $\lambda_M(R - \{z\}) = \lambda_{M\backslash z}(R - \{z\}) = 2$. This contradicts the fact that (R, B) is well coblocked by z. Both parts of the lemma follow from this contradiction. $\square$

2. Schematic diagrams

Intuition for matroids can be gained by giving geometric representations. Of course geometric representations are impossible for matroids of high rank. If a matroid has 2-separations or 3-separations some attempt can be made to give geometric insight by the use of *schematic diagrams*. Such diagrams are not infallible and are never a substitute for logic, but we find them invaluable as an aid to intuition in conducting research in matroid structure theory, and we believe that they can aid the reader as well. There are no hard and fast rules for their precise interpretation and ambiguity can always threaten. With that warning we give some examples to illustrate their use. Other examples are scattered throughout the paper; indeed Figs. 1.1 and 1.2 of the introduction are schematic diagrams.

Fig. 2.1 illustrates a matroid where the sets A, B, C and D are 3-separating. Note that $(A \cup B, C \cup D)$ is also a 3-separation. The sets A, B, C and D look like planes, but should simply be interpreted

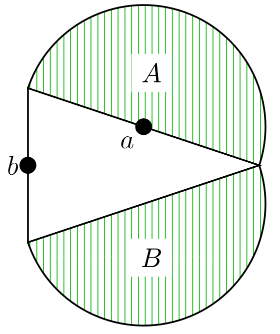

Fig. 2.2. Elements in the guts and coguts.

simply as sets having rank at least three. Note that $\sqcap(C, D) = 2$ and $\sqcap(A, B) = 1$, so that $r(M) = (r(C) + r(D) - 2) + (r(A) + r(B) - 1) - 2 = r(A) + r(B) + r(C) + r(D) - 5$. In particular, if A, B, C and D are in fact planes, then $r(M) = 7$.

Fig. 2.2 illustrates a matroid with a 3-separation $(A \cup \{a\}, B \cup \{b\})$. Note that a is in the guts of this 3-separation and b is in the coguts. This illustrates a case of Lemmas 2.17 and 2.18 as deleting b gives a schematic diagram of a matroid with a 2-separation $A \cup \{a\}$ and contracting a corresponds to projection from a and reveals a 2-separation $(A, B \cup \{b\})$. Note also that the 3-separating sets A, $A \cup \{a\}$, and $A \cup \{a, b\}$ are equivalent.

3. Blocking and bridging sequences

Blocking sequences for matroids were introduced by Geelen, Gerards and Kapoor in their proof of Rota's Conjecture for $GF(4)$ [7]. They have since proved to be a valuable technique in matroid theory. In the application for [7], matroids are given by standard matrix representations and hence come naturally with a fixed basis. Thus blocking sequences are defined relative to a fixed basis. In this section we give a basis-free version of blocking sequences that we call "bridging sequences".

Let N be a matroid with an exact k-separation (X_1, X_2), so that $\lambda_N(X_1, X_2) = k - 1$ and let M be a matroid with an N-minor. Recall that (X_1, X_2) is *bridged* in M if $\kappa_M(X_1, X_2) \geqslant k$. Let $V = (v_0, v_1, \ldots, v_p)$ be an ordering of the elements of $E(M) - E(N)$ and let S' and T' denote the elements of V with even and odd indices respectively. Then V is a *bridging sequence* for (X_1, X_2) if there is a permutation (S, T) of $\{S', T'\}$ such that the following hold:

(i) S is coindependent, T is independent and $N = M\backslash S/T$;
(ii) if $i \in \{0, 1, \ldots, p\}$, then $\lambda_M(X_1 \cup \{v_0, \ldots, v_i\}, X_2 \cup \{v_{i+1}, \ldots, v_p\}) = k$;
(iii) if $v_i \in S$, then $\lambda_{M\backslash v_i}(X_1 \cup \{v_0, \ldots, v_{i-1}\}, X_2 \cup \{v_{i+1}, \ldots, v_p\}) = k - 1$; and
(iv) if $v_i \in T$, then $\lambda_{M/v_i}(X_1 \cup \{v_0, \ldots, v_{i-1}\}, X_2 \cup \{v_{i+1}, \ldots, v_p\}) = k - 1$.

Lemma 2.24. *Let (X_1, X_2) be an exact k-separation for the matroid N and let M be a matroid having N as a minor such that there is an ordering $V = (v_0, v_1, \ldots, v_p)$ of $E(M) - E(N)$ that is a bridging sequence for (X_1, X_2). Then (X_1, X_2) is bridged in M.*

Proof. Assume that the lemma fails. Then there is a subset W of V such that $\lambda_M(X_1 \cup W) < k$. Assume that W is maximal with this property. Let i be the first integer such that $v_i \notin W$. Such an i certainly exists. By the definition of bridging sequence $\lambda(X_1 \cup \{v_0, v_1, \ldots, v_{i-1}\}) = \lambda(X_1 \cup \{v_0, v_1, \ldots, v_i\}) = k$. But then, by uncrossing, $\lambda(X_1 \cup W \cup \{v_i\}) < k$, contradicting the choice of W. $\square$

If V is a bridging sequence for (X_1, X_2) and (S, T) is the partition of V given in the definition of bridging sequence, then it is easily seen that there is a basis B of M that contains T and avoids S.

Readers familiar with blocking sequences will observe that V is a blocking sequence relative to this basis. Thus properties of bridging sequences can be derived from properties of blocking sequences. For example we immediately have

Lemma 2.25. *Let (X_1, X_2) be an exact k-separation in the matroid N and let M be a matroid in which (X_1, X_2) is bridged. Then there is a minor M' of M that has a bridging sequence for (X_1, X_2).*

If (X_1, X_2) is bridged in M and M has a bridging sequence V, then we say that M is a *bridging matroid* for (X_1, X_2). If no proper minor of M has this property, then we say that M is a *minimal bridging matroid* for (X_1, X_2) and that V is a *minimal bridging sequence* for (X_1, X_2).

The next lemma is essentially [11, Theorem 3.4].

Lemma 2.26. *If M is a minimal bridging matroid for the exact k-separation (X_1, X_2) of N, then there is a unique partition (S, T) of $E(M) - E(N)$ such that $N = M\backslash S/T$. The set S is coindependent and T is independent. Moreover, there exists an ordering of $E(M) - E(N)$ that is a bridging sequence for (X_1, X_2).*

Let M be a bridging matroid for the exact k-separation (X_1, X_2) of N and let V be a bridging sequence for (X_1, X_2). Let S and T be the associated partition of $E(M) - E(N)$ given in the definition of bridging sequence. We refer to the elements of S and T as the *delete* and *contract* elements of V respectively. If we are viewing from another perspective, then we may refer to them as *extension* and *coextension* elements. Evidently M^* is a bridging matroid for the exact k-separation (X_1, X_2) of N^* and V is a bridging sequence for this k-separation. If $Z \subseteq V$, then we denote the matroid $M\backslash(S - Z)/(T - Z)$ by $N[Z]$. In particular we will often denote the bridging matroid M by $N[V]$.

We will need just a few properties of bridging sequences.

Lemma 2.27. *Assume that $V = (v_0, \dots, v_p)$ is a bridging sequence for the k-separation (X_1, X_2) of the matroid M.*

(i) *If v_i is a delete element of V, then $v_i \notin \mathrm{cl}_{N[v_0,\dots,v_i]}(X_2)$.*
(ii) *If v_i is a contract element of V, then $v_i \notin \mathrm{cl}^*_{N[v_0,\dots,v_i]}(X_2)$.*

Proof. Assume otherwise. Then $v_i \in \mathrm{cl}_{N[v_1,\dots,v_p]}(X_2 \cup \{v_{i+1}, \dots, v_p\})$, contradicting the fact that v_i blocks the k-separation $(X_1 \cup \{v_1, \dots, v_{i-1}\}, X_2 \cup \{v_{i+1}, \dots, v_p\})$ of the matroid $N[v_1, \dots, v_p]\backslash v_i$. $\square$

The proof of the next lemma is even easier and we omit it.

Lemma 2.28. *Let $V = (v_0, v_1, \dots, v_p)$ be a bridging sequence for the k-separation (X, Y) of N. Say $i < p$. Then, in $N[v_0, \dots, v_i]$, we have $v_i \in \mathrm{cl}(X \cup \{v_0, \dots, v_{i-1}\})$ and $v_i \in \mathrm{cl}^*(X \cup \{v_0, \dots, v_{i-1}\})$.*

We will also use the next technical lemma.

Lemma 2.29. *Let (X_1, X_2) be an exact k-separation of the matroid N and let $V = (v_0, v_1, \dots, v_p)$ be a bridging sequence for (X_1, X_2) with associated bridging matroid $N[V]$. If (V_1, V_2) is a partition of V and there is no $i \in \{0, 1, \dots, p\}$ such that $V_1 = \{v_0, v_1, \dots, v_i\}$, then $\lambda_{N[V]}(X_1 \cup V_1, X_2 \cup V_2) > k$.*

Proof. Assume that $|V| = 2$, so that $V = (v_0, v_1)$. Consider $\lambda_{N[V]}(X \cup \{v_1\}, Y \cup \{v_0\})$. We may assume that v_1 is a delete element, so that v_0 is a contract element. We have $\lambda_{N[V]\backslash v_1}(X \cup \{v_0\}, Y) = k - 1$. But $v_0 \in \mathrm{cl}_{N[V]\backslash v_1}(X)$ and $v_0 \notin \mathrm{cl}_{N[V]\backslash v_1}(Y)$, so that $\lambda_{N[V]\backslash v_1}(X, Y \cup \{v_0\}) = k$. Now $v_1 \notin \mathrm{cl}_{N[V]}(X)$, so $\lambda_{N[V]}(X \cup \{v_1\}, Y \cup \{v_0\}) = k + 1$. Thus the lemma holds in this case.

Assume that $|V| > 2$. Under the hypotheses of the lemma, there is an $i \in \{0, 1, \dots, p\}$ such that $v_i \in V_1$, and $v_{i-1} \in V_2$. We may assume that v_{i-1} is a delete element of V. Let M' be the minor obtained by contracting all of the delete elements from $V - \{v_i, v_{i-1}\}$ and deleting all of the contract elements from $V - \{v_i, v_{i-1}\}$. It is an easy consequence of the definition of bridging sequence and Tutte's Linking Lemma that $\kappa_{M'}(X_1, X_2) = k$. Indeed (v_{i-1}, v_i) is a bridging sequence for the k-separation (X_1, X_2) of $M'\backslash v_{i-1}/v_i$. Thus $\lambda_{M'}(X_1 \cup \{v_i\}, X_2 \cup \{v_{i-1}\}) = k + 1$, and $\lambda_{N[V]}(X_1 \cup V_1, X_2 \cup V_2) > k$ as required. □

Lemma 2.30. *Let (P_1, P_2, P_3) be a partition of the ground set of a matroid N such that $\lambda(P_1) = \lambda(P_3) = \kappa(P_1, P_3) = k - 1$. Let (C, D) be a partition of P_2, where C is independent, D is coindependent and $\lambda_{N/C\backslash D}(P_1, P_3) = k - 1$. Let V be a bridging sequence for $(P_1, P_2 \cup P_3)$ such that $(P_1 \cup P_2, P_3)$ is also bridged in $N[V]$. Then the exact k-separation (P_1, P_3) of $N/C\backslash D$ is bridged in $N[V]/C\backslash D$ and V is a bridging sequence for this k-separation.*

Proof. Say $V = (v_0, v_1, \dots, v_p)$. Let e be an element of C.

2.30.1. *If $i \in \{0, 1, \dots, p\}$, then $\lambda_{N[V]/e}(P_1 \cup \{v_0, v_1, \dots, v_i\}) = k$ and $\lambda_{N[V]/e}(P_3 \cup \{v_{i+1}, v_{i+2}, \dots, v_p\}) = k$.*

Subproof. Assume that $e \in \mathrm{cl}_{N[V]}(P_1 \cup V)$. Then $e \in \mathrm{cl}_N(P_1)$ so that $\lambda_{N/e}(P_1) < \lambda_N(P_1) = k - 1$, contradicting the fact that $\lambda_{N/C\backslash D}(P_1) = k - 1$. Thus $e \notin \mathrm{cl}_{N[V]}(P_1 \cup V)$, and similarly $e \notin \mathrm{cl}_{N[V]}(P_3 \cup V)$.

It follows from the above that $r_{N[V]/e}(P_1 \cup \{v_0, v_1, \dots, v_i\}) = r_{N[V]}(P_1 \cup \{v_0, v_1, \dots, v_i\})$, and it follows that $\lambda_{N[V]/e}(P_1 \cup \{v_0, v_1, \dots, v_i\}) = \lambda_{N[V]}(P_1 \cup \{v_0, v_1, \dots, v_i\}) = k$. Similarly $\lambda_{N[V]/e}(P_3 \cup \{v_{i+1}, v_{i+2}, \dots, v_p\}) = k$. □

We now show that V is a bridging sequence for $(P_1, P_2 \cup P_3 - \{e\})$ in $N[V]/e$. Let T be the set of contract elements of V. As C is independent in N, we see that $C \cup T$ is independent in $N[V]$, so that T is independent in $N[V]$. Thus property (i) of bridging sequences holds. Property (ii) follows from 2.30.1. Say $i > 1$ and v_i is a delete element of V. Then $k - 1 = \lambda_{N[V]\backslash v_i}(P_1 \cup \{v_0, v_1, \dots, v_{i-1}\}) \geqslant \lambda_{N[V]/e\backslash v_i}(P_1 \cup \{v_0, v_1, \dots, v_{i-1}\}) \geqslant \lambda_{N/C\backslash D}(P_1) = k - 1$. Thus (iii), and similarly (iv), also hold and V is indeed a bridging sequence for P_1 in $N[V]/e$.

It follows from 2.30.1 and an argument similar to that of Lemma 2.24 that P_3 is bridged in $N[V]/e$. The lemma now follows from an obvious induction. □

4. Special structures

In this section we review properties of certain highly structured matroids and sets in matroids that play an important role in this paper. We begin by looking at 3-separating sets.

Sequential 3-separators Let M be a 3-connected matroid on E and let A be a 3-separating subset of E. Recall that $(A, E - A)$ is *sequential* if either $\mathrm{fcl}(A) = E$ or $\mathrm{fcl}(E - A) = E$. If the latter case holds we say that A is a *sequential* 3-separator.

Evidently any subset of E with at most two elements is a sequential 3-separator. If A is a sequential 3-separator, then there is an ordering $(a_1, a_2, \dots, a_n)$ of A such that, for all $i \in \{1, 2, \dots, n\}$, the set $\{a_1, a_2, \dots, a_i\}$ is 3-separating. Such an ordering is said to be a *sequential ordering* of A. Sequential orderings are typically far from unique. The next few lemmas summarise some elementary properties, most of which follow immediately from definitions. In all of the lemmas A is a sequential 3-separator of the 3-connected matroid M on E and $(a_1, a_2, \dots, a_n)$ is a sequential ordering of A. Recall that a *triangle* of a matroid is a 3-element circuit and a *triad* is a 3-element cocircuit.

Lemma 2.31.

(i) *If $n \geqslant 2$, then $A \subseteq \mathrm{fcl}(\{a_1, a_2\})$.*
(ii) *If $n \geqslant 3$, then $\{a_1, a_2, a_3\}$ is either a triangle or a triad.*
(iii) *If $i \in \{1, 2, \dots, n\}$, then $a_i \in \mathrm{cl}^{(*)}(\{a_{i+1}, a_{i+2}, \dots, a_n\} \cup (E - A))$.*
(iv) *If $i \in \{2, 3, \dots, n - 1\}$, then $a_{i+1} \in \mathrm{cl}^{(*)}(\{a_1, \dots, a_i\})$.*

Lemma 2.32. *If A is fully closed and $|A| \geqslant 4$, then either $M \backslash a_n$ or M/a_n is 3-connected. In particular,*

(i) *if $a_n \in \mathrm{cl}(E - A)$, then $M \backslash a_n$ is 3-connected, and*
(ii) *if $a_n \in \mathrm{cl}^*(E - A)$, then M/a_n is 3-connected.*

If A is a sequential 3-separation set, and B is an exactly 3-separating subset of A, then it is not necessarily the case that $\mathrm{fcl}(B)$ contains A, but the behaviour of such sets is not wild.

Lemma 2.33. *If B is a 3-separating subset of A, then B is sequential.*

Proof. As A is sequential, $\mathrm{fcl}(E(M) - A) = E(M)$. As $(E(M) - B) \supseteq (E(M) - A)$ we have $\mathrm{fcl}(E(M) - B) = E(M)$. Thus B is sequential. $\square$

Finally we note

Lemma 2.34. *If $x \in A$ and $M \backslash x$ is 3-connected, then $A - \{x\}$ is a sequential 3-separator of $M \backslash x$.*

Fans Let F be a set of elements of the 3-connected matroid M. Then F is a *fan* of M if it has an ordering $(f_1, f_2, \ldots, f_n)$ such that

(i) for all $i \in \{1, 2, \ldots, n-2\}$, the triple $\{f_i, f_{i+1}, f_{i+2}\}$ is either a triangle or a triad, and
(ii) if $i \in \{1, 2, \ldots, n-3\}$ then $\{f_i, f_{i+1}, f_{i+2}\}$ is a triangle if and only if $\{f_{i+1}, f_{i+2}, f_{i+3}\}$ is a triad.

An ordering of a fan satisfying (i) and (ii) is a *fan ordering* of F and we will refer to such an ordered set as an *ordered fan*. At times we may blur the distinction between a fan and an ordered fan. As triads and triangles are interchanged under duality, a fan in M is also a fan in M^*.

Fans are studied in some detail by Oxley and Wu in [26]. All of the lemmas of this section follow from results in [26], although note that the terminology of [26] differs significantly from that used here.

Lemma 2.35. *If F is a fan of a 3-connected matroid and $|E(M) - F| \geqslant 2$, then F is a sequential 3-separating set and any fan ordering is a sequential ordering of F.*

Proof. The lemma clearly holds if $|F| \leqslant 2$. Assume that $|F| \geqslant 3$. Let $(f_1, f_2, \ldots, f_n)$ be a fan ordering of F. Then $F - \{f_n\}$ is a fan and we may assume that the lemma holds for $F - \{f_n\}$. Now $\{f_{n-2}, f_{n-1}, f_n\}$ is either a triangle or triad of M. In the former case $f_n \in \mathrm{cl}(F - \{f_n\})$ and in the latter $f_n \in \mathrm{cl}^*(F - \{f_n\})$. The lemma follows by induction. $\square$

Note that a fan has many sequential orderings; indeed any triangle or triad of a fan can initiate such an ordering. Most are not fan orderings. If $F = (f_1, f_2, \ldots, f_m)$ is an ordered fan, and $i \in \{1, 2, \ldots, m\}$, then we say that $\{f_1, f_2, \ldots, f_i\}$ is an *initial section* of F, and $\{f_{i+1}, f_{i+2}, \ldots, f_m\}$ is a *terminal section* of F.

Lemma 2.36. *Let M be a 3-connected matroid with at least four elements. Then $E(M)$ is a fan if and only if M is a wheel or a whirl.*

Thus fans can, in general, be thought of as partial wheels or whirls. Once we are past degeneracies called by small size, the structure of fans becomes quite canonical. Subsets of size less than two are trivially fans. Fans of size three are either triangles or triads and any ordering is a fan ordering. Assume that F is a fan such that $|E(M) - F| \geqslant 2$. If $|F| \geqslant 5$, and $(f_1, f_2, \ldots, f_n)$ is a fan ordering of F, then the only other fan ordering of F is to reverse the order of the indices. Fans with four elements are not quite canonical. If (f_1, f_2, f_3, f_4) is a fan ordering of F, then (f_1, f_3, f_2, f_4) is also a fan ordering of F. This is a fact that is, at times, irritating in the minutiae of arguments.

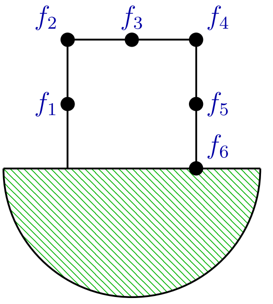

Fig. 2.3. A 6-element fan.

We generalise terminology for elements of wheels and whirls to fans as follows. Let $(f_1, f_2, \ldots, f_n)$ be a fan ordering of a fan F with at least five elements. If $\{f_1, f_2, f_3\}$ is a triangle, then the elements of $(f_1, f_2, \ldots, f_n)$ with odd indices are *spoke* elements and the elements with even indices are *rim* elements of F. If $\{f_1, f_2, f_3\}$ is a triad, then the elements with odd indices are *rim* elements and the elements with even indices are *spoke* elements. Evidently the above labelling of elements is independent of the fan ordering. If (f_1, f_2, f_3, f_4) is a 4-element fan, then f_1 is a spoke or a rim element according as to whether $\{f_1, f_2, f_3\}$ is a triangle or a triad respectively. This gives f_1 and f_4 labels, but we cannot assign canonical labels to f_2 and f_3.

One also sees that any fan with at least four elements has well-defined *end* elements and *internal* elements in an obvious way. We now recall a basic result of Tutte [30].

Lemma 2.37 *(Tutte's Triangle Lemma). Let T be a triangle of a 3-connected matroid with at least four elements. If T is not contained in a fan with at least four elements, then there are elements $t_1, t_2 \in T$ such that $M\backslash t_1$ and $M\backslash t_2$ are both 3-connected.*

Thus there is at most one element of a triangle whose deletion from M destroys 3-connectivity unless the triangle is in a larger fan. We can never remove an internal element of a fan to keep 3-connectivity. Nonetheless we can get close. The next lemma is a straightforward consequence of Bixby's Lemma.

Lemma 2.38. *Let M be a 3-connected matroid; let F be a fan of M with at least five elements; let $(f_1, f_2, \ldots, f_n)$ be a fan ordering of F; and let f_i be an internal element of F.*

(i) *If f_i is a spoke element of F, then $M\backslash f_i$ is 3-connected up to the single series pair $\{f_{i-1}, f_{i+1}\}$.*
(ii) *If f_i is a rim element of F, then M/f_i is 3-connected up to the single parallel pair $\{f_{i-1}, f_{i+1}\}$.*

A fan is *maximal* if it is not contained in a larger fan. If M is not a wheel or a whirl, then any maximal fan F is exactly 3-separating. The next lemma is not quite a special case of Lemma 2.32 as a maximal fan need not be fully closed.

Lemma 2.39. *Let M be a matroid that is not a wheel or a whirl and let F be a maximal fan of M with at least four elements. Let f be an end of F.*

(i) *If f is a spoke element of F, then $M\backslash f$ is 3-connected.*
(ii) *If f is a rim element of F, then M/f is 3-connected.*

Finally we note that while fans in matroids generalise the eponymous structures in graphs, it can be misleading for matroidal arguments to visualise them graphically. Fig. 2.3 is a schematic diagram of a 6-element fan in a matroid. The rim elements are $\{f_1, f_3, f_5\}$ and the spoke elements are $\{f_2, f_4, f_6\}$.

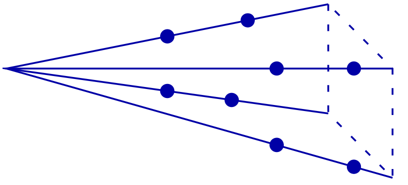

Fig. 2.4. Illustration of Λ_4.

Swirls Let $n \geqslant 3$ be an integer and N be a simple matroid whose ground set is the disjoint union of a basis $B = \{b_1, b_2, \dots, b_n\}$ and sets $P_1 = \{p_1, q_1\}, P_2 = \{p_2, q_2\}, \dots, P_n = \{p_n, q_n\}$ such that, for all $i \in \{1, 2, \dots, n-1\}$, we have $P_i \subseteq \mathrm{cl}(\{b_i, b_{i+1}\})$ and $P_n \subseteq \mathrm{cl}(\{b_n, b_1\})$. Then $M = N \backslash B$ is a rank-n *swirl*. We say that $\{P_1, P_2, \dots, P_n\}$ is the set of *legs* of the swirl. It is easily seen that swirls are 3-connected.

Evidently $P_i \cup P_{i+1}$ is a circuit of M for all i in the cyclic order on $\{1, 2, \dots, n\}$. Otherwise, if C is a non-spanning circuit of a swirl, then C is a transversal of the legs. If M has no such non-spanning circuits, then M is the rank-n *free swirl* and we denote it by Δ_n. As noted in the introduction one can obtain Δ_n by placing the elements of P_i freely on the line spanned by $\{b_i, b_{i+1}\}$. Fig. 1.1 in the introduction illustrates Δ_5. Evidently Δ_n is unique up to isomorphism. Note that Δ_n is self-dual.

It is shown in [25] that if q is a prime power that exceeds five and is not of the form 2^p, where $2^p - 1$ is prime, then Δ_n has at least 2^n inequivalent representations over $GF(q)$. In particular, for a prime field $GF(p)$ of size at least seven, we can obtain free swirls with an arbitrary number of inequivalent representations. Free swirls are structures that necessarily need to be dealt with in understanding inequivalent representations of matroids over prime fields.

Spikes Let $n \geqslant 3$ be an integer and let N be a rank-n matroid with ground set $\{t, p_1, q_1, p_2, q_2, \dots, p_n, q_n\}$ such that

(i) $\{t, p_i, q_i\}$ is a triangle for all $i \in \{1, 2, \dots, n\}$, and
(ii) $r(\bigcup_{j \in J} \{a_j, b_j\}) = |J| + 1$ for every proper subset J of $\{1, 2, \dots, n\}$.

Then the matroid $M = N \backslash t$ is a rank-n *spike*. Each pair $\{p_i, q_i\}$ is a *leg* of the spike. For distinct $i, j \in \{1, 2, \dots, n\}$, the set $\{p_i, q_i, p_j, q_j\}$ is a circuit of M. As with swirls, any other non-spanning circuit of a spike is a transversal of the legs. If all transversals of the legs are independent then M is the rank-n *free spike* and we denote is by Λ_n. Alternatively one can obtain Λ_n by taking $M(K_{2,n})$, a matroid that has rank $n+1$, and truncating it to rank n. Note that spikes are 3-connected and that Λ_n is self-dual. Fig. 2.4 illustrates Λ_4.

Spikes in general have turned out to be fundamental in matroid theory, frequently as sources of counterexamples to superficially reasonable conjectures. Free spikes are representable over all non-prime fields and it is shown in [25] that, for every non-prime field with more than four elements, Λ_n has at least 2^{n-1} inequivalent representations. Fortunately, it is shown in [12, Lemma 11.6] that if $p \geqslant 3$ is prime, then Λ_p is not $GF(p)$-representable so, if our eventual goal is to understand inequivalent representations over $GF(p)$, we can exclude Λ_p from consideration.

Other properties of spikes and swirls follow from the fact that the members of the partition of such a matroid into its legs form the petals of a spike-like or swirl-like flower respectively. Indeed, these are extremal examples of matroids with many pairwise-inequivalent mutually-crossing 3-separations and this is part of the reason for the key role that they play in matroid structure theory.

Quads A *quad* of a matroid is a 4-element set that is both a circuit and a cocircuit. It is easily checked that if D is a quad, then $\lambda(D) = 2$ so that D is exactly 3-separating. It is immediate from the definition that if D is a quad in M, then D is also a quad in M^*. Note that the complement in M of a quad is fully closed so that quads are cohesive. Let M be a 3-connected matroid with at least five elements. Then any 3-separating set of size three is sequential as it is either a triangle or a triad. If D

is a non-sequential 3-separating set with four elements, then it is easily checked that D is a quad. It is no doubt the fact that quads are the unique minimum-sized non-sequential 3-separating sets that account for their frequent appearance in proofs in this paper.

Note also that the union of any pair of legs of a spike is a quad, while the union of any consecutive pair of legs of a swirl is a quad.

Chapter 3. Flowers

Flowers arise when a matroid has crossing 3-separations. They turn out to be fundamental structures. Their study was initiated in [20] and continued in [21]. In these papers flowers were defined only for 3-connected matroids. The definition was extended to arbitrary matroids and, indeed, to structures that arise from higher-order crossing separations, by Aikin and Oxley [1]. This chapter begins by reviewing material from [20,1] and then developing further material that will be needed for this paper.

1. Definition and basic properties

We begin by defining a flower in a way that is slightly less restrictive than that of [20], and somewhat less general than that of [1]. Let M be a connected matroid, and let $\mathbf{F} = (P_1, P_2, \ldots, P_n)$ be a partition of $E(M)$. Then $\mathbf{F}$ is a *flower* in M with *petals* $P_1, P_2, \ldots, P_n$ if the following hold.

(i) If $n > 1$, then $\lambda(P_i) = 2$ for all $i \in \{1, 2, \ldots, n\}$.
(ii) If $n > 2$, then $\lambda(P_i \cup P_{i+1}) = 2$ for all $i \in \{1, 2, \ldots, n\}$ where all subscripts are interpreted modulo n.
(iii) If (X, Y) is a 2-separation of M, then for some petal P_i, either X or Y is a subset of P_i.

If M is 3-connected, then the definition given here is precisely the definition given in [20]. This is the case we are usually interested in, but, at times, it will facilitate arguments to allow flowers in matroids that are not 3-connected.

The ordering of the petals of a flower is always the cyclic order, so that subscripts should always be interpreted modulo n. With this in mind we say that a set of petals of $\mathbf{F}$ is *consecutive* if it is of the form $\{P_i, P_{i+1}, \ldots, P_{i+k}\}$, for some $i, k \in \{1, 2, \ldots, n\}$. The flower $\mathbf{F}$ is a *daisy* if the union of a set of petals is 3-separating if and only if the set of petals is consecutive. The flower $\mathbf{F}$ is an *anemone* if the union of any set of petals is 3-separating. The next theorem is a special case of [1, Theorem 1.1].

Theorem 3.1. *If* $\mathbf{F}$ *is a flower in the connected matroid* M*, then* $\mathbf{F}$ *is either an anemone or a daisy.*

We next introduce a structure related to flowers. Before doing that we settle some terminology for ordered partitions. Let $\mathbf{P} = (P_1, P_2, \ldots, P_n)$ be an ordered partition of a set S. Then the ordered partition $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_m)$ is a *concatenation* of $\mathbf{P}$ if there are indices $0 = k_0 < k_1 < \cdots < k_m = n$ such that $Q_i = P_{k_{i-1}+1} \cup \cdots \cup P_{k_i}$ for $i \in \{1, \ldots, m\}$. If $\mathbf{Q}$ is a concatenation of $\mathbf{P}$, then $\mathbf{P}$ is a *refinement* of $\mathbf{Q}$. Also, in any unexplained context we allow members of a partition to be empty sets and will often abuse notation and denote a singleton set $\{q\}$ by q.

Let $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_n)$ be a partition of the ground set of a connected matroid M. Then $\mathbf{Q}$ is a *quasi-flower* of M with *petals* $Q_1, Q_2, \ldots, Q_m$ if

(i) $\lambda(Q_i \cup Q_{i+1} \cup \cdots \cup Q_{i+k}) \leqslant 2$ for all $i, k \in \{1, 2, \ldots, n\}$ where subscripts are interpreted modulo n; and
(ii) if (X, Y) is a 2-separation of M, then for some petal Q_i, either X or Y is a subset of Q_i.

The next lemma is clear.

Lemma 3.2. *Let* $\mathbf{Q}$ *be a quasi-flower of the connected matroid* M *and let* $\mathbf{Q}' = (Q'_1, Q'_2, \ldots, Q'_m)$ *be a concatenation of* $\mathbf{Q}$*. If* $\lambda(Q'_i) \geqslant 2$ *for all* $i \in \{1, 2, \ldots, m\}$*, then* $\mathbf{Q}'$ *is a flower in* M*.*

The concatenations of a quasi-flower that are flowers are the flowers *displayed* by the quasi-flower. We also say that a flower or quasi-flower *displays* a 3-separating set X or a 3-separation (X, Y) if X is a union of petals. A 3-separating set X is *contained in a petal* of a flower or quasi-flower if $X \subseteq \mathrm{fcl}(P)$ for some petal P.

Now return attention to flowers. A *trivial* flower has just one petal. A flower with two petals is nothing more than a partition (P_1, P_2) of $E(M)$ for which $\lambda(P_1) = 2$. If $n = 3$, there is no distinction between an anemone and a daisy. Say $n \geqslant 3$. Then the anemone $(P_1, \dots, P_n)$ is

(i) a *paddle* if $\sqcap(P_i, P_j) = 2$ for all distinct $i, j \in \{1, 2, \dots, n\}$;
(ii) a *copaddle* if $\sqcap(P_i, P_j) = 0$ for all distinct $i, j \in \{1, 2, \dots, n\}$; and
(iii) *spike-like* if $n \geqslant 4$, and $\sqcap(P_i, P_j) = 1$ for all distinct $i, j \in \{1, 2, \dots, n\}$.

A daisy $(P_1, \dots, P_n)$ is

(i) *swirl-like* if $n \geqslant 4$ and $\sqcap(P_i, P_j) = 1$ for all consecutive i and j, and $\sqcap(P_i, P_j) = 0$ for all non-consecutive i and j; and
(ii) *Vámos-like* if $n = 4$ and $\sqcap(P_i, P_j) = 1$ for all consecutive i and j, while $\{\sqcap(P_1, P_3), \sqcap(P_2, P_4)\} = \{0, 1\}$.

A flower is *unresolved* if $n = 3$ and $\sqcap(P_i, P_j) = 1$ for all distinct $i, j \in \{1, 2, 3\}$. Due to the presence of possible additional structure, some unresolved flowers are best regarded as spike-like and others as swirl-like.

Theorem 3.3. *If* $\mathbf{F}$ *is a flower, then* $\mathbf{F}$ *is either a paddle, a copaddle, spike-like, Vámos-like, or swirl-like.*

Note that $\mathbf{F}$ is a flower in M if and only if $\mathbf{F}$ is a flower in M^*. Indeed if $\mathbf{F}$ is spike-like in M, then $\mathbf{F}$ is spike-like in M^*, and if M is swirl-like in M, then $\mathbf{F}$ is swirl-like in M^*. Moreover $\mathbf{F}$ is a paddle in M if and only if $\mathbf{F}$ is a copaddle in M^*.

2. Equivalent flowers

We have defined flowers for connected matroids and there will be a number of occasions where we will consider flowers in matroids that are not 3-connected. Having said this the majority of the time we are interested only in flowers in 3-connected matroids. The more detailed structural descriptions that we give in the remainder of this chapter apply only to flowers in 3-connected matroid. It would certainly be possible to generalise these notions to matroids that are not 3-connected, but at the cost of additional technicalities.

Let M be a 3-connected matroid. Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be flowers of M, then $\mathbf{F}_1 \preccurlyeq \mathbf{F}_2$ if every non-sequential 3-separation displayed by $\mathbf{F}_1$ is equivalent to one displayed by $\mathbf{F}_2$. Clearly $\preccurlyeq$ is a quasi-order on the collection of flowers of M. The flowers $\mathbf{F}_1$ and $\mathbf{F}_2$ are *equivalent*, denoted $\mathbf{F}_1 \cong \mathbf{F}_2$, if $\mathbf{F}_1 \preccurlyeq \mathbf{F}_2$ and $\mathbf{F}_2 \preccurlyeq \mathbf{F}_1$. Thus equivalent flowers display, up to equivalence of 3-separations, exactly the same non-sequential 3-separations. The flower $\mathbf{F}_1$ is *maximal* if whenever $\mathbf{F}_1 \preccurlyeq \mathbf{F}_2$, then $\mathbf{F}_1 \cong \mathbf{F}_2$. The *order* of a flower $\mathbf{F}$ is the minimum number of petals in a flower equivalent to $\mathbf{F}$.

We give here some examples to illustrate some of the ideas developed so far and also to motivate some of the future material. Let M be a rank-n wheel or a whirl, with fan ordering $(a_1, a_2, \dots, a_{2n})$ of the elements of M. Then $(\{a_1, a_2\}, \{a_3, a_4\}, \dots, \{a_{2n-1}, a_{2n}\})$, is a swirl-like flower of M. However, wheels and whirls have no non-sequential 3-separations, so this flower is equivalent to the trivial flower $(E(M))$. Thus there is a sense in which flowers obtained from wheels and whirls are degenerate.

On the other hand, let $\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}, \{a_4, b_4\}$ be the set of legs, in order, of a rank-4 swirl. Then $(\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}, \{a_4, b_4\})$ is a swirl-like flower of order 4. To see this, note that if $i \in \{1, 2, 3, 4\}$, then the partition $(\{a_i, b_i, a_{i+1}, b_{i+1}\}, \{a_{i+2}, b_{i+2}, a_{i+3}, b_{i+3}\})$ is a non-sequential 3-separation; indeed both sides of the separation are quads. Certainly no flower with fewer petals displays all these 3-separations. This example generalises to larger swirls and to spikes as well.

Let $\mathbf{F}$ be a flower of M. An element e of M is *loose* in $\mathbf{F}$ if $e \in \mathrm{fcl}(P_i)$ for some petal P_i. An element that is not loose is *tight*. The petal P_i is *loose* if all elements of P_i are loose, otherwise it is *tight*. A flower is *tight* if all of its petals are tight. The next lemma summarises a number of results from [20].

Lemma 3.4. *Let $\mathbf{F}$ be a flower of the matroid M.*

(i) *If P is a tight petal of $\mathbf{F}$, then P has at least two tight elements.*
(ii) *If $\mathbf{F}' \cong \mathbf{F}$, then $\mathbf{F}$ and $\mathbf{F}'$ have the same set of tight elements.*
(iii) *If $\mathbf{F}$ has order at least 3, then the order of $\mathbf{F}$ is the number of petals in any tight flower equivalent to $\mathbf{F}$.*

The condition that $\mathbf{F}$ has order at least 3 in Lemma 3.4 is important as degeneracies can occur for flowers of low order. Consider $U_{3,6}$. Note that $U_{3,6} \cong \Delta_3 \cong \Lambda_3$. Any partition of the elements of $U_{3,6}$ into 2-element subsets gives a tight 3-petal flower. This flower is equivalent to the trivial flower as it displays no non-sequential 3-separations. Thus the flower has order 1.

For another example of a similar degenerate situation that we converge to in case analyses, let (C, D) be a non-sequential 3-separation of the 3-connected matroid M where D is a quad. Let (D_1, D_2) be an arbitrary partition of D into 2-element subsets. Then (C, D) and (C, D_1, D_2) are equivalent flowers as both display just one non-sequential 3-separation, namely (C, D). Evidently this equivalence class of flowers has order 2. But (C, D_1, D_2) is a tight flower, so that (iii) does not hold in this case either.

3. Structure of tight flowers

We begin by giving more detail from [20] about the structure of equivalence classes of flowers. Our primary interest is in swirl-like flowers, but at several places through the paper we need to understand the structure of other types of flowers. If $(P_1, P_2, \dots, P_n)$ is an anemone, then any permutation of the petals gives an equivalent flower, while, if it is a daisy, then any permutation of the petals that corresponds to a symmetry of the regular n-gon gives an equivalent flower. If one flower can be obtained from another by such a permutation then the two flowers are alternative descriptions of the same underlying object and we will say that the flowers are *equal up to labels*. In fact we use the term "up to labels" to cover a multitude of sins in this paper and will always mean something like "by an appropriate relabelling" or "by an appropriate reordering of the indices".

We now clarify the status of 3-petal spike-like and swirl-like flowers. Recall that we called such flowers "unresolved". Let $\mathbf{F} = (P_1, P_2, P_3)$ be an unresolved flower. If $\mathbf{F}$ has no loose elements, then we can regard it as both spike-like and swirl-like. Assume that there are loose elements. If P_1 contains loose elements, then there is an element $x \in P_1$ such that, up to labels, $x \in \mathrm{cl}^{(*)}(P_2)$. If $x \in \mathrm{cl}^{(*)}(P_3)$, then we say that the flower is *spike-like* and if $x \notin \mathrm{cl}^{(*)}(P_3)$, then we say that the flower is *swirl-like*. It is shown in [20] that this definition is consistent in that, if $\mathbf{F}$ has loose elements, then $\mathbf{F}$ is not both spike-like and swirl-like.

Vámos-like flowers are easily described. The next theorem is [20, Theorem 6.1].

Theorem 3.5. *Let $\mathbf{F}$ be a Vámos-like flower of the 3-connected matroid M. Then $\mathbf{F}$ has no loose elements and any flower equivalent to $\mathbf{F}$ is equal to $\mathbf{F}$ up to labels.*

Anemone structure A set S of elements of a 3-connected matroid M is a *segment* if either $|S| \leqslant 1$ or $|S| \geqslant 2$ and $M|S \cong U_{2,|S|}$. Equivalently S is a segment if every 3-element subset of S is a triangle. The set S is a *cosegment* of M if S is a segment of M^*. The following is [20, Theorem 7.1].

Theorem 3.6. *Let M be a 3-connected matroid and let $\mathbf{F}$ be a tight flower of M of order $n \geqslant 3$ that is a paddle, a copaddle, or is spike-like. Let T and L denote the sets of tight and loose elements of $\mathbf{F}$ respectively. For each petal P_i of $\mathbf{F}$, let $T_i = P_i \cap T$.*

(i) *If $\mathbf{F}$ is a paddle, then L is a segment, $r(T_i) \geqslant 3$, and $L \subseteq \mathrm{cl}(T_i)$ for all $i \in \{1, 2, \dots, n\}$.*

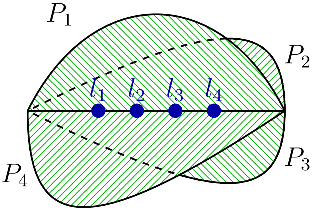

Fig. 3.1. A paddle.

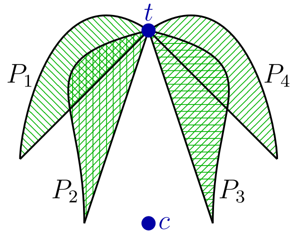

Fig. 3.2. Spike-like structure.

(ii) *If* $\mathbf{F}$ *is a copaddle, then* L *is a cosegment,* $r^*(T_i) \geqslant 3$*, and* $L \subseteq \mathrm{cl}^*(T_i)$ *for all* $i \in \{1, 2, \ldots, n\}$.

(iii) *If* $\mathbf{F}$ *is spike-like, then* $|L| \leqslant 2$. *If* L *contains a single element, then that element is either in the closure of* T_i *for each* i*, or is in the coclosure of* T_i *for each* i. *If* $|L| = 2$*, then one member of* L *is contained in the closure of each* T_i*, while the other is contained in the coclosure of each* T_i.

Moreover, up to arbitrary permutations of the petals, the tight flowers equivalent to $\mathbf{F}$ *are precisely the partitions of the form* $(T_1 \cup L_1, T_2 \cup L_2, \ldots, T_n \cup L_n)$*, where* $(L_1, L_2, \ldots, L_n)$ *is a partition of* L.

A loose element of a spike-like flower that is in the closure of each petal is called the *tip* of the spike-like flower and a loose element that is in the coclosure of each petal is called the *cotip*.

We illustrate Theorem 3.6 with some examples. Fig. 3.1 illustrates the structure corresponding to an equivalence class of paddles. The set of loose elements is $\{l_1, l_2, l_3, l_4\}$. One flower in the equivalence class is $(P_1, P_2 \cup \{l_3\}, P_3 \cup \{l_1, l_2, l_4\}, P_4)$.

Fig. 3.2 illustrates a spike-like flower with tip t and cotip c. Note that if c is deleted, then the petals P_1, P_2, P_3 and P_4 are 2-separating. If c is contracted we have a spike-like flower without a cotip.

Swirl-like flower structure The structure of swirl-like flowers is of particular interest to us. For a petal P of a flower let $\check{P}$ denote the set of tight elements of P and let $\hat{P} = \mathrm{fcl}(P)$. For petals P_i and P_{i+1} of a swirl-like flower, let $P_i^+ = P_{i+1}^- = \hat{P}_i \cap \hat{P}_{i+1}$.

Theorem 3.7. *Let* M *be a* 3*-connected matroid and let* $\mathbf{F} = (P_1, P_2, \ldots, P_n)$ *be a tight swirl-like flower of* M *of order at least* 3. *Then* $P_1^+ \cup P_2^+ \cup \cdots \cup P_n^+$ *is the set of loose elements of* $\mathbf{F}$. *Moreover, there is a partition*

$$\mathbf{P} = \left(\check{P}_1, p_1^1, \ldots, p_1^{k_1}, \check{P}_2, p_2^1, \ldots, p_2^{k_2}, \check{P}_3, \ldots, \check{P}_n, p_n^1, \ldots, p_n^{k_n}\right)$$

of $E(M)$ *having the following properties.*

(i) $\mathbf{P}$ *is a quasi-flower.*

(ii) $(p_i^1, p_i^2, \ldots, p_i^{k_i})$ *is an ordered fan in* M *and* $\{p_i^1, p_i^2, \ldots, p_i^{k_i}\} = P_i^+$ *for all* $i \in \{1, 2, \ldots, n\}$.

(iii) *The partition* $(P'_1, \ldots, P'_n)$ *of* $E(M)$ *is a tight flower equivalent to* $\mathbf{F}$ *if and only if* $(P'_1, P'_2, \ldots, P'_n)$ *is a concatenation of* $\mathbf{P}$ *such that* $\check{P}_i \subseteq P'_i$ *for all* $i \in \{1, 2, \ldots, n\}$.

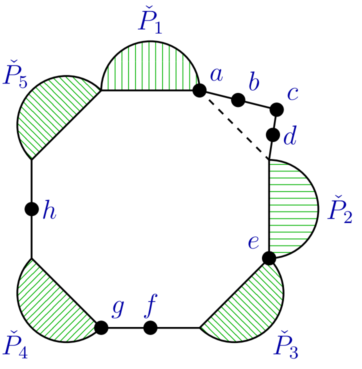

Fig. 3.3. Swirl-like structure.

Fig. 3.3 illustrates Theorem 3.7. The quasi-flower corresponding to the diagram is

$$(\check{P}_1, a, b, c, d, \check{P}_2, e, \check{P}_3, f, g, \check{P}_4, h, \check{P}_5).$$

A flower in the equivalence class corresponding to the above quasi-flower is

$$\big(\check{P}_1 \cup \{a\}, \{b, c, d\} \cup \check{P}_2 \cup \{e\}, \check{P}_3, \{f, g\} \cup \check{P}_4, \{h\} \cup \check{P}_5\big).$$

Blooms In this subsection we develop further terminology for swirl-like flowers to facilitate arguments in proofs. We will call the partition of $E(M)$ given by Theorem 3.7 a *bloom* of M. Note that, associated with a bloom, is a particular ordering of the fans of loose elements between petals. From now on, when we refer to such a set, we will typically regard it as being endowed with the ordering induced from a bloom. To avoid clumsy notation we will often denote a bloom simply by $(\hat{P}_1, \hat{P}_2, \dots, \hat{P}_n)$ noting that it follows from Theorem 3.7 that, if $n \geqslant 3$, then the full bloom can be recovered using structure in the underlying matroid.

For a bloom $(\hat{P}_1, \hat{P}_2, \dots, \hat{P}_n)$ we use the following facts and notational conventions freely throughout this paper: $P_i^+ = \hat{P}_i \cap \hat{P}_{i+1}$, $P_i^- = P_{i-1}^+$, and $\check{P}_i = \hat{P}_i - (P_i^- \cup P_i^+)$.

Recall that blooms are quasi-flowers and that the concatenations of a quasi-flower that are flowers are the flowers *displayed* by that bloom. In particular a 3-separation is *displayed* by a bloom if it is a concatenation of that bloom. A 3-separating set is *contained in a petal* of a bloom if $A \subseteq \hat{P}_i$ for some i. Of course two blooms are *equivalent* if they display the same flowers and it is easily seen that equivalent blooms are equal up to permutations of the n-gon.

The partial order on swirl-like flowers extends easily to include blooms. Thus, if $\mathbf{F}$ is a swirl-like flower and $\mathbf{B}$ is bloom of M, then $\mathbf{F} \preccurlyeq \mathbf{B}$ if there is a flower $\mathbf{F}'$ displayed by $\mathbf{B}$ such that $\mathbf{F} \preccurlyeq \mathbf{F}'$.

We could easily have broadened the notion of blooms to cover all types of flowers, but we will have no need for this more general notion in this paper. The next lemma is easy.

Lemma 3.8. *If $\mathbf{B}$ is a maximal bloom, and $\mathbf{F} \preccurlyeq \mathbf{B}$ is a flower, then $\mathbf{F}$ is a concatenation of a flower displayed by $\mathbf{B}$.*

Loose elements in swirl-like flowers The next lemma says that, most of the time, elements in the guts or coguts of 3-separations displayed by a swirl-like flower are loose elements.

Lemma 3.9. *Let $\mathbf{F} = (P_1, P_2, \dots, P_n)$ be a tight swirl-like flower of the 3-connected matroid M and let i be an integer in $\{1, 2, \dots, n-2\}$. Then the following hold.*

(i) *If $x \notin P_1 \cup P_2 \cup \cdots \cup P_i$, then $x \in \mathrm{cl}^{(*)}(P_1 \cup P_2 \cup \cdots \cup P_i)$ if and only if $x \in \mathrm{cl}^{(*)}(P_1)$ or $x \in \mathrm{cl}^{(*)}(P_i)$.*

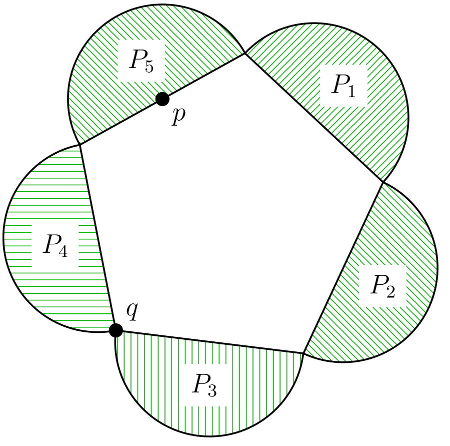

Fig. 3.4. Illustration of Lemma 3.9.

(ii) $\mathrm{fcl}(P_1 \cup P_2 \cup \cdots \cup P_i) = (P_1 \cup P_2 \cup \cdots \cup P_i) \cup (\hat{P}_1 \cup \hat{P}_i) = \hat{P}_1 \cup \hat{P}_2 \cup \cdots \cup \hat{P}_i$.
(iii) $\hat{P}_1 \cup \hat{P}_2 \cup \cdots \cup \hat{P}_i$ *is fully closed.*

Note that the condition that $i \leqslant n-2$ in Lemma 3.9 is necessary. To see this consider the flower illustrated in Fig. 3.4. The element p is in the closure of $P_1 \cup P_2 \cup P_3 \cup P_4$, but is not a loose element of the flower. On the other hand the element q is in the closure of $P_1 \cup P_2 \cup P_3$ and is guaranteed to be a loose element.

It follows from Lemma 3.9 that, if $(P_1, P_2, \ldots, P_n)$ is a tight swirl-like flower of M, then $\hat{P}_1 \cup P_2 \cup \cdots \cup P_{i-1} \cup \hat{P}_i = \hat{P}_1 \cup \hat{P}_2 \cup \cdots \cup \hat{P}_{i-1} \cup \hat{P}_i$ for any $i \in \{1, 2, \ldots, n\}$, so no harm is done by blurring this distinction.

Let $\mathbf{F} = (P_1, P_2, \ldots, P_n)$ be a swirl-like flower in the 3-connected matroid M. Consider the ordered set $P_i^+ = (p_1, p_2, \ldots, p_k)$. By Theorem 3.7, this ordered set is a fan. We say that it is the fan of loose elements *between* $\check{P}_i$ *and* $\check{P}_{i+1}$. We say that the element p_i is a *spoke* element if $p_i \in \mathrm{cl}(\check{P}_i \cup \{p_1, p_2, \ldots, p_{i-1}\})$, and p_i is a *rim* element if $p_i \in \mathrm{cl}^*(\check{P}_i \cup \{p_1, p_2, \ldots, p_{i-1}\})$. It is easily seen that elements of P_i^+ alternate between rim and spoke elements. Note that, while a fan with less than five elements does not have a canonical ordering, the structure induced by the bloom does induce a canonical ordering on the elements of fans of loose elements between petals of a swirl-like flower regardless of the number of elements they have.

Lemma 3.10. *Let* $\mathbf{F} = (P_1, P_2, \ldots, P_n)$ *be a swirl-like flower of the* 3*-connected matroid* M *and say that* $1 \leqslant i \leqslant n-2$*. Then* $x \in \mathrm{cl}(P_1 \cup P_2 \cup \cdots \cup P_i)$ *if and only if either*

(i) *x is the last element of P_1^- not in P_1, and x is a spoke element of P_1^-, or*
(ii) *x is the first element of P_i^+ not in P_1, and x is a spoke element of P_i^+.*

Now consider adjacent petals of a swirl-like flower of order at least 3. Label these petals P_1 and P_2. Assume that x is the first element of P_1^+ not in P_1. Then $x \in \mathrm{cl}^{(*)}(P_1)$. Thus x is either in the guts or coguts of the 3-separating set P_1. By Lemma 3.10, if x is in the guts of P_1, then x is a spoke element of P_1^+ and by the dual of that lemma, if x is in the coguts of P_1, then x is a rim element of P_1^+. For this reason we will frequently refer to the rim or spoke element x as being in the *guts* or *coguts* of (P_1, P_2). We may also refer to x as being a loose guts or coguts element *between* P_1 and P_2. The next lemma is essentially a rephrasing of Lemma 3.10.

Lemma 3.11. *Let* $\mathbf{F} = (P_1, P_2, \ldots, P_n)$ *be a swirl-like flower of the* 3*-connected matroid* M*. If* $1 \leqslant i \leqslant n-2$*, and* $x \notin (P_1 \cup P_2 \cup \cdots \cup P_i)$*, then x is in the guts of the* 3*-separation* $(P_1 \cup P_2 \cup \cdots \cup P_i, P_{i+1} \cup P_{i+2} \cup \cdots \cup P_n)$ *if and only if either*

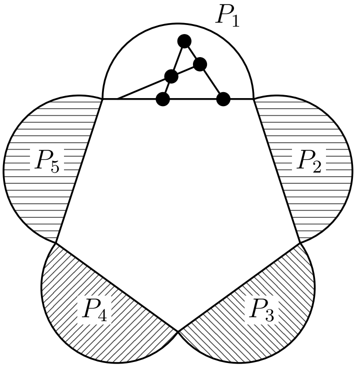

Fig. 3.5. Not all fans are loose.

(i) *x is in the guts of (P_n, P_1), and x is the first element of P_1^- not in P_1, or*
(ii) *x is in the guts of (P_i, P_{i+1}), and x is the first element of P_i^+ not in P_i.*

The next lemma describes one situation in which a swirl-like flower is induced in an extension of a matroid.

Lemma 3.12. *Let M and $M\backslash x$ be 3-connected matroids and let $\mathbf{F} = (P_1, P_2, \dots, P_n)$ be a swirl-like flower of $M\backslash x$. Say $n \geqslant j > i + 1 \geqslant 1$, and x is in both $\mathrm{cl}(P_1 \cup P_2 \cup \dots \cup P_i)$ and $\mathrm{cl}(P_j \cup P_{j+1} \cup \dots \cup P_n)$. Then $(P_1 \cup \{x\}, P_2, \dots, P_n)$ is a flower of M and, in this flower, x is in the guts of $(P_1 \cup \{x\}, P_2)$.*

Ends of fans of loose elements between petals are good choices for removal without losing 3-connectivity, or indeed, as we shall see later, other types of connectivity.

Lemma 3.13. *Let $\mathbf{P}$ be a swirl-like flower of order at least three in the 3-connected matroid M. Let $(f_1, f_2, \dots, f_n)$ be a maximal ordered fan of loose elements between consecutive petals of $\mathbf{P}$ with ordering induced by a bloom associated with $\mathbf{P}$.*

(i) *If f_i is a spoke element, then $M\backslash f_i$ is 3-connected if $i \in \{1, n\}$ and $M\backslash f_i$ is 3-connected up to the single series pair $\{f_{i-1}, f_{i+1}\}$ otherwise.*
(ii) *If f_i is a rim element, then M/f_i is 3-connected if $i \in \{1, n\}$ and M/f_i is 3-connected up to the single parallel pair $\{f_{i-1}, f_{i+1}\}$ otherwise.*

Given the fact that the loose elements of swirl-like flowers form fans, the reader may be tempted to think that the converse holds in that a petal whose elements form a fan consists entirely of loose elements, that is, is a loose petal. This is not the case. For example the legs of a swirl are trivially fans and their elements are not loose. Another example is illustrated in Fig. 3.5. It all depends on how the elements of the fan align with the rest of the matroid.

4. A grab bag of flower lemmas

This section consists of an unordered collection of lemmas on flowers. The criteria used for inclusion in this section is that the lemma is needed somewhere in this paper and that it didn't have a natural home elsewhere in this chapter.

A 3-separation (R, B) is *well displayed* by a bloom $\mathbf{F}$ if it is displayed by $\mathbf{F}$ and neither R nor B is contained in a petal of $\mathbf{F}$. The next lemma is important and used many times, often without reference.

Lemma 3.14. *If* $\mathbf{F}$ *is a bloom of* M *and the* 3*-separation* (R, B) *is well displayed by* $\mathbf{F}$*, then* (R, B) *is non-sequential.*

Proof. Say that (R, B) is well displayed. Then for some tight 4-petal flower (P_1, P_2, P_3, P_4) displayed by $\mathbf{F}$, we have $(R, B) = (P_1 \cup P_2, P_3 \cup P_4)$. It follows from Lemma 3.9, that $\mathrm{fcl}(P_1 \cup P_2) \neq E(M)$ and $\mathrm{fcl}(P_3 \cup P_4) \neq E(M)$, so that (R, B) is non-sequential. □

Recall that disjoint sets A and B of a matroid M are *fully skew* if $\sqcap(A, B) = \sqcap^*(A, B) = 0$. The next lemma follows from the definition of a swirl-like flower and the fact that duals of swirl-like flowers are swirl-like flowers.

Lemma 3.15. *If* P_i *and* P_j *are non-consecutive petals of a (not necessarily tight) swirl-like flower, then* P_i *and* P_j *are fully skew.*

The next lemma follows from results in [20].

Lemma 3.16. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be inequivalent maximal flowers of a* 3*-connected matroid* M *of order at least three. Then there is a petal* P *of* $\mathbf{P}$ *and a petal* Q *of* $\mathbf{Q}$ *such that* $E(M) - P$ *is contained in* $\mathrm{fcl}(Q)$.

As an easy consequence of Lemma 3.16 we have

Corollary 3.17. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be inequivalent maximal swirl-like flowers of a* 3*-connected matroid, then there is a petal* $P \in \mathbf{P}$ *and a petal* $Q \in \mathbf{Q}$ *such that* $\check{P} \subseteq Q$.

One might expect that Vámos-like flowers are potentially problematic, particularly as we make no assumptions about representability until the final chapter. However, it turns out that they cause no difficulties for us. One reason for this is that a Vámos-like flower is never a concatenation of a flower of order greater than four. Another reason is captured by the following lemma.

Lemma 3.18. *Let* M *and* $M\backslash x$ *be* 3*-connected matroids, let* $\mathbf{P} = (P_1, P_2, \ldots, P_n)$ *be a flower of* $M\backslash x$*. Assume that, for some* $i \in \{3, 4, \ldots, n\}$*, we have* $x \in \mathrm{cl}(P_i \cup P_{i+1} \cup \cdots \cup P_n)$ *and let* $\mathbf{P}' = (P_1, P_2, \ldots, P_{i-1}, P_i \cup P_{i+1} \cup \cdots \cup P_n \cup \{x\})$*. Then* $\mathbf{P}'$ *is a flower in* M *of the same type as* $\mathbf{P}$.

Another easy lemma with a similar flavour to Lemma 3.18 is

Lemma 3.19. *Let* $\mathbf{P} = (P_1, P_2, \ldots, P_n)$ *be a flower in the connected matroid* M *and let* M' *be a* 3*-connected minor of* M *on* E' *with the property that* $|P_i \cap E'| \geqslant 2$ *for all* $i \in \{1, 2, \ldots, n\}$*. Then* $(P_1 \cap E', P_2 \cap E', \ldots, P_n \cap E')$ *is a flower in* M' *of the same type as* $\mathbf{P}$.

The next lemma is proved in [20].

Lemma 3.20. *Let* $\mathbf{F}$ *be a flower of the* 3*-connected matroid* M*, and let* (R, B) *be a* 3*-separation of* M *such that:*

(i) *neither* R *nor* B *is contained in a petal of* $\mathbf{F}$; *and*
(ii) *if* (R, B) *crosses a petal* P*, then* $|P \cap R|, |P \cap B| \geqslant 2$.

Then there is a flower that refines $\mathbf{F}$ *and displays* (R, B).

Spikes and swirls show that it is possible for a petal of a tight flower to have two elements, but this does not happen for all flowers.

Lemma 3.21. *Let* P *be a petal of a tight flower* $\mathbf{F}$ *of a* 3*-connected matroid. If* $|P| = 2$*, then* $\mathbf{F}$ *is either swirl-like, spike-like or Vámos-like.*

We frequently need to test that we have a flower of a certain type. One way to do this is by local connectivity between petals. Another way is by the behaviour of loose elements. These tests are obvious consequences of the structure theorems for flowers. We note one here.

Lemma 3.22. *Let* $\mathbf{F} = (P_1, P_2, \dots, P_n)$ *be a flower in the* 3*-connected matroid* M *where* $n \geqslant 3$*. Assume that both* $\mathrm{cl}(P_1) \cap P_2$ *and* $\mathrm{cl}^*(P_1) \cap P_2$ *are nonempty. Then* $\mathbf{F}$ *is spike-like.*

The next fact on loose elements in swirl-like flowers is used many times.

Lemma 3.23. *Let* $\mathbf{F} = (P_1, P_2, \dots, P_n)$ *be a swirl-like flower in the* 3*-connected matroid* M*, where* $n \geqslant 3$*. Then* $|\mathrm{cl}(P_1) \cap P_2| \leqslant 1$*, and if* $\mathrm{cl}(P_1) \cap P_2 \neq \emptyset$*, then* $\mathrm{cl}^*(P_1) \cap P_2 = \emptyset$*.*

Neither of the next two facts is surprising.

Lemma 3.24. *Let* $\mathbf{F}$ *be a flower. Then there is a maximal flower* $\mathbf{F}'$ *such that* $\mathbf{F} \preccurlyeq \mathbf{F}'$ *and* $\mathbf{F}'$ *displays* $\mathbf{F}$*.*

Lemma 3.25. *Let* (X, Y) *be a* 3*-separation of the* 3*-connected matroid* M *and let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be maximal flowers of* M *that display* (X, Y)*. If* (X, Y) *is well displayed by* $\mathbf{P}$*, then* $\mathbf{P} \cong \mathbf{Q}$*.*

Finally we note that if a loose coguts element of a swirl-like flower is in a triangle, then that triangle is part of a fan of loose elements.

Lemma 3.26. *Let* $\mathbf{P} = (P_1, \{b\}, P_2, \dots, P_n)$ *be a swirl-like quasi-flower of the* 3*-connected matroid* M *of order at least four, where* b *is a loose element of* $\mathbf{P}$ *in the coguts of* P_1*. If* $\{a, b, c\}$ *is a triangle of* M*, then, up to labels,* $a \in P_1$*,* $c \in P_2$*, and both* a *and* c *are loose elements in the guts of* P_1 *and* P_2 *respectively.*

5. Flowers and modularity

Let A and B be sets of elements in a matroid M. Recall that A and B form a *modular pair* if $r(A) + r(B) = r(A \cup B) + r(A \cap B)$. The next lemma is elementary but fundamental.

Lemma 3.27. *Let* e *be an element of the matroid* M *and let* A *and* B *be a modular pair of sets of* M*. If* $e \in \mathrm{cl}(A)$ *and* $e \in \mathrm{cl}(B)$*, then* $e \in \mathrm{cl}(A \cap B)$*.*

The next lemma provides a useful connection between the connectivity function and modularity.

Lemma 3.28. *Let* A *and* B *be sets of elements of the matroid* M*. If* $\lambda(A) + \lambda(B) = \lambda(A \cup B) + \lambda(A \cap B)$*, then* A *and* B *are a modular pair in* M*.*

Proof. Let $A' = E(M) - A$ and $B' = E(M) - B$. Since $\lambda(A) + \lambda(B) = \lambda(A \cup B) + \lambda(A \cap B)$, we have

$$r(A) + r\big(A'\big) + r(B) + r\big(B'\big) = r(A \cup B) + r\big(A' \cap B'\big) + r(A \cap B) + r\big(A' \cup B'\big),$$

so that

$$r(A) + r(B) - r(A \cup B) - r(A \cap B) = r\big(A' \cup B'\big) + r\big(A' \cap B'\big) - r\big(A'\big) - r\big(B'\big).$$

The lemma now follows from the submodularity of the rank function. □

The next lemma follows from Lemma 3.28 and the definition of flower.

Lemma 3.29. *Let* $\mathbf{P}$ *be a flower in the connected matroid* M. *Let* A *and* B *be sets displayed by* $\mathbf{P}$ *such that* $A \cap B$ *contains at least one petal of* $\mathbf{P}$ *and* $A \cup B$ *avoids at least one petal of* $\mathbf{P}$. *Then* A *and* B *form a modular pair.*

As an immediate corollary of Lemma 3.27 and Lemma 3.29, we have

Corollary 3.30. *Let* x *be an element of the matroid* M *such that* $M\backslash x$ *is connected with a flower* $\mathbf{P}$. *Let* A *and* B *be sets displayed by* $\mathbf{P}$ *such that* $A \cap B$ *contains at least one petal of* $\mathbf{P}$ *and* $A \cup B$ *avoids at least one petal of* $\mathbf{P}$. *If* $x \in \mathrm{cl}(A)$ *and* $x \in \mathrm{cl}(B)$, *then* $x \in \mathrm{cl}(A \cap B)$.

The next lemma is a strengthening of Corollary 3.30 for blooms. We apply it in many places.

Lemma 3.31. *Let* M *be a matroid with an element* x *such that* $M\backslash x$ *is* 3*-connected with a bloom* $\mathbf{F}$ *of order at least* 3. *Assume that the* 3*-separating sets* A *and* B *are displayed by* $\mathbf{F}$ *and that* $|A \cap B| \geqslant 2$. *Assume further that* $x \in \mathrm{cl}(A)$ *and* $x \in \mathrm{cl}(B)$.

(i) *If* $|E(M\backslash x) - (A \cup B)| \geqslant 2$, *then* $x \in \mathrm{cl}(A \cap B)$.
(ii) *If* $|E(M\backslash x) - (A \cup B)| = 1$ *and this set consists of a loose coguts element of* $\mathbf{F}$, *then* $x \in \mathrm{cl}(A \cap B)$.

Proof. Let $C = E(M\backslash x) - (A \cup B)$. Assume that (i) holds. Then $(A - B, A \cap B, B - A, C)$ is a flower in $M\backslash x$ and the result holds by Corollary 3.30.

Assume that (ii) holds. Say $C = \{c\}$. As $(A - B, A \cap B, (B - A) \cup \{c\})$ is a flower in $M\backslash x$ and c is a loose coguts element, we see that c is in the coguts of the 3-separation $(A - B, B \cup \{c\})$ of $M\backslash x$. Thus $\sqcap(A - B, B) = \sqcap(A - B, B \cup \{c\}) - 1 = 1$. Hence

$$r(A \cup B) = r(B) + r(A - B) + 1.$$

Also, as $(A - B, A \cap B, (B - A) \cup \{c\})$ is a swirl-like flower in $M\backslash x$, we have $\sqcap(A - B, A \cap B) = 1$. Hence

$$r(A) = r(A \cap B) + r(A - B) + 1.$$

It follows from the two displayed equations that A and B form a modular pair in $M\backslash x$. Now $x \in \mathrm{cl}_M(A \cap B)$ by Lemma 3.27. $\square$

6. Separations crossing blooms

In this section we analyse the precise ways in which a 3- or 4-separation can interact with a maximal bloom. The results of this section are lemmas for results in later sections.

Let $\mathbf{F} = (\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_n)$ be a maximal bloom of the matroid M. Then a set S is *consecutive* in $\mathbf{F}$ if it is a union of consecutive petals in the quasi-flower $\mathbf{F}$. In other words, S is consecutive if, for some $i, j \in \{1, \ldots, n\}$, $S = L_i^- \cup \check{P}_i \cup \hat{P}_{i+1} \cup \cdots \cup \hat{P}_{j-1} \cup \check{P}_j \cup L_j^+$, where L_i^- is a terminal section of P_i^-, and L_j^+ is an initial section of P_j^+. Note that a 3-separation (A, B) is displayed by $\mathbf{F}$ if and only if A is consecutive. Recall that when we say that a set S is contained in a petal of $\mathbf{F}$, we mean that $S \subseteq \hat{P}_i$ for some i.

Lemma 3.32. *Let* M *be a* 3*-connected matroid, let* $\mathbf{F}$ *be a maximal bloom of order at least* 4, *and let* (R, B) *be a* 3*-separation of* M. *If* (R, B) *is not displayed by* $\mathbf{F}$, *then either* R *or* B *is contained in a petal of* $\mathbf{F}$.

Proof. If (R, B) is non-sequential, then the result follows from [20, Theorem 8.1]. Thus we may assume that either R or B is sequential. Say that R is sequential. Then there is a triangle or triad $T = \{a, b, c\}$ of R such that $R \subseteq \mathrm{fcl}(T)$. Up to labels, there is an $i \in \{1, 2, \ldots, n - 2\}$ such that $a \in \hat{P}_1$, $b \in \hat{P}_i$, and $c \in \hat{P}_j$ for some $j \geqslant i$. As $i \leqslant n - 2$, by Lemma 3.9, $\hat{P}_1 \cup \hat{P}_2 \cup \cdots \cup \hat{P}_i$ is fully closed and either $c \in \hat{P}_1$ or $c \in \hat{P}_i$. Up to labels we may assume the former holds. Then $\{a, c\} \subseteq \hat{P}_1$ and $\hat{P}_1$ is fully closed, so $T \subseteq \hat{P}_1$ and indeed, $\mathrm{fcl}(T) \subseteq \hat{P}_1$. Thus R is contained in a petal. $\square$

The next lemma gives a more precise description of the way that a 3-separation can cross a bloom.

Lemma 3.33. *Let M be a 3-connected matroid, let $\mathbf{F} = (\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_n)$ be a maximal bloom of M of order at least 4, and let (R, B) be a 3-separation of M that is not displayed by $\mathbf{F}$. Then the following hold.*

(i) *Up to labels, $R \subseteq \hat{P}_1$.*
(ii) *$|R \cap P_1^-|, |R \cap P_1^+| \leqslant 1$, and if $R \cap P_1^- \neq \emptyset$, then $R \cap P_1^-$ is the last element of P_1^-, and if $R \cap P_1^+ \neq \emptyset$, then $R \cap P_1^+$ is the first element of P_1^+.*
(iii) *If $|R \cap \check{P}_1| = 1$, then R is a triangle or a triad.*
(iv) *If $|B \cap \check{P}_1| = 1$, then R is equivalent to the displayed 3-separation $R \cup \check{P}_1$.*

Proof. Part (i) is just a restatement of Lemma 3.32. Consider (ii). Assume that $R \cap \check{P}_1 = \emptyset$. If $R \subseteq P_1^-$, or $R \subseteq P_1^+$, then R is clearly a consecutive set of loose elements and is displayed by $\mathbf{F}$, although a flower displayed by $\mathbf{F}$ that displayed R would not be tight. Assume that $R \cap P_1^- \neq \emptyset$ and $R \cap P_1^+ \neq \emptyset$. In this case an appropriate concatenation of $\mathbf{F}$ and Lemma 3.15 show that $R \cap P_1^-$ and $R \cap P_1^+$ are fully skew contradicting Lemma 2.20.

Thus we may assume that $R \cap \check{P}_1 \neq \emptyset$. Say that $|R \cap L_1^-| \geqslant 2$. By uncrossing R with $\hat{P}_n$, we see that $\lambda(\hat{P}_n \cup R) = 2$. But $R \cap \check{P}_1 \neq \emptyset$, so $\hat{P}_n \cup R$ is not displayed by $\mathbf{F}$ and neither it nor its complement is contained in a petal, contradicting Lemma 3.32. Hence $|R \cap P_1^-| \leqslant 1$. Assume that $R \cap L_1^- = \{f\}$, where $\{f\}$ is not the last element of P_1^-. By Lemma 3.10, $f \notin \mathrm{cl}^{(*)}(R - \{f\})$, so R is split and by Lemma 2.20, (R, B) is not a 3-separation. Thus, if $R \cap P_1^-$ is nonempty, then $R \cap P_1^-$ is the last element of P_1^-. The same argument holds for $R \cap P_1^+$. Thus (ii) holds. Parts (iii) and (iv) are easy. □

We now consider how 4-separations cross swirl-like flowers and blooms. Observe that an unsplit exactly 4-separating set in a matroid has at least four elements.

Lemma 3.34. *Let $\mathbf{F} = (P_1, \ldots, P_n)$ be a maximal swirl-like flower of the 3-connected matroid M such that $\mathbf{F}$ has order at least 5, and $|E(M)| \geqslant 11$. Let (R, B) be an unsplit exact 4-separation of M. Assume that, amongst all equivalent flowers, $\mathbf{F}$ has a minimum number of petals crossed by (R, B). Then, up to labels, either*

(i) *for some $i \in \{1, 2, \ldots, n\}$ we have $P_1 \cup P_2 \cup \cdots \cup P_{i-1} \subseteq B$, the partition (R, B) crosses P_i, and $P_{i+1} \cup P_{i+2} \cup \cdots \cup P_n \subseteq R$; or*
(ii) *B crosses P_1 and P_2, and $B \subseteq P_1 \cup P_2$.*

Proof. Assume that the lemma fails and that (R, B) satisfies neither (i) nor (ii). We first show

3.34.1. *There is a petal of $\mathbf{F}$ that is not crossed by (R, B).*

Subproof. Assume that all petals of $\mathbf{F}$ are crossed by (R, B). By uncrossing, either $R \cap (P_1 \cup P_2 \cup P_3)$, or $B \cap (P_4 \cup P_5 \cup \cdots \cup P_n)$ is 3-separating. Assume the former case holds. Then, by Lemma 3.33, there is an equivalent flower in which this set crosses at most one petal. Such a flower has fewer petals crossed by (R, B), contradicting the choice of $\mathbf{F}$. Thus the latter case holds. But we may apply the previous argument to $B \cap (P_4 \cup P_5 \cup \cdots \cup P_n)$ unless this set has at most two elements. This means that $n = 5$, and $|B \cap P_4| = |B \cap P_5| = 1$. This argument extends to show that $|R \cap P_i| = |B \cap P_i| = 1$ for all $i \in \{1, \ldots, 5\}$, so that $|E(M)| = 10$ contradicting the fact that M has at least 11 elements. □

Thus we may assume that there is at least one petal contained in R. Up to labels, we may assume that $\{P_2, P_3, \ldots, P_{i-1}\}$ is a maximal consecutive set of petals each member of which is contained in R. Also we may assume that there are at least two crossed petals as otherwise, under the assumption that the lemma fails, we can routinely deduce from Lemma 3.15, that either R or B is split.

3.34.2. *Both P_1 and P_i are crossed by (R, B).*

Subproof. If neither P_1 nor P_i is crossed, then, by Lemma 3.15, R is split. Thus we may assume that P_1 is crossed. Assume that $P_i \subseteq B$ and let $\{P_i, P_{i+1}, \dots, P_j\}$ be a maximal set of consecutive petals each member of which is contained in B. If $P_{j+1} \subseteq R$, then, by Lemma 3.15, B is split. Thus P_{j+1} is crossed. A similar argument shows that P_l is crossed for all $l \in \{j+1, j+2, \dots, n\}$. In particular, P_n is crossed. If $P_1 = P_{j+1}$, then the lemma holds. Thus $j+1 \leqslant n$. By uncrossing, either $\lambda(P_1 \cup P_2 \cup \cdots \cup P_{i-1}) = 2$ or $\lambda(P_i \cup P_{i+1} \cup \cdots \cup P_n) = 2$. In either case, by Lemma 3.33, there is a flower equivalent to $\mathbf{F}$ that displays that 3-separation. But again by Lemma 3.33, such a flower has fewer petals crossed by (R, B). Note that for this to be true, we do need to know that P_n is crossed. $\square$

We now know that P_1 and P_i both cross (R, B). If $i = n$, then the lemma holds, so $i < n$. If $l \in \{i+1, i+2, \dots, n\}$, then $P_l \cap B \neq \emptyset$, otherwise, by Lemma 3.15, B is split. As B is not split, $|B| \geqslant 4$. Thus, either $|(P_i \cup P_{i+1} \cup \cdots \cup P_n) \cap B| \geqslant 3$, or $|(P_{i+1} \cup P_{i+2} \cup \cdots \cup P_n \cup P_1) \cap B| \geqslant 3$. Up to labels we may assume the latter holds. Clearly $|(P_2 \cup P_3 \cup \cdots \cup P_i) \cap R| \geqslant 3$. By uncrossing, one of these sets is 3-separating. But again, in either case, it follows straightforwardly from Lemma 3.33 that a flower equivalent to $\mathbf{F}$ crosses (R, B) in fewer petals. This final contradiction to the choice of $\mathbf{F}$ completes the proof. $\square$

We will say that the unsplit exact 4-separation (R, B) is 1-*crossing* or 2-*crossing* for $\mathbf{F}$ according as to whether Lemma 3.34(i) or (ii) holds. We now refine Lemma 3.34 to give a more precise description of the way that an unsplit 4-separation crosses a bloom.

Lemma 3.35. *Let $\mathbf{F} = (\hat{P}_1, \hat{P}_2, \dots, \hat{P}_n)$ be a bloom of the 3-connected matroid M, where $|E(M)| \geqslant 11$ and $n \geqslant 5$, and let (R, B) be an exact unsplit 4-separation of M. If (R, B) is 1-crossing, then, up to labels, $|R \cap \hat{P}_1|$, $|B \cap \hat{P}_1| \geqslant 2$, and one of the following holds.*

(i) $R \subseteq \hat{P}_1$.
(ii) *For some $i \in \{2, 3, \dots, n-1\}$, $R = (\hat{P}_1 \cap R) \cup \check{P}_2 \cup \hat{P}_3 \cup \cdots \cup \hat{P}_{i-1} \cup \check{P}_i \cup L_i^+$, where L_i^+ is an initial section of P_i^+. Moreover, $R \cap P_1^-$ is either empty or the last element of P_1^-, and $B \cap P_1^+$ is either empty or the first element of P_1^+.*

Proof. By Lemma 3.34, there is a flower $(P_1, P_2, \dots, P_n)$ displayed by $\mathbf{F}$ such that (R, B) crosses P_1 and $R = (P_1 \cap R) \cup P_2 \cup \cdots \cup P_i$ for some $i \in \{1, 2, \dots, n\}$. Thus, (R, B) crosses P_1. Assume that $|R \cap P_1| = 1$, say $R \cap P_1 = \{r\}$. Then the set $R - \{r\}$ is displayed in the flower $(P_1, P_2, \dots, P_n)$ so that $\lambda(R - \{r\}) = 2$. Hence $r \notin \mathrm{cl}^{(*)}(R - \{r\})$ implying that R is split. Therefore $|R \cap \hat{P}_1| \geqslant 2$ and similarly $|B \cap \hat{P}_1| \geqslant 2$.

Assume that (i) does not hold. Then, as (R, B) is 1-crossing, we may assume up to labels that $R = (\hat{P}_1 \cap R) \cup \check{P}_2 \cup \hat{P}_3 \cup \cdots \cup \hat{P}_{i-1} \cup \check{P}_i \cup L_i^+$, where L_i^+ is an initial section of P_i^+, and $i \in \{2, 3, \dots, n-1\}$. In particular $\check{P}_2 \subseteq R$ and $\check{P}_n \subseteq B$. Let $P_1^+ = (f_1, f_2, \dots, f_m)$. Assume for a contradiction that $P_1^+ \cap B \notin \{\emptyset, \{f_1\}\}$. Let k be the greatest integer such that $f_k \in B$. By assumption, $k > 1$. If $f_{k-1} \in B$, then R is split by Lemma 3.15. Thus $f_{k-1} \in R$. By Lemma 3.10, $f_k \notin \mathrm{cl}^{(*)}(\hat{P}_3 \cup \hat{P}_4 \cup \cdots \cup \hat{P}_n \cup \check{P}_1 \cup \{f_1, f_2, \dots, f_{k-2}\})$ and this set contains $B - \{f_k\}$. Thus f_k is isolated in $B - \{f_k\}$, contradicting the fact that B is unsplit. Hence $P_1^+ \cap B \in \{\emptyset, \{f_1\}\}$ and symmetrically $P_1^- \cap R$ is either empty or the last element of P_1^- so that (ii) holds. $\square$

Lemma 3.36. *Let $\mathbf{F} = (P_1, P_2, \dots, P_n)$ be a maximal tight swirl-like flower of order at least 5 in the 3-connected matroid M and let (R, B) be an unsplit exact 4-separation of M that is 2-crossing for $\mathbf{F}$. Assume that $R \subseteq P_1 \cup P_2$. Then the following hold.*

(i) $\lambda(P_1 \cap R) = \lambda(P_2 \cap R) = 2$.
(ii) $\{\sqcap(P_1 \cap R, P_2 \cap R), \sqcap^*(P_1 \cap R, P_2 \cap R)\} = \{0, 1\}$.

Proof. Assume that $|P_1 \cap R| = 1$; say $P_1 \cap R = \{x\}$. As (R, B) is not split, $x \in \mathrm{cl}^{(*)}(P_2 \cap R)$, and hence $x \in \mathrm{cl}(P_2)$, so that $(P_1 - \{x\}, P_2 \cup \{x\}, P_3, \ldots, P_n)$ is a flower equivalent to **F**, and (R, B) crosses only one petal of this flower, contradicting the fact that (R, B) is 2-crossing. Thus $|P_1 \cap R| \geqslant 2$, and hence $\lambda(P_1 \cap R) \geqslant 2$. Assume that $\lambda(P_1 \cap R) > 2$. Then, by uncrossing, $\lambda(P_2 \cup P_3 \cup \cdots \cup P_n \cup (P_1 \cap B)) = 2$. But, then, by Lemma 3.32, we can obtain a flower equivalent to **F** that crosses (R, B) in fewer petals. Hence $\lambda(P_1 \cap R) = 2$. Similarly $\lambda(P_2 \cap R) = 2$, so that (i) holds.

By Lemma 2.12 and (i), we have

$$\begin{aligned}3 = \lambda(R) &= \lambda(R \cap P_1) + \lambda(R \cap P_2) - \sqcap(R \cap P_1, R \cap P_2) - \sqcap^*(R \cap P_1, R \cap P_2)\\ &= 4 - \sqcap(R \cap P_1, R \cap P_2) - \sqcap^*(R \cap P_1, R \cap P_2).\end{aligned}$$

Part (ii) follows from this equation. □

Chapter 4. *k*-Coherent matroids

At last we can introduce the key connectivity notion for this paper. Let $k \geqslant 5$ be an integer. Note that if a matroid has a flower of order $n > k$, then it also has a flower of order k. A matroid M is *k-coherent* if it is 3-connected and has no swirl-like flowers of order k. The 3-connected matroid M is *k-fractured* if it has a swirl-like flower of order k. A *k-fracture* of M is a swirl-like flower in M of order at least k. This chapter is devoted to developing properties of k-coherent matroids, focussing particularly on issues related to preserving k-coherence in minors. In any unexplained context, if we say that M is a k-coherent matroid, then it will always be implicit that M is 3-connected and that k is an integer greater than 4.

An earlier draft of this paper had a number of purpose built proofs that focussed solely on k-coherence, but it soon became apparent that they were special cases of more general theorems related to the broader issue of when it is possible to remove elements without exposing 3-separations. The study of this question was undertaken in [22–24]. We now derive a number of our results as corollaries of theorems in [22–24].

We frequently illustrate situations for k-coherent matroids with schematic diagrams. In such diagrams it is always assumed that $k = 5$.

1. Exposure and *k*-coherence

We begin by recalling some terminology. Let x be an element of the 3-connected matroid M. Assume that $M\backslash x$ is 3-connected. Then a 3-separation (A, B) of $M\backslash x$ is *well blocked* by x if every 3-separation of $M\backslash x$ equivalent to (A, B) is blocked by x. In the case that (A, B) is well blocked by x we say that (A, B) is *exposed* by x or that *x exposes (A, B) in $M\backslash x$*. On the other hand, if M/x is 3-connected and (A, B) is a 3-separation in this matroid. Then (A, B) is *well coblocked* by x if every 3-separation of M/x equivalent to (A, B) is coblocked by x. In this case we say that *x exposes (A, B) in M/x*. Recall also that, if (R, B) is a 3-separation displayed by a flower **F** of a matroid, then (R, B) is *well displayed* if both R and B contain at least two tight petals of **F**. The first task of this chapter is to prove that if M is k-coherent, but $M\backslash x$ is 3-connected and k-fractured, then x exposes a 3-separation in $M\backslash x$. The next lemma is clear.

Lemma 4.1. *Let M and $M\backslash x$ be 3-connected matroids. If x is in the closure of a petal of a k-fracture of $M\backslash x$, then M is not k-coherent.*

The next lemma shows that we cannot lose k-coherence without exposing a 3-separation.

Lemma 4.2. *Let x be an element of the k-coherent matroid M such that $M\backslash x$ is 3-connected and k-fractured. Then x exposes a 3-separation in $M\backslash x$. Moreover, if **F** is a k-fracture of $M\backslash x$, then there is a 3-separation of $M\backslash x$ that is well displayed by **F** and is exposed by x.*

Proof. Assume that the lemma fails. Say $\mathbf{F} = (\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_m)$ is a maximal bloom that k-fractures $M\backslash x$. If (R, B) is a well-displayed 3-separation in $\mathbf{F}$, then either $x \in \mathrm{cl}_M(\mathrm{fcl}_{M\backslash x}(R))$ or $x \in \mathrm{cl}_M(\mathrm{fcl}_{M\backslash x}(B))$, as otherwise this is a 3-separation that is well blocked by x. Assume that, amongst all such 3-separations (R, B) is chosen so that $x \in \mathrm{cl}_M(\mathrm{fcl}_{M\backslash x}(R))$ and that $\mathrm{fcl}_{M\backslash x}(R)$ contains a minimum number of petals of $\mathbf{F}$. By Lemma 3.9, we may assume, up to labels, that $\mathrm{fcl}_{M\backslash x}(R) = \hat{P}_1 \cup \hat{P}_2 \cup \cdots \cup \hat{P}_i$, for some $i \in \{2, 3, \ldots, m-2\}$. But now, either $x \in \mathrm{cl}(\hat{P}_m \cup \hat{P}_1)$, or $x \in \mathrm{cl}(\hat{P}_2 \cup \hat{P}_3 \cup \cdots \cup \hat{P}_{m-1})$, as otherwise we again have a 3-separation exposed by x. By Lemma 3.31 either $x \in \mathrm{cl}(\hat{P}_1)$ or $x \in \mathrm{cl}(\hat{P}_2)$, contradicting the choice of (R, B). □

The following theorem is [24, Theorem 1.1].

Theorem 4.3. *Let M be a 3-connected matroid that is not a wheel or a whirl. Then M has an element x such that either $M\backslash x$ or M/x is 3-connected and does not expose any 3-separations.*

The next corollary follows immediately from Theorem 4.3 and Lemma 4.2.

Corollary 4.4. *Let M be a k-coherent matroid. If M is neither a wheel nor a whirl, then M has an element x such that either $M\backslash x$ or M/x is k-coherent.*

2. Some local wins

While it is good to know that there is almost always an element somewhere that can be removed to keep k-coherence, we also need to identify specific locations where we may remove elements and keep k-coherence. In this section we describe a number of such situations. Most often we focus on finding situations in which removing an element does not expose any 3-separations. We may omit the obvious corollary for preserving k-coherence. We begin by describing some straightforward cases. The next lemma is [24, Lemma 2.10]. The proof is short, so we give it here.

Lemma 4.5. *Let M be a 3-connected matroid and let A be a sequential 3-separating set of M with at least four elements. If $x \in A$ and $M\backslash x$ is 3-connected, then x does not expose any 3-separations in $M\backslash x$.*

Proof. By Lemma 2.34, $A - \{x\}$ is a sequential 3-separator of $M\backslash x$. By Lemma 2.31 $A - \{x\}$ has a triangle or triad T such that $\mathrm{fcl}_{M\backslash x}(T) \supseteq A - \{x\}$. Let (R, B) be a 3-separation in $M\backslash x$. Up to labels we may assume that $|R \cap T| \geqslant 2$. Then either $T \subseteq \mathrm{cl}_{M\backslash x}(R \cap T)$, or $T \subseteq \mathrm{cl}^*_{M\backslash x}(R \cap T)$, so that $A - \{x\} \subseteq \mathrm{fcl}_{M\backslash x}(R)$. In $M\backslash x$, the 3-separation (R, B) is equivalent to $(\mathrm{fcl}_{M\backslash x}(R), B - \mathrm{fcl}_{M\backslash x}(R))$. But $A \subseteq \mathrm{fcl}_{M\backslash x}(R)$, and $x \in \mathrm{cl}_M(A - \{x\})$, so that $(\mathrm{fcl}_{M\backslash x}(R), B - \mathrm{fcl}_{M\backslash x}(R))$ is induced in M and (R, B) is not exposed by x. □

For k-coherent matroids we have

Corollary 4.6. *Let M be a k-coherent matroid and let A be a sequential 3-separating set in M such that $|A| \geqslant 4$.*

(i) *If x is an element of A such that $M\backslash x$ is 3-connected, then $M\backslash x$ is k-coherent.*
(ii) *If $(a_1, a_2, \ldots, a_n)$ is a sequential ordering of A, then either $M\backslash a_n$ or M/a_n is k-coherent.*

Proof. Part (i) follows from Lemmas 4.2 and 4.5. Part (ii) follows from (i) and Lemma 2.32. □

Another easy case, also proved in [24], is provided by quads. The proof is similar to that of Lemma 4.5, but easier.

Lemma 4.7. *Let D be a quad of the 3-connected matroid M and let d be an element of D that is not in a triad. Then $M\backslash d$ is 3-connected and d does not expose any 3-separations in $M\backslash d$.*

Thus an element of a quad can be removed one way or the other to keep 3-connectivity and not expose 3-separations unless it is in both a triangle and triad, that is, unless it is in a 4-element fan. One may expect it to be difficult for every element of a quad to be in a 4-element fan, but it can happen. Indeed, let M be a spike with a tip t and cotip c and let $\{p_1, q_1\}$ and $\{p_2, q_2\}$ be legs of the spike. Then $\{p_1, q_1, p_2, q_2\}$ is a quad and (t, p_1, q_1, c) is a maximal fan with ends t and c.

Lemma 4.8. *Let D be a quad of the 3-connected matroid M. If D has an element that is in both a triangle and a triad, then every element of D is in a 4-element fan and there is a partition (D_1, D_2, C) of $E(M)$ with $D_1 \cup D_2 = D$ such that the following hold.*

(i) *(D_1, D_2, C) is a spike-like flower with tip $t \in C$ and cotip $c \in C$.*
(ii) *$D_1 \cup \{c, t\}$ and $D_2 \cup \{c, t\}$ are 4-element fans of M.*

Proof. Say $d \in D$, and that d belongs to a 4-element fan F. As d is in both a triangle and a triad, d is an internal element of F. Let $F = (t, d, x, c)$, where $\{t, d, x\}$ is a triangle and $\{d, x, c\}$ is a triad. As D is both a circuit and a cocircuit we see that both $\{t, d, x\}$ and $\{d, x, c\}$ contain exactly two elements of D so that $x \in D$ and $\{t, c\} \subseteq E(M) - D$. Let $D_1 = \{d, x\}$ and $D_2 = D - D_1$. Consider the flower $\mathbf{F} = (C, D_1, D_2)$. Note that t and c are loose elements of this flower, indeed, $t \in \mathrm{cl}(D_1)$, and $c \in \mathrm{cl}^*(D_1)$. By Lemma 3.22, $\mathbf{F}$ is spike-like, so that $t \in \mathrm{cl}(D_2)$ and $c \in \mathrm{cl}^*(D_2)$. The lemma now follows routinely. □

The next theorem is [24, Theorem 7.1]. Its proof is quite substantial.

Theorem 4.9. *Let $(P, \{a, b\}, Q)$ be a tight flower of a 3-connected matroid M where $\{a, b\}$ is fully closed and both P and Q have at least three elements. Then the following hold.*

(i) *If a is in a triangle, then $M\backslash a$ is 3-connected and has no 3-separations exposed by a.*
(ii) *If a is in a triad, then M/a is 3-connected and has no 3-separations exposed by a.*
(iii) *If a is in neither a triangle nor a triad, then both $M\backslash a$ and M/a are 3-connected.*

Moreover, if a is in neither a triangle nor a triad and both $M\backslash a$ and M/a have 3-separations exposed by a, then $|P| = |Q| = 4$, both $M\backslash b$ and M/b are 3-connected, and neither $M\backslash b$ nor M/b has a 3-separation exposed by b.

Combining Theorem 4.9 with other facts we obtain the following consequence.

Corollary 4.10. *Let $(P, \{a, b\}, Q)$ be a tight flower of the k-coherent matroid M where $\{a, b\}$ is fully closed. Then exactly one of the following holds.*

(i) *Either $M\backslash a$ or M/a is k-coherent.*
(ii) *Up to labels, $P \cup \{a, b\}$ is a quad. Moreover, there is a labelling p_1, p_2 of the elements of P such that $(\{p_1, a\}, \{p_2, b\}, Q)$ is a spike-like flower with a tip and a cotip.*

Proof. Note that a 3-connected matroid with at most nine elements is k-coherent. By Theorem 4.9 and Lemma 4.2 part (i) holds unless either $|P| = 2$ or $|Q| = 2$. Assume that $|P| = 2$. Then, as the flower is tight, $P \cup \{a, b\}$ is a quad in M. If a is not in both a triangle and a triad, then it follows from Lemma 4.7 that (i) holds. Otherwise it follows from Lemma 4.8 that (ii) holds. □

We illustrate Corollary 4.10 with some examples. Consider the matroid M illustrated in Fig. 4.1. Let $P = P_1 \cup P_2$ and $Q = P_3 \cup P_4 \cup P_5$. Then the flower $(P, \{a, b\}, Q)$ satisfies the hypotheses of Corollary 4.10. Assume that M is 5-coherent. Note that $(P_1 \cup \{b\}, P_2, P_3, P_4, P_5)$ is a 5-fracture of $M\backslash a$, so that $M\backslash a$ is not 5-coherent. Nonetheless, M/a is 5-coherent. Due to the way that a and b are

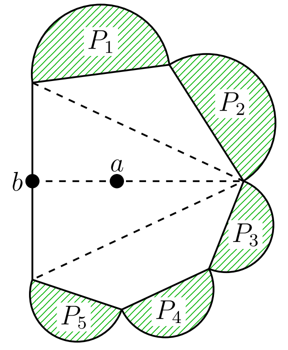

Fig. 4.1. Outcome (i) of Corollary 4.10.

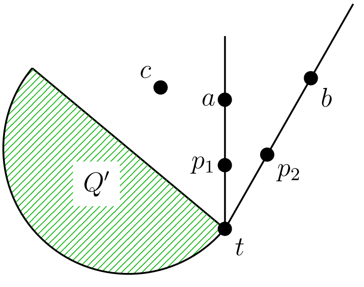

Fig. 4.2. Outcome (ii) of Corollary 4.10.

aligned in this particular case, $(P_1, P_2 \cup \{b\}, P_3, P_4, P_5)$ is a 5-fracture of M/b. In this case $M\backslash b$ is 5-coherent.

The next example illustrates the need for outcome (ii) in Corollary 4.10. Consider the matroid illustrated in Fig. 4.2. Let $Q = Q' \cup \{c, t\}$. Note that $(\{b, p_2\}, \{a, p_1\}, Q)$ is a respectable spike-like flower with tip t and cotip c. Unfortunately, we were given the perverse flower $(\{p_1, p_2\}, \{a, b\}, Q)$ in which $\{a, b\}$ is tight and fully closed.

Lemma 4.11. *Let M be a k-coherent matroid, and let $\mathbf{R} = (R_1, R_2, \ldots, R_m)$ be a flower of M, where $m \geqslant 4$ and $|R_1| \in \{2, 3, 4\}$. If $p \in R_1$ and $M\backslash p$ is 3-connected, then $M\backslash p$ is k-coherent.*

Proof. Assume that $|R_1| = 2$, say $R_1 = \{p, p'\}$. Let $\mathbf{P} = (\{p, p'\}, P_1, P_2, P_3)$ be a 4-petal concatenation of $\mathbf{R}$. As $M\backslash p$ is 3-connected, $p \in \mathrm{cl}_M(P_1 \cup \{p'\})$ and $p \in \mathrm{cl}_M(P_3 \cup \{p'\})$. Clearly $(\{p'\} \cup P_1, P_2, P_3)$ and $(P_1, P_2, P_3 \cup \{p'\})$ are flowers in $M\backslash p$.

Assume that the lemma fails and let $\mathbf{Q} = (\hat{Q}_1, \hat{Q}_2, \ldots, \hat{Q}_n)$ be a bloom of $M\backslash p$ that k-fractures $M\backslash p$. Assume that $P_1 \cup \{p'\}$ is not displayed by $\mathbf{F}$. Then, by Lemma 3.33, either $P_1 \cup \{p'\}$ or $P_2 \cup P_3$ is contained in $\hat{Q}_i$ for some $i \in \{1, 2, \ldots, n\}$. The former case implies that $p \in \mathrm{cl}_M(\hat{Q}_i)$, so that, by Lemma 4.1, M is k-fractured. Consider the latter case. Evidently $p' \in \mathrm{cl}^{(*)}_{M\backslash p}(P_3)$ so that $p' \in \mathrm{cl}^{(*)}_{M\backslash p}(\hat{Q}_i)$, a fully-closed set. Thus $P_3 \cup \{p'\} \subseteq \hat{Q}_i$. Again we see that $p \in \mathrm{cl}_M(\hat{Q}_i)$ and again we obtain a contradiction to the fact that M is k-coherent.

Thus $P_1 \cup \{p'\}$ and, similarly $P_3 \cup \{p'\}$ are both displayed in $\mathbf{Q}$ and neither is contained in a petal of $\mathbf{Q}$. It is now clear that we have a quasi-flower $(\{p'\}, Q'_1, Q'_2, Q'_3, Q'_4)$ displayed by $\mathbf{Q}$, where $P_1 = Q'_1 \cup Q'_2$, $P_2 = Q'_3$ and $P_3 = Q'_4$, and having the further property that, for some $i \in \{1, 2, \ldots, n\}$ we have $\check{Q}_i \subseteq Q'_1 \subseteq Q'_1 \cup \{p'\} \subseteq \hat{Q}_i$. Now $p \in \mathrm{cl}_M(\{p'\} \cup Q'_1 \cup Q'_2)$ and $p \in \mathrm{cl}_M(Q'_4 \cup Q'_1 \cup \{p'\})$, so, by Lemma 3.31, $p \in \mathrm{cl}_M(\{p'\} \cup Q'_1)$. Thus $p \in \mathrm{cl}(\hat{P}_i)$. Again we obtain the contradiction that M is k-fractured. Thus the lemma holds in the case that $|R_1| = 2$.

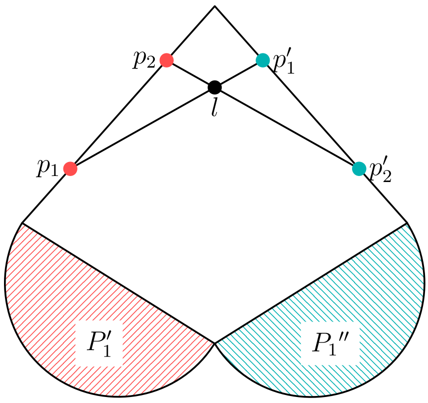

Fig. 4.3. Illustration for Lemma 4.13.

A similar, but easier, analysis establishes the lemma in the case when R_1 has three elements. The case when $|R_1| = 4$ is too easy to resist. In this case, if R_1 is sequential, then it follows from Corollary 4.6 that $M\backslash x$ is k-coherent. On the other hand, if R_1 is non-sequential, then R_1 is a quad, and it follows from Lemma 4.7 that $M\backslash x$ is k-coherent. $\square$

If a triangle T is not in a 4-element fan then, as we shall see in the next section, it is not always possible to find an element of T that can be deleted to preserve k-coherence. Nonetheless things often work out well. The next lemma is [22, Theorem 4.2].

Lemma 4.12. *Let $\{a, b, c\}$ be a triangle in a 3-connected matroid M having at least four elements. Assume that $\{a, b, c\}$ is not contained in a 4-element fan, and that $M\backslash b$ is not 3-connected. If $z \in \{a, b\}$ then $M\backslash z$ is 3-connected and z does not expose a 3-separation in $M\backslash z$.*

We now consider loose elements in swirl-like or spike-like flowers. It turns out that, apart from some almost degenerate situations we can remove loose elements of such flowers and preserve k-coherence. The degeneracy is again related to the fact that we may partition quads arbitrarily into 2-element subsets to obtain 3-petal flowers. Fig. 4.3 illustrates the needs for the constraints in part (iii) of Lemma 4.13. Here $P_1 = \{p_1, p'_1\}$, $P_2 = \{p_2, p'_2\}$ and $P_3 = P'_3 \cup P''_3$. Deleting l exposes the 3-separation $(\{p_1, p_2\} \cup P'_3, \{p'_1, p'_2\} \cup P''_3)$.

Lemma 4.13. *Let M be a 3-connected matroid and let $\mathbf{F} = (P_1 \cup \{l\}, P_2, \dots, P_n)$ be a spike-like or swirl-like flower of M with at least three petals and $|P_1| \geqslant 2$. Assume that l is a loose element between $P_1 \cup \{l\}$ and P_2 and that $M\backslash l$ is 3-connected.*

(i) *If either $P_1 \cup \{l\}$ or P_2 is a loose petal, then l does not expose any 3-separations in $M\backslash l$.*
(ii) *If $n \geqslant 4$, then l does not expose any 3-separations in $M\backslash l$.*
(iii) *If $n = 3$ and at most one member of $\{P_1, P_2, P_3\}$ has two elements, then l does not expose any 3-separations in $M\backslash l$.*

Proof. We first show that (i) holds. Assume that $P_1 \cup \{l\}$ is a loose petal. Then, as $|P_1 \cup \{l\}| \geqslant 3$, it must be that case that $\mathbf{F}$ is swirl-like. By Theorem 3.7(ii), $P_1 \cup \{l\}$ is a fan. If $|P_1 \cup \{l\}| \geqslant 4$, then it follows from Corollary 4.6 that l does not expose a 3-separation in $M\backslash l$. Thus we may assume that $|P_1 \cup \{l\}| = 3$. Say that, written as an ordered fan, $P_1 \cup \{l\} = (f_1, f_2, l)$. In this case $\{f_1, f_2, l\}$ is a triangle and is not contained in a 4-element fan as otherwise we may again apply Corollary 4.6. As l is loose, $l \in \mathrm{cl}^{(*)}(P_2)$. If $l \in \mathrm{cl}^*(P_2)$, then $M\backslash l$ is not 3-connected. Thus $l \in \mathrm{cl}(P_2)$. It now follows from Lemma 3.10 that $f_2 \in \mathrm{cl}^*(P_2 \cup \{l\})$, so that $M\backslash f_2$ is not 3-connected. In this case l does not expose any 3-separations in $M\backslash l$ by Lemma 4.12. The same argument applies in the case that P_2 is loose and (i) holds.

Consider (ii). Assume that $\mathbf{F}$ has at least four petals. Assume that (R, B) is a 3-separation of $M\backslash l$ that is exposed by l. By (i), both $P_1 \cup \{l\}$ and P_2 are tight petals. Let $\mathbf{F}'' = (P_1, P_2, P'_3, P'_4)$ be a concatenation of $(P_1, P_2, \dots, P_n)$ and let $\mathbf{F}'$ be a maximal flower such that $\mathbf{F}'' \preccurlyeq \mathbf{F}'$ and such that $\mathbf{F}'$ displays $\mathbf{F}''$. Such a flower exists by Lemma 3.24. Say

$$\mathbf{F}' = (P_{11}, \dots, P_{1i}, P_{21}, \dots, P_{2j}, P_{31}, \dots, P_{3k}, P_{41}, \dots, P_{4m}),$$

where $P_1 = P_{11} \cup \dots \cup P_{1i}$, $P_2 = P_{21} \cup \dots \cup P_{2j}$, $P'_3 = P_{31} \cup \dots \cup P_{3k}$, and $P'_4 = P_{41} \cup \dots \cup P_{4m}$. Then $l \in \mathrm{cl}(P_1)$ and $l \in \mathrm{cl}(P_2)$, so, by Lemma 3.12, $l \in \mathrm{cl}(P_{1i})$ and $l \in \mathrm{cl}(P_{21})$. As $\mathbf{F}'$ is maximal, (R, B) conforms with $\mathbf{F}'$. Thus there is a 3-separation (R', B'), equivalent to (R, B) such that either (R', B') is displayed by $\mathbf{F}'$, or one of R' or B' is contained in a petal of $\mathbf{F}'$. In either case we deduce that either P_{1i} or P_{21} does not cross (R', B'). But then, as $l \in \mathrm{cl}(P_{1i})$ and $l \in \mathrm{cl}(P_{21})$, we see that either $l \in \mathrm{cl}(R')$ or $l \in \mathrm{cl}(B')$, contradicting the assumption that (R, B) is well blocked by l. Thus (ii) holds.

Consider (iii). Here $\mathbf{F} = (P_1 \cup \{l\}, P_2, P_3)$. Again assume that (R, B) is a 3-separation of $M\backslash l$ exposed by l. Consider the flower (P_1, P_2, P_3) of $M\backslash l$. As $l \in \mathrm{cl}(P_1)$, and $l \in \mathrm{cl}(P_2)$, (R, B), and any 3-separation of $M\backslash l$ equivalent to (R, B), crosses both P_1 and P_2. As at most one member of $\{P_1, P_2, P_3\}$ has two elements, we may assume up to labels that $|P_1| > 2$ and that $|P_1 \cap R| > 2$.

4.13.1. $|P_1 \cap B| \geqslant 2$.

Subproof. Assume that $|B \cap P_1| = 1$; let $\{b\} = B \cap P_1$. We have $|B - P_1| \geqslant 2$, so that $R \cup P_1$ avoids at least two elements of $E(M\backslash l)$. As $|R \cap P_1| \geqslant 2$, by uncrossing we see that $R \cup P_1$ is 3-separating. But $R \cup P_1 = R \cup \{b\}$. Hence $(R, B) \cong (R \cup \{b\}, B - \{b\})$, but this latter 3-separation does not cross P_1. □

A similar uncrossing argument to that of 4.13.1 shows that $|P_2 \cap R| = 1$ if and only if $|P_2 \cap B| = 1$. Using an uncrossing argument and possibly moving to a 3-separation equivalent to (R, B) we may also assume that $|P_3 \cap R| = 1$ if and only if $|P_3 \cap B| = 1$.

4.13.2. $|P_i \cap R| = |P_i \cap B| = 1$ *for some* $i \in \{2, 3\}$.

Subproof. If neither P_2 nor P_3 has the property of the sublemma, then we may apply Lemma 3.20 and obtain a flower $\mathbf{F}'$ that refines (P_1, P_2, P_3) and displays a 3-separation equivalent to (R, B). As $l \in \mathrm{cl}(P_1)$ and $l \in \mathrm{cl}(P_2)$, it follows from Lemma 3.12, that, for some petal P of $\mathbf{F}'$, we have $l \in \mathrm{cl}(P)$. But, as (R, B) is displayed, either $P \subseteq R$ or $P \subseteq B$. In either case we contradict the assumption that (R, B) is well blocked by l. □

4.13.3. $|P_2 \cap R| = |P_2 \cap B| = |P_3 \cap R| = |P_3 \cap B| = 1$.

Subproof. Let (Q_2, Q_3) be a permutation of (P_2, P_3) and assume for a contradiction that $|R \cap Q_2| \geqslant 2$. By 4.13.2, $|Q_3 \cap R| = |Q_3 \cap B| = 1$. Let $(r, b) = (Q_3 \cap R, Q_3 \cap B)$.

Assume that $Q_2 \cap B = \emptyset$. Then, by uncrossing, $(R \cup \{b\}, B - \{b\})$ is a 3-separation in $M\backslash l$. But $(R \cup \{b\}, B - \{b\}) \cong (R, B)$ and $B - \{b\} \subseteq P_1$, so $(R \cup \{b\}, B - \{b\})$ is not blocked by l, contradicting the assumption that (R, B) is well blocked by l. Thus $Q_2 \cap B \neq \emptyset$ and, as $|Q_2 \cap R| \geqslant 2$, we deduce that $|Q_2 \cap B| \geqslant 2$.

Now $|E(M\backslash l) - (B \cup (P_1 \cup Q_3))| \geqslant 2$ and $|B \cap (P_1 \cup Q_3)| \geqslant 2$. So by uncrossing, $\lambda_{M\backslash l}(B \cap (P_1 \cup Q_3)) = 2$. Note that $B \cap (P_1 \cup Q_3) = (B \cap P_1) \cup \{b\}$. A similar uncrossing argument shows that $\lambda_{M\backslash l}(B \cap P_1) = 2$. As $B \cap P_1$ and $(B \cap P_1) \cup \{b\}$ are both exactly 3-separating in $M\backslash l$, it follows that $b \in \mathrm{cl}^{(*)}_{M\backslash l}(B \cap P_1)$ and hence, in M, we have $b \in \mathrm{cl}^{(*)}_M(P_1 \cup \{l\})$. By symmetry we also deduce that $r \in \mathrm{cl}^{(*)}_M(P_1 \cup \{l\})$. Thus r and b are loose elements of the flower $\mathbf{F}$, and by the structure of loose elements in swirl-like or spike-like flowers, we deduce that, up to labels, we have $b \in \mathrm{cl}_M(P_1 \cup \{l\})$. As $l \in \mathrm{cl}_M(P_1)$, we have

$\{b,l\} \subseteq \mathrm{cl}_M(P_1)$. But, in the flower $(P_1, P_2 \cup \{l\}, P_3)$, the pair $\{b,l\}$ is a subset of the closure of a single petal and, by the structure of swirl-like and spike-like flowers, there is at most one element in the petal $P_2 \cup \{l\}$ contained in the closure of P_1. This contradiction establishes the sublemma. □

By 4.13.3, $|P_2| = |P_3| = 2$ contradicting the assumption that at most one of these sets has two elements and (iii) follows. □

3. *k*-Wild triangles

Let $\{a,b,c\}$ be a triangle of the k-coherent matroid M that is not in a 4-element fan. We have seen in Lemma 4.12 that if $M\backslash b$ is not 3-connected, then both $M\backslash a$ and $M\backslash c$ are k-coherent. Unfortunately the case arises where $M\backslash a$, $M\backslash b$, and $M\backslash c$ are 3-connected and k-fractured. The remainder of this section is devoted to describing the structure of a matroid relative to such a triangle. The results of this section are essentially corollaries of results in [22].

We begin by recalling the $\Delta - Y$ operation for matroids. Let $\Delta = \{a,b,c\}$ be a triangle of the matroid M and take a copy of $M(K_4)$ having Δ as a triangle and $\{a',b',c'\}$ as the complementary triad, labelled such that $\{a,b',c'\}$, $\{a',b,c'\}$ and $\{a',b',c\}$ are triangles. Let $P_\Delta(M(K_4),M)$ denote the generalised parallel connection of $M(K_4)$ and M. We write ΔM for $P_\Delta(M(K_4),M)\backslash\Delta$ and say that ΔM is obtained from M by a $\Delta - Y$ *exchange* on Δ. As is common practice, we relabel a', b' and c' as a, b and c so that M and ΔM have the same ground set. The matroid N is obtained from M by performing a $Y - \Delta$ exchange on a triad $\{a,b,c\}$ if N^* is obtained from M^* by performing a $\Delta - Y$ exchange on the triangle $\{a,b,c\}$ of M^*.

A triangle T of the 3-connected matroid M is *wild* if, for all $t \in T$, either $M\backslash t$ is not 3-connected, or $M\backslash t$ is 3-connected and exposes a 3-separation in M. The structure of wild triangles is described in [22], and we will later outline the results from there that we need. First we describe the particular type of wild triangle that is problematic from the perspective of k-coherence.

Let M be a k-coherent matroid and $T = \{a,b,c\}$ be a triangle of M. Then T is *k-wild* if $M\backslash t$ is 3-connected and k-fractured for all t in T. If T is a k-wild triangle, then a *k-wild display* for T is a partition

$$(A_1, A_2, \dots, A_{k-2}, B_1, B_2, \dots, B_{k-2}, C_1, C_2, \dots, C_{k-2})$$

of $E(M) - T$ such that the following hold, where $A = A_1, A_2 \cup \dots \cup A_{k-2}$, $B = B_1, B_2 \cup \dots \cup B_{k-2}$ and $C = C_1, C_2 \cup \dots \cup C_{k-2}$.

(i) $(A_1, A_2, \dots, A_{k-2}, B \cup C \cup T)$, $(B_1, B_2, \dots, B_{k-2}, A \cup C \cup T)$, and $(C_1, C_2, \dots, C_{k-2}, A \cup B \cup T)$ are tight swirl-like flowers of M.
(ii) $(A_1, A_2, \dots, A_{k-2}, B \cup \{b\}, C \cup \{c\})$, $(A \cup \{a\}, B_1, B_2, \dots, B_{k-2}, C \cup \{c\})$, and $(A \cup \{a\}, B \cup \{b\}, C_1, C_2, \dots, C_{k-2})$ are k-fractures of $M\backslash a$, $M\backslash b$ and $M\backslash c$ respectively.

Moreover, T is a *standard* k-wild triangle if it has a k-wild display such that $(A \cup \{a\}, B \cup \{b\}, C \cup \{c\})$ is a swirl-like flower of M. Fig. 4.4 illustrates a standard 5-wild triangle.

We now describe another type of k-wild triangle obtained from a $\Delta - Y$ exchange. We first note an elementary lemma. We omit the easy proof.

Lemma 4.14. *Let T be a triangle of the matroid M; let ΔM be the matroid obtained by performing a $\Delta - Y$ exchange on T; and let A be a set of elements of M. Then the following hold.*

(i) *If $T \subseteq A$, then $r_{\Delta M}(A) = r_M(A) + 1$.*
(ii) *If $T \cap A = \emptyset$, then $r_{\Delta M}(A) = r_M(A)$.*
(iii) *If $T \subseteq A$, then $\lambda_{\Delta M}(A) = \lambda_M(A)$.*
(iv) *If $t \in T$, then $M\backslash t \cong \Delta M/t$ where the isomorphism is obtained by switching the labels of the other elements of T.*

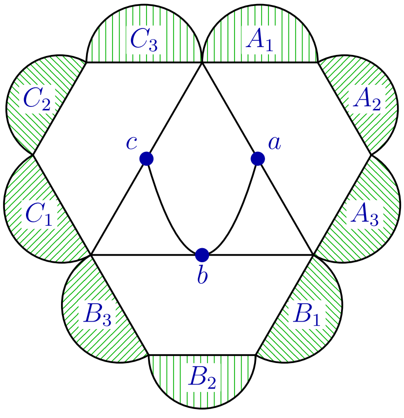

Fig. 4.4. A standard 5-wild triangle.

Lemma 4.15. *Let T be a triangle of the matroid M such that T is not contained in a 4-element fan and let M' be the matroid obtained by first performing a $\Delta - Y$ exchange on T and then taking the dual. Then the following hold.*

(i) *M' is 3-connected if and only if M is.*
(ii) *Assume that M is 3-connected, that $\mathbf{P} = (P_1, P_2, \ldots, P_m)$ is a swirl-like flower of M of order at least 3, and T is contained in a petal of $\mathbf{P}$. Then $\mathbf{P}$ is tight in M if and only if $\mathbf{P}$ is tight in M'.*
(iii) *M' is k-coherent if and only if M is.*
(iv) *Assume that M is k-coherent. Then T is k-wild in M if and only if T is k-wild in M'.*

Proof. First note that, by definition, M is obtained from M' by first taking the dual and then performing a $Y - \Delta$ exchange. Again by definition this means that we first perform a $\Delta - Y$ exchange on M' and then take the dual. Thus the operation we are considering is an involution and we need only prove the parts of the lemma in one direction. The straightforward proof of (i) is given in [22, Lemma 8.2] and we omit it here.

Consider (ii). Assume that $T = \{a, b, c\}$ and $T \subseteq P_2$. It is easily seen that $\mathbf{P}$ is a swirl-like flower of M'. The only problem that could happen is that P_2 either ceases to be tight or becomes tight. (Indeed this could happen if $\mathbf{P}$ were a copaddle, so the issue needs addressing.) By the observation at the start of the proof we may assume that P_2 is a loose petal of $\mathbf{P}$ in M. Say $P_2 \neq T$. Then T belongs to a 4-element fan in M' and performing a $\Delta - Y$ exchange on a triangle in a 4-element fan destroys 3-connectivity. Thus we may assume that $T = P_2$. We now prove that P_2 is a loose petal of $\mathbf{P}$ in M'. Up to labels, we have $a \in \mathrm{cl}_M(P_1)$, $b \in \mathrm{cl}^*_M(P_1 \cup \{a\})$ and $c \in \mathrm{cl}_M(P_3)$. Note that, $\{a', b', c\}$ is a triangle in $P_\Delta(M(K_4), M)$. Thus $r_{\Delta M}(E(M) - (P_1 \cup \{c\})) = r_M(E(M) - P_1)$. Evidently $r_{\Delta M}(P_1 \cup \{c\}) = r_M(P_1) + 1$. Hence $\lambda_{\Delta M}(P_1 \cup \{c\}) = 2$ so that $c \in \mathrm{cl}^*_{\Delta M}(P_1)$. It follows easily that $b \in \mathrm{cl}_{\Delta M}(P_1 \cup \{c\})$. By symmetry $a \in \mathrm{cl}^*_{\Delta M}(P_3)$ and part (ii) follows.

Consider (iii). Assume that M is not k-coherent. Let $(P_1, P_2, \ldots, P_m)$ be a tight flower that k-fractures M. By Lemma 3.33 we may assume up to labels that $T \subseteq P_1$. By (iii), $(P_1, P_2, \ldots, P_m)$ is also a tight flower of M' so that M' is not k-coherent.

Assume that M is k-coherent and T is k-wild. Then it has a k-wild display. By the earlier parts of this lemma, M' is also k-coherent and, indeed, the k-wild display is also a k-wild display for T in M'. Thus T is k-wild in M'. $\square$

The k-wild triangle T of the k-coherent matroid M is a *costandard* k-wild triangle if T is a standard wild triangle in the matroid M' constructed in Lemma 4.15. It does not seem easy to produce a schematic diagram for a costandard k-wild triangle that is at all insightful.

Another type of triangle that has no element that can be deleted to preserve k-coherence is one that is an internal triangle in a fan. Note that such a triangle has two elements a and c such that $\mathrm{co}(M\backslash a)$ and $\mathrm{co}(M\backslash c)$ are both k-coherent. We can now state the main result of this section.

Theorem 4.16. *Let T be a triangle of the k-coherent matroid M. Assume that $M\backslash t$ is not k-coherent for all $t \in T$. Then T is either an internal triangle in a fan of M, a standard k-wild triangle, or a costandard k-wild triangle.*

To prepare for the proof of Theorem 4.16 we recall material from [22]. Evidently a k-wild triangle is wild. If T is a wild triangle of the 3-connected matroid M, then $\{a,b,c\}$ is a *standard* wild triangle if there is a partition $\mathbf{P} = (P_1, P_2, \dots, P_6)$ of $E(M) - \{a,b,c\}$ such that $|P_i| \geqslant 2$ for all i and the following hold.

(i) $M\backslash a$, $M\backslash b$, and $M\backslash c$ are 3-connected, $M\backslash a,b,c$ is connected and $\mathrm{co}(M\backslash a,b,c)$ is 3-connected.
(ii) $(P_1 \cup P_2 \cup \{a\}, P_3 \cup P_4 \cup \{b\}, P_5 \cup P_6 \cup \{c\})$ is a flower in M.
(iii) $(P_2 \cup P_3 \cup P_4 \cup \{b\}, P_5 \cup P_6 \cup P_1 \cup \{c\})$, $(P_4 \cup P_5 \cup P_6 \cup \{c\}, P_1 \cup P_2 \cup P_3 \cup \{a\})$ and $(P_6 \cup P_1 \cup P_2 \cup \{a\}, P_3 \cup P_4 \cup P_5 \cup \{b\})$ are 3-separations exposed in M by a, b, and c respectively.

A partition $\mathbf{P}$ satisfying these conditions is a partition *associated* to $\{a,b,c\}$.

If the wild triangle T is obtained from a standard wild triangle in a 3-connected matroid by first doing a $\Delta - Y$ exchange and dualising in the manner of Lemma 4.15, then T is a *costandard* wild triangle. One other type of wild triangle needs to be described.

Let $R = \{a,b,c,s,t,u,v\}$ be a 3-separating set in a 3-connected matroid M, where $\{a,b,c\}$ is a triangle. Then R is a *trident* with wild triangle $\{a,b,c\}$ if $\{t,s,u,b\}$, $\{t,u,v,c\}$, and $\{t,s,v,a\}$ are exposed quads in $M\backslash a$, $M\backslash b$ and $M\backslash c$ respectively. The main theorem of [22] is

Theorem 4.17. *Let T be a wild triangle of a 3-connected matroid M with at least twelve elements. Then T is either a standard wild triangle, a costandard wild triangle, a triangle in a trident of M, or an internal triangle of a fan of M.*

We will also need the next theorem from [22].

Theorem 4.18. *Let $\{a,b,c\}$ be a standard wild triangle in a 3-connected matroid M where $|E(M)| \geqslant 12$ and let (X_1, X_2), (Y_1, Y_2), and (Z_1, Z_2) be 3-separations exposed by a, b and c, respectively, with $a \in Y_2 \cap Z_1$, $b \in Z_2 \cap X_1$, and $c \in X_2 \cap Y_1$. Then (X_1, X_2), (Y_1, Y_2) and (Z_1, Z_2) can be replaced by equivalent 3-separations such that $(X_2 \cap Y_2, Z_1 \cap X_1, Y_2 \cap Z_2, X_1 \cap Y_1, Z_2 \cap X_2, Y_1 \cap Z_1)$ is a partition associated to $\{a,b,c\}$.*

Lemma 4.19. *Let $T = \{a,b,c\}$ be a k-wild triangle of the k-coherent matroid M. If T is a standard wild triangle of M, then T is a standard k-wild triangle of M.*

Proof. As T is k-wild, there are 3-separations (X_1, X_2), (Y_1, Y_2) and (Z_1, Z_2) exposed by a, b and c respectively that are displayed in k-fractures of $M\backslash a$, $M\backslash b$ and $M\backslash c$ respectively. We may replace these by any equivalent 3-separations. So by Theorem 4.18, we may assume that $(X_2 \cap Y_2, Z_1 \cap X_1, Y_2 \cap Z_2, X_1 \cap Y_1, X_2 \cap Z_2, Y_1 \cap Z_1)$ is a partition associated to $\{a,b,c\}$.

Let $P_1 = X_2 \cap Y_2$, $P_2 = X_1 \cap Z_1$, $P_3 = Y_2 \cap Z_2$, $P_4 = X_1 \cap Y_1$, $P_5 = X_2 \cap Z_2$ and $P_6 = Y_1 \cap Z_1$. Then, by the definition of an associated partition, $(P_1 \cup P_2 \cup \{a\}, P_3 \cup P_4 \cup \{b\}, P_5 \cup P_6 \cup \{c\})$ is a flower in M, so that $(P_1 \cup P_2, P_3 \cup P_4 \cup \{b\}, P_5 \cup P_6 \cup \{c\})$ is a flower in $M\backslash a$. The 3-separation $(X_1, X_2) = (P_2 \cup P_3 \cup P_4 \cup \{b\}, P_5 \cup P_6 \cup P_1 \cup \{c\})$ crosses this flower so by Lemma 3.20 $(P_1, P_2, P_3 \cup P_4 \cup \{b\}, P_5 \cup P_6 \cup \{c\})$ is a flower in $M\backslash a$. As (X_1, X_2) is displayed in a k-fracture of $M\backslash a$, this flower refines to a k-fracture $\mathbf{A}$ of $M\backslash a$. This shows that the above flower is swirl-like.

We now show that $\mathbf{A}$ is obtained by refining P_1 and P_2. Assume otherwise. Then we may assume that there is a partition (P', P'') of $P_3 \cup P_4 \cup \{b\}$ such that $(P_1, P_2, P', P'', P_5 \cup P_6 \cup \{c\})$ is a flower in $M\backslash a$. Certainly $a \in \mathrm{cl}(P_1 \cup P_2)$, so that $(P_1 \cup P_2 \cup \{a\}, P', P'', P_5 \cup P_6 \cup \{c\})$ is a flower in M. We may assume that b is in P''. Then, as $b \in \mathrm{cl}(\{a, c\})$, we see by Lemma 3.10 that, up to labels, $(P_1 \cup P_2 \cup \{a\}, P', P'' - \{b\}, P_5 \cup P_6 \cup \{b, c\})$ is a flower in M and, indeed, $(P_1 \cup P_2, P', P'' - \{b\}, P_5 \cup P_6 \cup \{a, b, c\})$ is a flower in M. This contradicts the fact that a exposes the 3-separation (X_1, X_2).

We deduce that there is a refinement $(A_1, \dots, A_m)$ of $P_1 \cup P_2$ such that $(A_1, \dots, A_m, P_3 \cup P_4 \cup \{b\}, P_5 \cup P_6 \cup \{c\})$ is a k-fracture of $M\backslash a$. As M is k-coherent, and a does not block $A_1 \cup \dots \cup A_m$, we see that $m = k - 2$. The lemma follows by repeating the above argument in $M\backslash b$ and $M\backslash c$. $\square$

While triangles in tridents are a problem in the general case, they cause no difficulties in the k-coherent case.

Lemma 4.20. *Let T be a triangle in a trident of the k-coherent matroid M. Then T is not k-wild.*

Proof. Assume that $\{a, b, c\}$ is in the trident $X = \{a, b, c, t, s, u, v\}$ where the labelling accords with that given in the definition of a trident. Assume that $\{a, b, c\}$ is k-wild. Let **A** be a maximal k-fracture of $M\backslash a$. Certainly $X - \{a\}$ is not contained in a petal of **A**, as otherwise M is k-fractured. It is now readily verified that the quad $\{t, s, u, b\}$ of $M\backslash a$ is displayed in **A**. This shows that, for some $m \geqslant 2$, there is a partition $(P_1, P_2, \dots, P_m)$ of $E(M)$ such that $(X, P_1, P_2, \dots, P_m)$ is a maximal swirl-like flower of M, and $(\{c, v\}, \{t, s, u, b\}, P_1, P_2, \dots, P_m)$ is a swirl-like flower in $M\backslash a$. (It is conceivable that this flower is not maximal in that $\{t, s, u, b\}$ could refine to a pair of petals. In fact this does not happen, but we don't need to establish this fact.) We now know that

$$\sqcap\big(\{c, v\}, P_m\big) = 1 \quad \text{and} \quad \sqcap\big(\{s, t, u, b\}, P_m\big) = 0.$$

Repeating the above argument for $M\backslash c$, and using Lemma 3.25 we deduce that either $(\{b, u\}, \{s, t, a, v\}, P_1, P_2, \dots, P_m)$ or $(\{s, t, a, v\}, \{b, u\}, P_1, P_2, \dots, P_m)$ is a swirl-like flower in $M\backslash c$. As $\sqcap(\{s, t, u, b\}, P_m) = 0$, it must be the former, and we deduce that $\sqcap(\{s, t, a, v\}, P_m) = 0$ so that

$$\sqcap\big(\{a, s\}, P_m\big) = 0.$$

Repeating for $M\backslash b$ and using the above fact, we deduce that $(\{a, s\}, \{v, u, t, c\}, P_1, P_2, \dots, P_m)$ is a swirl-like flower in $M\backslash b$. But this means that $\sqcap(\{v, u, t, c\}, P_m) = 0$ so that $\sqcap(\{c, v\}, P_m) = 0$ contradicting the fact that $\sqcap(\{c, v\}, P_m) = 1$. $\square$

The proof of Theorem 4.16 is now just a matter of summing up.

Proof of Theorem 4.16. Let T be a k-wild triangle of M. Then certainly T is wild. By Lemma 4.20, T is not a triangle in a trident. If T is a standard wild triangle of M, then T is a standard k-wild triangle of M by Lemma 4.19. Assume that T is a costandard wild triangle. Let M' be the matroid obtained by doing a $\Delta - Y$ exchange on T and then taking the dual. By Lemma 4.15, T is a k-wild triangle of M'. By definition T is a standard wild triangle of M' so that T is a standard k-wild triangle of M'. Now, by the definition of a costandard k-wild triangle we deduce that T is indeed a costandard k-wild triangle of M. The theorem now follows from Theorem 4.17. $\square$

If T is a triad of the k-coherent matroid M, then T is *k-wild* if T is a k-wild triangle of M^*. If T is a k-wild triad, then T is a *standard* (respectively *costandard*) k-wild triad if T is a standard (respectively costandard) k-wild triangle of M^*. A partition of $E(M)$ is a *k-wild display* for the triad T of M if it is a k-wild display for T in M^*.

We conclude this section by giving some elementary properties of k-wild triangles. We will use the following lemma on $\Delta - Y$ exchanges.

Lemma 4.21. *Let (X, Y) be a 3-separation of the 3-connected matroid M, let $\{a, b, c\}$ be a triangle of M contained in Y and let N be the matroid obtained by performing a $\Delta - Y$ exchange on $\{a, b, c\}$. Assume that*

$M\backslash a$, $M\backslash b$, and $M\backslash c$ are 3-connected and that $\{a,b,c\} \cap \mathrm{cl}_M(X) \neq \emptyset$. Then $\mathrm{cl}_N(X) \cap \{a,b,c\} = \mathrm{cl}^*_N(X) \cap \{a,b,c\} = \emptyset$.

Proof. Up to labels we may assume that $a \in \mathrm{cl}_M(X)$. By Lemma 4.14(ii), $r_N(X) = r_M(X)$, $r_N(Y) = r_M(Y) + 1$, and $r(N) = r(M) + 1$. Thus, $\lambda_N(X) = 2$. Assume that $\mathrm{cl}^{(*)}_N(X)$ contains the element t in $\{a,b,c\}$. The set $\{a,b,c\}$ is a cocircuit of N so that $t \notin \mathrm{cl}_N(X)$. Hence $t \in \mathrm{cl}^*_N(X)$, so that $\lambda_{N\backslash t}(X) = 1$. Say $t = a$. Then $\lambda_{N\backslash a/b}(X) = 1$. By Lemma 4.14(iv), $N\backslash a/b \cong M\backslash a,b$. So $\lambda_{M\backslash a,b}(X) = 1$. But $a \in \mathrm{cl}(X)$ so that $\lambda_{M\backslash b}(X) = 1$ contradicting the fact that $M\backslash b$ is 3-connected. If $b \in \mathrm{cl}^*_N(X)$, then we see that $\lambda_{N\backslash b/a}(X) = 1$. But $N\backslash b/a \cong M\backslash a,b$ and we arrive at the same contradiction. $\square$

The next lemma highlights subtle differences in the behaviour of standard and costandard k-wild triangles. To avoid a cumbersome statement we omit obvious symmetric statements.

Lemma 4.22. *Let $\{a,b,c\}$ be a k-wild triangle of the k-coherent matroid M with k-wild display $(A_1, A_2, \ldots, A_{k-2}, B_1, B_2, \ldots, B_{k-2}, C_1, C_2, \ldots, C_{k-2})$. Let $A = A_1 \cup A_2 \cup \cdots \cup A_{k-2}$. Then the following hold.*

(i) *If T is standard, then $a \in \mathrm{cl}(A)$.*
(ii) *If T is costandard, then $a \notin \mathrm{cl}(A)$.*
(iii) *If T is standard, then $\mathrm{si}(M/a)$ is not 3-connected.*
(iv) *If T is costandard, then $\mathrm{si}(M/a)$ is 3-connected.*

Proof. Part (i) is clear. Part (ii) follows from Lemma 4.21. As $a \in \mathrm{cl}(A)$ we see that a is in the guts of a vertical 3-separation of M and (iii) follows. While routine to prove, we simply note here that (iv) follows from [22, Corollary 3.3(iii)]. $\square$

The fact that any element of a standard k-wild triangle is on the guts of a vertical 3-separation is helpful for proving that certain triangles are not standard k-wild. The next lemma is useful for certifying that a triangle is not a costandard k-wild triangle. For ease of proof we state the dual form.

Lemma 4.23. *Let $\{a,b,c\}$ be a costandard k-wild triad of the k-coherent matroid M. Then $\mathrm{si}(M/a,b)$ is not 3-connected.*

Proof. Let M' be the matroid obtained by performing a $Y - \Delta$ exchange on M. Observe that T is a standard k-wild triangle of M. By Lemma 4.14(iv) $M/a,b \cong M'\backslash a/b$ and it is readily verified that $\mathrm{si}(M'\backslash a/b)$ is not 3-connected. $\square$

This next lemma gives some elementary properties that are common to both types of k-wild triangle.

Lemma 4.24. *Let $\{a,b,c\}$ be a k-wild triangle of the k-coherent matroid M with k-wild display $(A_1, A_2, \ldots, A_{k-2}, B_1, B_2, \ldots, B_{k-2}, C_1, C_2, \ldots, C_{k-2})$. Let $A = A_1 \cup A_2 \cup \cdots \cup A_{k-2}$. Then the following hold.*

(i) *$\lambda(A \cup \{a,b\}) > 2$ and $\lambda(A \cup \{a,b,c\}) > 2$.*
(ii) *If $i \in \{1,2,\ldots,k-2\}$, then neither a nor b is in the full closure of A_i.*

Proof. Consider (i). Assume that T is standard. If $\lambda(A \cup \{a,b,c\}) = 2$, then, as $c \in \mathrm{cl}(C_1 \cup C_2 \cup \cdots \cup C_{k-2})$, we have $\lambda(A \cup \{a,b\}) = 2$ so it suffices to prove that $\lambda(A \cup \{a,b\}) > 2$. Assume otherwise. Then $b \in \mathrm{cl}^{(*)}(A \cup \{a\})$. But then $(A \cup \{a,b\}, B_1, B_2, \ldots, B_{k-2}, C_1, C_2 \cup \cdots \cup C_{k-2} \cup \{c\})$ is a k-fracture of M contradicting the fact that M is k-coherent. Thus (i) holds in the case that T is standard.

Assume that T is costandard. It follows from (i) and Lemma 4.14 that $\lambda(A \cup \{a,b,c\}) > 2$. If $\lambda(A \cup \{a,b\}) = 2$, then, as $c \in \mathrm{cl}(\{a,b\})$ we have $\lambda(A \cup \{a,b,c\}) = 2$. Thus (i) also holds in the case that T is costandard.

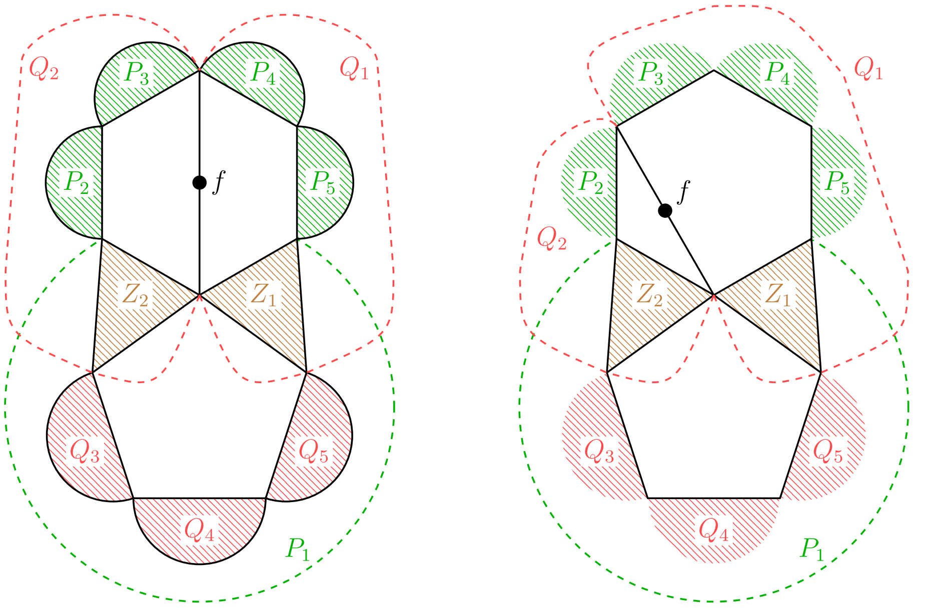

Fig. 4.5. Some feral elements.

If (ii) fails then we easily see that either a or b is in the full closure of a petal of a k-fracture of $M\backslash a$ or $M\backslash b$ respectively. This contradicts the fact that M is k-coherent. $\square$

4. Feral elements

Throughout this section we assume that M is a k-coherent matroid. An element f of M is *feral* if both $M\backslash f$ and M/f are 3-connected and k-fractured. The goal of this section is to gain insight into the structure of a matroid relative to a feral element.

Let $(P_1, P_2, \dots, P_m)$ and $(Q_1, Q_2, \dots, Q_k)$ be partitions of $E(M) - \{f\}$. Then these partitions form a *feral display* for f if there is an $i \in \{2, 3, \dots, m-1\}$ such that the following hold.

(i) $(P_1, P_2, \dots, P_m)$ and $(Q_1, Q_2, \dots, Q_k)$ are k-fractures of $M\backslash f$ and M/f respectively.
(ii) $\{P_2, P_3, \dots, P_m, Q_3, Q_4, \dots, Q_k, Z_1 = Q_1 \cap P_1, Z_2 = Q_2 \cap P_1\}$ partitions $E(N) - \{f\}$ into nonempty sets, with the exception that one of Z_1 or Z_2 may be empty.
(iii) $P_1 = Q_3 \cup Q_4 \cup \dots \cup Q_k \cup Z_1 \cup Z_2$.
(iv) $Q_1 = P_{i+1} \cup P_{i+2} \cup \dots \cup P_m \cup Z_1$.
(v) $Q_2 = Z_2 \cup P_2 \cup P_3 \cup \dots \cup P_i$.
(vi) $(Q_1 \cup Q_2 \cup \{f\}, Q_3, \dots, Q_k)$ is a swirl-like flower of order $(k-1)$ in M.
(vii) $(P_2, P_3, \dots, P_i, P_{i+1} \cup P_{i+2} \cup \dots \cup P_m \cup P_1 \cup \{f\})$ is a swirl-like flower of order i in M.
(viii) $(P_{i+1}, P_{i+2}, \dots, P_m, P_1 \cup P_2 \cup \dots \cup P_i \cup \{f\})$ is a swirl-like flower of order $(m-i+1)$ in M.
(ix) Either $Z_1 \neq \emptyset$ or $Z_2 \neq \emptyset$ and $\lambda_M(Z_1), \lambda_M(Z_2) \leqslant 3$.
(x) f blocks P_1 and f coblocks Q_1.

Figs. 4.5 and 4.6 illustrate some different cases of feral displays associated with a feral element f. In each of the cases of these figures, both Z_1 and Z_2 are nonempty. But the case that one of these sets is empty does arise. Such a case is illustrated by "bogan couples" which are defined and discussed in Chapter 5.

The primary goal is to prove that feral elements are characterised by the existence of a feral display. In other words we prove

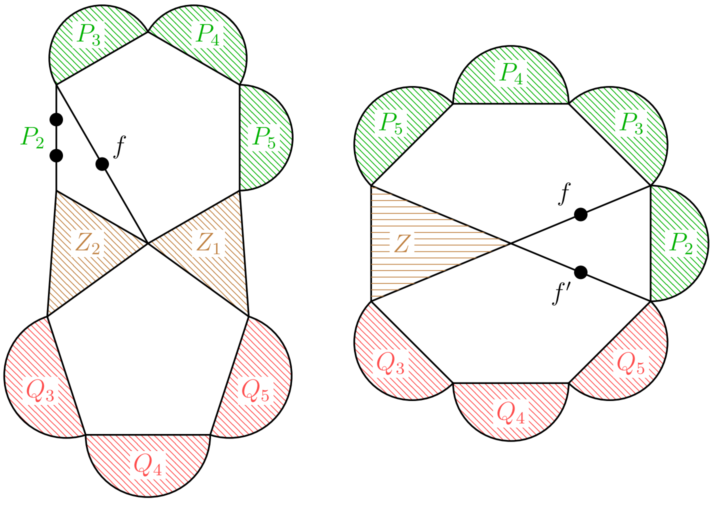

Fig. 4.6. Some more feral elements.

Theorem 4.25. *Let f be a feral element of the matroid M and let $\mathbf{F}_d$ and $\mathbf{F}_c$ be blooms that fracture $M\backslash f$ and M/f respectively. Then there are flowers displayed by $\mathbf{F}_d$ and $\mathbf{F}_c$ that form a feral display for f in at least one of M or M^*.*

In what follows we assume that f is indeed a feral element of M and that $\mathbf{F}_d = (\hat{P}_1, \hat{P}_2, \dots, \hat{P}_m)$ and $\mathbf{F}_c = (\hat{Q}_1, \hat{Q}_2, \dots, \hat{Q}_n)$ are maximal blooms that fracture $M\backslash f$ and M/f respectively. We say that f is 1-*blocking* for $\mathbf{F}_d$ if there is a flower $(P_1, P_2, \dots, P_m)$ displayed by $\mathbf{F}_d$ such that, f blocks P_i and no other petal for some $i \in \{1, 2, \dots, m\}$. Moreover, f is 2-*spanned* by $\mathbf{F}_d$ if $f \in \mathrm{cl}(\hat{P}_i \cup \hat{P}_{i+1})$ for some $i \in \{1, 2, \dots, m\}$. Dually, we say that f is 1-*coblocking* for $\mathbf{F}_c$ if f is 1-blocking for $\mathbf{F}_c$ in the matroid M^*, and f is 2-*cospanned* by $\mathbf{F}_c$ if f is 2-spanned by $\mathbf{F}_c$ in M^*.

Let (R, B) be a *well-coblocked* 3-separation of M/f. Recall that this means that f coblocks every 3-separation equivalent to (R, B). By Corollary 2.23, (R, B) is an unsplit 4-separation in $M\backslash f$. By Lemma 3.34, (R, B) crosses either one or two members of $(\check{P}_1, \check{P}_2, \dots, \check{P}_m)$. Recall that (R, B) is 1-*crossing* or 2-*crossing* for $\mathbf{F}_d$ according as to which case holds. Note that, if (R, B) is 1-crossing, there is a flower $(P_1, P_2, \dots, P_m)$, displayed by $\mathbf{F}_d$ such that $R = (P_1 \cap R) \cup P_2 \cup \dots \cup P_i$ for some $i \in \{1, 2, \dots, m\}$. As $f \in \mathrm{cl}_M(R)$ and $f \in \mathrm{cl}_M(B)$, we have $i \in \{2, 3, \dots, m-1\}$, as otherwise M is k-fractured.

Lemma 4.26.

(i) *Up to labels $\hat{P}_1$ is blocked by f.*
(ii) *If some well-coblocked* 3*-separation displayed by $\mathbf{F}_c$ is* 1*-crossing for $\mathbf{F}_d$, then f is* 1*-blocking for $\mathbf{F}_d$.*
(iii) *If some well-coblocked* 3*-separation displayed by $\mathbf{F}_c$ is* 2*-crossing for $\mathbf{F}_d$, then f is* 2*-spanned by $\mathbf{F}_d$.*
(iv) *f is either* 1*-blocking in $\mathbf{F}_d$ or is* 2*-spanned by $\mathbf{F}_d$.*

Proof. By Lemma 4.2, there is a well-coblocked 3-separation (R, B) displayed by $\mathbf{F}_c$. By the observations made prior to the lemma, (R, B) is either 1-crossing or 2-crossing for $\mathbf{F}_d$.

4.26.1. *If (R, B) is* 2*-crossing, then* (i) *and* (iii) *hold.*

Subproof. Assume that (R, B) is 2-crossing. By Lemma 3.34, up to labels, $R \subseteq \hat{P}_1 \cup \hat{P}_2$. As $f \in \mathrm{cl}(R)$, the element f is 2-spanned by $\mathbf{F}_d$. Thus (iii) holds. Let $(\hat{P}_1, P'_2, P'_m) = (\hat{P}_1, \hat{P}_2 - P_2^-, E(M\backslash f) - (\hat{P}_1 \cup \hat{P}_2))$. Assume that f blocks P'_2. If f does not block $\hat{P}_2$, then $f \in \mathrm{cl}(\hat{P}_2)$ and we deduce that M is

k-fractured. Thus f blocks $\hat{P}_2$ and the result holds by an appropriate relabelling of $\mathbf{F}_d$. Thus we may assume that f does not block P'_2. Also f does not block P'_m. If f does not block $\hat{P}_1$, then we again contradict the fact that M is k-coherent. Hence (i) holds. □

4.26.2. *If* (R, B) *is* 1*-crossing, then* (i) *and* (ii) *hold.*

Proof. Assume that (R, B) is 1-crossing. As observed prior to the lemma, there is a flower $(P_1, P_2, \ldots, P_m)$ displayed by $\mathbf{F}_d$ such that $R = (R \cap P_1) \cup P_2 \cup \cdots \cup P_i$ for some $i \in \{2, 3, \ldots, m-1\}$. As $f \in \mathrm{cl}(R)$ and $f \in \mathrm{cl}(B)$ we see that $f \in \mathrm{cl}(P_1 \cup P_2 \cup \cdots \cup P_i)$ and $f \in \mathrm{cl}(P_{i+1} \cup P_{i+2} \cup \cdots \cup P_m \cup P_1)$. Thus, if $i \neq 1$, the element f does not block P_i. Now let $(\hat{P}_1, P'_2, P'_m) = (\hat{P}_1, (P_2 \cup P_3 \cup \cdots \cup P_i) - \hat{P}_1, (P_{i+1} \cup P_{i+2} \cup \cdots \cup P_m) - \hat{P}_1)$. As $f \in \mathrm{cl}(\hat{P}_1 \cup P'_m)$ and $f \in \mathrm{cl}(\hat{P}_1 \cup P'_2)$, the element f does not block P'_2 or P'_m.

Assume that f does not block $\hat{P}_1$. It is easily checked that $f \notin \mathrm{cl}(\hat{P}_1), \mathrm{cl}(P'_2), \mathrm{cl}(P'_m)$ and that $f \in \mathrm{cl}(\hat{P}_1 \cup P'_2), \mathrm{cl}(\hat{P}_1 \cup P'_m), \mathrm{cl}(P'_2 \cup P'_m)$. As $(\hat{P}_1, P'_2, P'_m)$ is a swirl-like flower in $M\backslash f$, we have $\sqcap(\hat{P}_1, P'_2) = \sqcap(\hat{P}_1, P'_m) = \sqcap(P'_2, P'_m) = 1$. Thus, by Lemma 2.10, $\sqcap_{M/f}(\hat{P}_1, P'_2) = \sqcap_{M/f}(\hat{P}_1, P'_m) = \sqcap_{M/f}(P'_2, P'_m) = 2$, so that $(\hat{P}_1, P'_2, P'_m)$ is a paddle in M/f. But, by Lemmas 3.35 and 3.36, $|R \cap \hat{P}_1|$, $|B \cap \hat{P}_1| \geqslant 2$. Thus, by Lemma 3.20, there is a paddle in M/f that refines $(\hat{P}_1, P'_2, P'_m)$ and displays (R, B). But, (R, B) is also displayed in the swirl-like flower $\mathbf{F}_c$ of this matroid. However (R, B) cannot be displayed in both a paddle and a swirl-like flower. This contradiction shows that f blocks $\hat{P}_1$ and the sublemma follows. □

Parts (i), (ii) and (iii) of the lemma follow from 4.26.1 and 4.26.2. Part (iv) follows from parts (ii) and (iii) and Lemma 3.34. □

Lemma 4.27. *Assume that f is 1-blocking but not 2-spanned. Then there is a flower $(P_1, P_2, \ldots, P_m)$ displayed by $\mathbf{F}_d$, and an $i \in \{3, 4, \ldots, m-2\}$ such that the following hold.*

(i) *$P_1 = \hat{P}_1$ and f blocks P_1.*
(ii) *$f \in \mathrm{cl}_M(P_1 \cup P_2 \cup \cdots \cup P_i)$ and $f \in \mathrm{cl}_M(P_{i+1} \cup P_{i+2} \cup \cdots \cup P_m \cup P_1)$.*
(iii) *If $3 \leqslant s \leqslant i < t \leqslant m-1$, then $P_s \cup P_{s+1} \cup \cdots \cup P_t$ is well blocked by f.*
(iv) *If (S, T) is a well-coblocked 3-separation of M/f, displayed by $\mathbf{F}_c$, then there is a 3-separation (S', T') of M/f, equivalent to (S, T) such that $\{S' - P_1, T' - P_1\} = \{P_2 \cup P_3 \cup \cdots \cup P_i, P_{i+1} \cup P_{i+2} \cup \cdots \cup P_m\}$.*
(v) *If (S, T) is a well-coblocked 3-separation of M/f, displayed by $\mathbf{F}_c$, then there is a flower $(P'_1, P'_2, \ldots, P'_m)$, displayed by $\mathbf{F}_d$, such that $\{S - P'_1, T - P'_1\} = \{P'_2 \cup P'_3 \cup \cdots \cup P'_i, P'_{i+1} \cup P'_{i+2} \cup \cdots \cup P'_m\}$.*

Proof. Let (R, B) be a 3-separation of M/f displayed by $\mathbf{F}_c$ that is well coblocked by f. Then, as f is not 2-spanned, it follows from Lemma 4.26 that (R, B) is 1-crossing. Thus, for some $i \in \{2, 3, \ldots, m-1\}$, we have, up to labels, that $R = (\hat{P}_1 \cap R) \cup \check{P}_2 \cup \hat{P}_3 \cup \cdots \cup \hat{P}_{i-1} \cup \check{P}_i \cup L_i^+$, where L_i^+ is an initial segment of P_i^+. As f is not 2-spanned, $i \in \{3, 4, \ldots, m-2\}$. By Lemma 4.26, f blocks $\hat{P}_1$. It is now clear that there is a flower $(P_1, P_2, \ldots, P_m)$ displayed by $\mathbf{F}_d$ such that $P_1 = \hat{P}_1$, $P_1 \cup P_2 \cup \cdots \cup P_i = R \cup P_1$, and $P_{i+1} \cup P_{i+2} \cup \cdots \cup P_m \cup P_1 = B \cup P_1$. One consequence of this is that (i) holds. As f coblocks (R, B) we see that $f \in \mathrm{cl}_M(R)$ and $f \in \mathrm{cl}_M(B)$. Hence $f \in \mathrm{cl}_M(P_1 \cup P_2 \cup \cdots \cup P_i)$ and $f \in \mathrm{cl}_M(P_{i+1} \cup P_{i+2} \cup \cdots \cup P_m \cup P_1)$, so that (ii) holds.

Consider (iii). Assume that $3 \leqslant u \leqslant i < v \leqslant m-1$. Let P be a 3-separating set that is equivalent to $P_u \cup P_{u+1} \cup \cdots \cup P_v$. Assume that f does not block P. Then either $f \in \mathrm{cl}_M(P)$, or $f \in \mathrm{cl}_M(E(M\backslash f) - P)$. Assume that the former holds. Note that $\check{P}_m \cap P = \emptyset$ and $\check{P}_i \subseteq P$. Thus by Lemma 3.31, $f \in \mathrm{cl}(P \cap (P_1 \cup P_2 \cup \cdots \cup P_i))$. But $P \cap (P_1 \cup P_2 \cup \cdots \cup P_i) \subseteq \hat{P}_u \cup \hat{P}_{u+1} \cup \cdots \cup \hat{P}_i$. Hence $f \in \mathrm{cl}(\hat{P}_u \cup \hat{P}_{u+1} \cup \cdots \cup \hat{P}_i)$. But we also have $f \in \mathrm{cl}(P_i \cup \cdots \cup P_m \cup P_1)$; so again using Lemma 3.31, we see that $f \in \mathrm{cl}(\hat{P}_i)$ contradicting the fact that M is k-coherent. Assume that the latter case holds, so that $f \in \mathrm{cl}(E(M\backslash f) - P)$. Arguing as above, we deduce that $f \in \mathrm{cl}(\hat{P}_1 \cup \hat{P}_2 \cup \cdots \cup \hat{P}_{u-1})$ and, as $f \in \mathrm{cl}(P_{i+1} \cup \cdots \cup P_m \cup P_1)$, we conclude that $f \in \mathrm{cl}(\hat{P}_1)$, contradicting the fact that x blocks $\hat{P}_1$. Thus (iii) holds.

Consider (iv). Note that the 3-separation (R, B) at the start of the proof determines a labelling of a flower displayed by the bloom $\mathbf{F}_d$. Of course the same conclusions also hold for (S, T), but we need to reconcile the labellings. As f is not 2-spanned, it follows from Lemma 4.26, that, up to the choice of labels S and T, that $S = (\hat{P}_s \cap S) \cup \check{P}_{s+1} \cup \hat{P}_{s+2} \cup \cdots \cup \hat{P}_{t-1} \cup \check{P}_t \cup K_t^+$, where K_t^+ is an initial segment of P_t^+, and $3 \leqslant t - s \leqslant m - 3$. Note that $\hat{P}_s$ is blocked by f.

4.27.1. $s \in \{m, 1, 2\}$.

Subproof. Say $s \notin \{m, 1, 2\}$. Up to symmetry, we may assume that $s \leqslant i$. If $s < i$, then $(E(M\backslash f) - \hat{P}_s) \supseteq B$, so $\hat{P}_s$ is not blocked by f. Thus we may assume that $s = i$. Say that $\check{P}_m \subseteq T$. Then, as $f \in \mathrm{cl}(\hat{P}_i \cup \hat{P}_{i+1} \cup \cdots \cup \hat{P}_t)$ and $f \in \mathrm{cl}(\hat{P}_1 \cup \hat{P}_2 \cup \cdots \cup \hat{P}_i)$, we see that $f \in \mathrm{cl}(\hat{P}_i)$ contradicting the fact that M is k-coherent. Thus $\check{P}_m \subseteq S$. A similar argument shows that $\check{P}_2 \subseteq T$. Assume that $\check{P}_1 \subseteq T$. In this case $S \cup B = (\hat{P}_i \cap S) \cup \check{P}_{i+1} \cup \hat{P}_{i+2} \cup \cdots \cup \hat{P}_{m-1} \cup \check{P}_m \cup (\hat{P}_1 \cap B)$. But by uncrossing we have $\lambda_{M/f}(S \cup B) = 2$, so $\lambda_{M\backslash f}(S \cup B) \leqslant 3$. Observe that in the case that $\lambda_{M\backslash f}(S \cup B) = 3$, the 4-separation $(S \cup B, E(M\backslash f) - (S \cup B))$ is unsplit. However $S \cup B$ crosses both $\check{P}_1$ and $\hat{P}_i$, the set $S \cup B$ does not have the form of Lemma 3.34 or 3.33. Essentially the same argument holds in the case that $\check{P}_1 \subseteq T$. □

Up to symmetry we may assume that $t \leqslant i$.

4.27.2. $t = i$.

Subproof. Assume that $t < i$. As $f \in \mathrm{cl}(S)$, we have $f \in \mathrm{cl}(\hat{P}_m \cup \cdots \cup \hat{P}_t)$. But then $\hat{P}_m \cup \hat{P}_1 \cup \cdots \cup \hat{P}_{i-1}$ is not blocked by f, contradicting (iii). Thus $t = i$. □

We may now relabel K_t^+ by K_i^+.

4.27.3. *Up to 3-separations of M/f equivalent to (S, T), we may assume that $K_i^+ = L_i^+$.*

Subproof. Assume that the claim fails. Then, up to symmetry, we may assume that K_i^+ is a proper subset of L_i^+. Let y be the first element of $L_i^+ - K_i^+$. Then either $y \in \mathrm{cl}(\check{P}_i \cup K_i^+)$ or $y \in \mathrm{cl}^*(\check{P}_i \cup K_i^+)$, so that either $y \in \mathrm{cl}_{M\backslash f}(S)$ or $y \in \mathrm{cl}^*_{M\backslash f}(S)$. In the former case it is clear that $y \in \mathrm{cl}_{M/f}(S)$. Consider the latter case. Note that $B \subseteq E(M\backslash f) - (\check{P}_i \cup K_i^+)$, so y is a coloop of $M\backslash(\check{P}_i \cup K_i^+)$, and thus y is a coloop $M/f\backslash(\check{P}_i \cup K_i^+)$. Thus $y \in \mathrm{cl}^*_{M/f}(\check{P}_i \cup K_i^+)$ and hence $y \in \mathrm{cl}^*_{M/f}(S)$.

We now have that in M/f the 3-separation (S, T) is equivalent to $(S \cup \{y\}, T - \{y\})$. The sublemma follows by an obvious induction □

We may now assume that $P_3 \cup P_4 \cup \cdots \cup P_i \subseteq S$ and $P_{i+1} \cup P_{i+2} \cup \cdots \cup P_{m-1} \subseteq T$.

4.27.4. $s = 1$.

Subproof. Assume that the sublemma fails. Up to symmetry we may assume that $s = 2$, so that $S \subseteq \hat{P}_2 \cup P_3 \cup \cdots \cup P_i$. Recall that $B \subseteq P_{i+1} \cup P_{i+2} \cup \cdots \cup P_m \cup \hat{P}_1$. Say that $B \cap P_1^+ = \emptyset$. Then $B \subseteq P_{i+1} \cup P_{i+2} \cup \cdots \cup P_m \cup (\hat{P}_1 - P_1^+)$. And, as $f \in \mathrm{cl}_M(B)$ and $f \in \mathrm{cl}_M(S)$, we see that f is in the guts of a 3-separation of M contradicting the fact that M/f is 3-connected. Thus $B \cap P_1^+ \neq \emptyset$. By Lemma 3.34, $B \cap P_i^+$ consists of a single element b, and this element is the first element of P_i^+. If $b \in \mathrm{cl}_{M\backslash f}(\check{P}_1)$, then $f \in \mathrm{cl}_M(B - \{b\})$ and we again obtain the contradiction that M/f is not 3-connected. Thus $b \in \mathrm{cl}^*_{M\backslash f}(B - \{b\})$. By symmetry there is a single element $p \in S \cap P_1^+$ and p is the last element of P_1^+. If $b \neq p$, then we again obtain a 3-separation (X, Y) of $M\backslash f$ with $B \subseteq X$ and $S \subseteq Y$, again contradicting the fact that M/f is 3-connected. Thus $B_1^+ = \{b\}$.

Now $f \in \mathrm{cl}_M(S)$, that is $f \in \mathrm{cl}_M((S - \{b\}) \cup \{b\})$. Hence $b \in \mathrm{cl}_M((S - \{b\}) \cup \{f\})$. But $S - \{b\} \subseteq R$, so that $b \in \mathrm{cl}_M(R \cup \{f\})$ and $b \in \mathrm{cl}_{M/f}(R)$ Thus, in M/f we have $(R, B) \cong (R \cup \{b\}, B - \{b\})$. As B is well blocked, $f \in \mathrm{cl}_M(B - \{b\})$. But $B - \{b\} \subseteq P_{i+1} \cup P_{i+2} \cup \cdots \cup P_m \cup \check{P}_1$, and $S \subseteq \hat{P}_2 \cup \cdots \cup P_{i-1} \cup P_i$, and again we contradict the fact the M/f is 3-connected. □

Part (iv) of the lemma now follows immediately. The proof of (v) follows from the fact that the equivalence moves performed in the proof of (iv) could also have been regarded as equivalence moves in the flower. □

We now consider the case when f is 2-spanned. Note that if $f \in \mathrm{cl}(\hat{P}_1 \cup \hat{P}_2)$, then $(\hat{P}_1 \cup \hat{P}_2 \cup \{f\}, \hat{P}_3, \ldots, \hat{P}_m)$ induces a swirl-like flower of M of order $m-1$. Thus $m = k$.

Lemma 4.28. *Assume that f is 2-spanned by $\mathbf{F}_d$, say $f \in \mathrm{cl}(\hat{P}_1 \cup \hat{P}_2)$. If $(P_1, P_2, \ldots, P_k)$ is any flower displayed by $\mathbf{F}_d$ with $P_1 \cup P_2 = \hat{P}_1 \cup \hat{P}_2$, then the following hold.*

(i) *If $i \in \{3, 4, \ldots, k-1\}$, then $P_2 \cup P_3 \cup \cdots \cup P_i$ is a well-blocked 3-separation of $M\backslash f$.*
(ii) *If (R, B) is a well-coblocked 3-separation of M/f, then, up to labels, $R \subseteq P_1 \cup P_2$.*

Proof. Consider (i). Say $i \in \{3, 4, \ldots, k-1\}$. Assume that $P_2 \cup P_3 \cup \cdots \cup P_i$ is not well blocked by f. Then either $f \in \mathrm{cl}(\hat{P}_2 \cup \hat{P}_3 \cup \cdots \cup \hat{P}_i)$, or $f \in \mathrm{cl}(\hat{P}_{i+1} \cup \hat{P}_{i+2} \cup \cdots \cup \hat{P}_k)$. Up to symmetry we may assume that $f \in \mathrm{cl}(\hat{P}_2 \cup \hat{P}_3 \cup \cdots \cup \hat{P}_i)$. But $f \in \mathrm{cl}(\hat{P}_1 \cup \hat{P}_2)$ and $(\hat{P}_1 \cup \hat{P}_2) \cup (\hat{P}_2 \cup \hat{P}_3 \cup \cdots \cup \hat{P}_i)$ avoids $\check{P}_m$. So, by Lemma 3.31, $f \in \mathrm{cl}(\hat{P}_2)$ contradicting the fact that M is k-coherent. Thus (i) holds.

Consider (ii). Say (R, B) is a well-coblocked 3-separation of M/f. If (R, B) is 2-crossing, then, up to labels, $R \subseteq \hat{P}_i \cup \hat{P}_{i+1}$ for some i. If $i \neq 1$, then we obtain a contradiction by an application of Lemma 3.31. A similar easy argument shows that (ii) holds in the case that (R, B) is 1-crossing with one exceptional case that we focus on now. In this case, up to labels, we have $R = (\hat{P}_k \cap R) \cup \check{P}_1 \cup P_1^+ \cup \check{P}_2 \cup L_2^+$, where L_2^+ is an initial segment of P_2^+. Let $P_2' = \check{P}_2 \cup L_2^+$. Assume we are in this case.

4.28.1. $f \in \mathrm{cl}(\hat{P}_1 \cup P_2')$.

Subproof. As $f \in \mathrm{cl}(R)$, we have $f \in \mathrm{cl}(\hat{P}_k \cup \hat{P}_1 \cup P_2')$. Also $f \in \mathrm{cl}(\hat{P}_1 \cup \hat{P}_2)$, so by Lemma 3.31, $f \in \mathrm{cl}(\hat{P}_1 \cup P_2')$. □

4.28.2. $\hat{P}_1 \subseteq R$.

Subproof. Assume the sublemma fails. Then, by Lemma 3.35, there is a single element $b \in B \cap \hat{P}_1$. Moreover, $(\hat{P}_1 \cup P_2') \cap B = \{b\}$. By 4.28.1, $f \in \mathrm{cl}(\hat{P}_1 \cup P_2')$. Also $f \notin \mathrm{cl}((\hat{P}_1 \cup P_2') - \{b\})$, as otherwise M/f is not 3-connected. Hence $b \in \mathrm{cl}(((\hat{P}_1 \cup P_2') - \{b\}) \cup \{f\})$, so that $b \in \mathrm{cl}_M(R \cup \{f\})$ and $b \in \mathrm{cl}_{M/f}(R)$. Thus, in M/f, the 3-separation (R, B) is equivalent to $(R \cup \{b\}, B - \{b\})$. As (R, B) is well coblocked, this means that $f \in \mathrm{cl}_M(B - \{b\})$. But, as $f \in \mathrm{cl}(\hat{P}_1 \cup P_2')$, and $(B - \{b\}) \cap (\hat{P}_1 \cup P_2') = \emptyset$, we have again contradicted the fact that M/f is 3-connected. □

Now $B \subseteq E(M\backslash f) - (\hat{P}_1 \cup P_2')$, $f \in \mathrm{cl}(B)$, and $f \in \mathrm{cl}(\hat{P}_1 \cup P_2')$, and again we contradict the fact that M/f is 3-connected. Part (ii) of the lemma follows from this final contradiction. □

Lemma 4.29. *Either $\mathbf{F}_d$ is 1-blocked, or $\mathbf{F}_c$ is 1-coblocked.*

Proof. Assume the lemma fails. Then, by Lemma 4.26(iv), we obtain the following up to labels. For $\mathbf{F}_d$ we have $f \in \mathrm{cl}(\hat{P}_1 \cup \hat{P}_2)$, f blocks $\hat{P}_1$, and f blocks $\hat{P}_2$. For $\mathbf{F}_c$ we have $f \in \mathrm{cl}^*(\hat{Q}_1 \cup \hat{Q}_2)$, f coblocks $\hat{Q}_1$ and f coblocks $\hat{Q}_2$. Let $(P_1, P_2, \ldots, P_m)$ be a flower displayed by $\mathbf{F}_d$, where $P_1 \cup P_2 = \hat{P}_1 \cup \hat{P}_2$ and let $(Q_1, Q_2, \ldots, Q_n)$ be a flower displayed by $\mathbf{F}_c$ where $Q_1 \cup Q_2 = \hat{Q}_1 \cup \hat{Q}_2$. As $(P_1 \cup P_2 \cup \{f\}, P_3, \ldots, P_m)$ and $(Q_1 \cup Q_2 \cup \{f\}, Q_3, \ldots, Q_n)$ are swirl-like flowers in M, we have $m = n = k$. Note that f blocks P_1 and P_2 and f coblocks Q_1 and Q_2, otherwise $\mathbf{F}_d$ is 1-blocked, or $\mathbf{F}_c$ is 1-coblocked.

By Lemma 4.28(i), $Q_2 \cup Q_3$ is a well-coblocked 3-separation of M/f. Thus, by Lemma 4.28(ii), either $Q_2 \cup Q_3 \subseteq P_1 \cup P_2$ or $Q_4 \cup Q_5 \cup \cdots \cup Q_k \cup Q_1 \subseteq P_1 \cup P_2$. In the latter case $Q_k \cup Q_1 \subseteq P_1 \cup P_2$

so that up to labels we may assume that $Q_2 \cup Q_3 \subseteq P_1 \cup P_2$. As $f \in \mathrm{cl}(Q_2)$, the set Q_2 is not contained in either P_1 or P_2, so $Q_2 \cap P_1 \neq \emptyset$ and $Q_2 \cap P_2 \neq \emptyset$. As $(Q_1 \cup Q_2 \cup \{f\}, Q_3, \dots, Q_n)$ are swirl-like flowers in M we see that $(Q_1 \cup Q_2, Q_3 \cup Q_4 \cup \cdots \cup Q_k)$ is a 3-separation of $M\backslash f$. As either P_1 or P_2 crosses this 3-separation, it is not displayed in $(P_1, P_2, \dots, P_k)$. As Q_1 meets both P_1 and P_2, the set $Q_1 \cup Q_2$ is not contained in a petal of $(P_1, P_2, \dots, P_k)$. Thus $Q_3 \cup Q_4 \cup \cdots \cup Q_k$ is contained in a petal of $(P_1, P_2, \dots, P_k)$. As $Q_3 \subseteq P_1 \cup P_2$, either $Q_3 \cup Q_4 \cup \cdots \cup Q_k \subseteq \hat{P}_1$ or $Q_3 \cup Q_4 \cup \cdots \cup Q_k \subseteq \hat{P}_2$. Up to labels we may assume that the latter case holds. Evidently we may also assume that $P_2 = \hat{P}_2$. Thus $Q_3 \cup Q_4 \cup \cdots \cup Q_k \subseteq P_2$.

Altogether we have $Q_2 \subseteq P_1 \cup P_2$, $Q_3 \cup Q_4 \cup \cdots \cup Q_k \subseteq P_2$ and $Q_1 \supseteq P_3 \cup \cdots \cup P_k$. If $Q_1 \cap P_2 = \emptyset$, then, as $f \in \mathrm{cl}(Q_1)$, the element f does not block P_2. Thus $Q_1 \cap P_2 \neq \emptyset$ and similarly $Q_1 \cap P_1 \neq \emptyset$.

Let $R = Q_3 \cup Q_4 \cup \cdots \cup Q_k \cup (Q_2 \cap P_2)$. By Lemma 3.36, $\lambda_{M\backslash f}(R) = 2$. But $f \in \mathrm{cl}_M(E(M) - (\{f\} \cup R))$, so $\lambda_{M/f}(R) = 2$.

4.29.1. *In M/f we have $R \ncong R \cup Q_2$ and $R \ncong R - Q_2$.*

Subproof. If $R \cong R \cup Q_2$, then $Q_1 \cong Q_1 \cup (Q_2 \cap P_1)$ and, as $f \notin \mathrm{cl}_M(P_2)$, we see that f is 1-blocking for $\mathbf{F}_d$. Assume that $R \cong R - Q_2$. In this case, as f is not 1-blocking for $\mathbf{F}_d$, we see that either $f \in \mathrm{cl}(Q_2 \cap P_1)$, contradicting the fact that f blocks P_1, or $f \in \mathrm{cl}_M(Q_3 \cup Q_4 \cup \cdots \cup Q_k \cup (P_2 \cup Q_2))$, that is, $f \in \mathrm{cl}_M(P_2)$, contradicting the fact that f blocks P_2. □

By 4.29.1 R is not displayed in $\mathbf{F}_c$ and neither R nor its complement are contained in a petal of $\mathbf{F}_c$. This contradicts the fact that $\mathbf{F}_c$ is a maximal bloom of M/f and the lemma follows. □

Lemma 4.30. *Either f is 2-spanned by $\mathbf{F}_d$ or f is 2-cospanned by $\mathbf{F}_c$.*

Proof. Assume that the lemma fails so that f is not 2-spanned by $\mathbf{F}_d$ and is not 2-cospanned by $\mathbf{F}_c$. Then there is a flower $(Q_1, Q_2, \dots, Q_n)$, displayed by $\mathbf{F}_c$, and an $i \in \{3, 4, \dots, n-2\}$ such that the dual of Lemma 4.27 holds. By that lemma $f \in \mathrm{cl}^*(Q_1 \cup Q_2 \cup \cdots \cup Q_i)$ and $f \in \mathrm{cl}^*(Q_{i+1} \cup Q_{i+2} \cup \cdots \cup Q_m \cup Q_1)$. Also $Q_i \cup Q_{i+1}$ is well coblocked by f. It is easily seen that we may assume that $Q_i \cup Q_{i+1} = \hat{Q}_i \cup \hat{Q}_{i+1}$.

Also by Lemma 4.27, there is flower $(P_1, P_2, \dots, P_m)$, displayed by $\mathbf{F}_d$, and a $j \in \{3, 4, \dots, m-2\}$, such that

$$Q_i \cup Q_{i+1} = \big(P_1 \cap (Q_i \cup Q_{i+1})\big) \cup P_2 \cup \cdots \cup P_j.$$

Thus $P_2 \cup P_3 \cup \cdots \cup P_j \subseteq Q_i \cup Q_{i+1}$.

As $Q_i \cup Q_{i+1}$ is coblocked by f, we have $f \in \mathrm{cl}(P_{j+1} \cup P_{j+2} \cup \cdots \cup P_m \cup P_1)$. Thus, $\lambda_M(P_2 \cup P_3 \cup \cdots \cup P_j) = 2$ and hence $\lambda_{M/f}(P_2 \cup P_3 \cup \cdots \cup P_j) = 2$. Clearly $P_{j+1} \cup \cdots \cup P_m \cup P_1$ is not contained in a petal of $\mathbf{F}_c$. Thus $P_2 \cup P_3 \cup \cdots \cup P_j$ is either contained in a petal of $\mathbf{F}_c$ or it is displayed by more than one petal of $\mathbf{F}_c$. As $P_2 \cup P_3 \cup \cdots \cup P_j \subseteq Q_i \cup Q_{i+1}$, the latter case implies that $Q_i \cup Q_{i+1} \cong P_2 \cup P_3 \cup \cdots \cup P_j$, contradicting the fact that $Q_i \cup Q_{i+1}$ is well coblocked. Hence $P_2 \cup P_3 \cup \cdots \cup P_j$ is contained in a petal of $\mathbf{F}_c$. Assume without loss of generality that $P_2 \cup P_3 \cup \cdots \cup P_j \subseteq \hat{Q}_i$.

Consider $P_j \cup P_{j+1}$. This is one side of a well-blocked 3-separation of $M\backslash f$. We may assume that $P_j \cup P_{j+1} = \hat{P}_j \cup \hat{P}_{j+1}$. By Lemma 4.27, either $P_j \cup P_{j+1} \supseteq Q_2 \cup Q_3 \cup \cdots \cup Q_i$, contradicting the fact that $P_2 \subseteq Q_i$, or $P_j \cup P_{j+1} \supseteq Q_{i+1} \cup Q_{i+2} \cup \cdots \cup Q_n$ contradicting the fact that $P_j \subseteq Q_i$ and the lemma follows. □

Lemma 4.31. *Up to duality, f is 1-blocking for $\mathbf{F}_d$ and 2-cospanned by $\mathbf{F}_c$.*

Proof. By Lemma 4.29, we may assume that f is 1-blocking for $\mathbf{F}_d$. Assume that f is not 2-cospanned by $\mathbf{F}_c$. By Lemma 4.30, f is 2-spanned by $\mathbf{F}_d$, and by Lemma 4.26(iii), f is 1-coblocking for $\mathbf{F}_c$. The lemma now follows by taking the dual. □

Lemma 4.32. *If f is* 1*-blocking for* $\mathbf{F}_d$ *and f is* 2*-cospanned by* $\mathbf{F}_c$*, then there is a feral display for f obtained from flowers displayed by* $\mathbf{F}_d$ *and* $\mathbf{F}_c$*.*

Proof. As f is 2-cospanned by $\mathbf{F}_c$, we may assume that $f \in \mathrm{cl}(\hat{Q}_1 \cup \hat{Q}_2)$. By Lemma 4.26(i), we may assume that $\hat{Q}_1$ is coblocked by f. Let $(Q_1, Q_2, \ldots, Q_k)$ be a flower displayed by $\mathbf{F}_c$ where $Q_1 \cup Q_2 = \hat{Q}_1 \cup \hat{Q}_2$.

We split the proof into two cases. For the first case assume that f is not 2-spanned by $\mathbf{F}_d$. Then Lemma 4.27 applies. Let $(P_1, P_2, \ldots, P_m)$ be a flower displayed by $\mathbf{F}_d$ satisfying Lemma 4.27. By Lemma 4.28, $Q_2 \cup Q_3$ is well coblocked by f. Thus, up to labels, there is an $i \in \{3, 4, \ldots, m-2\}$ and a 3-separating set R_3 of M/f, equivalent to $Q_2 \cup Q_3$, such that $R_3 = P_{i+1} \cup P_{i+2} \cup \cdots \cup P_m \cup (R_3 \cap P_1)$. Indeed, by replacing $(Q_1, Q_2, \ldots, Q_k)$ by an equivalent flower, we may assume that $R_3 = Q_2 \cup Q_3$. Also $Q_2 \cup Q_3 \cup \cdots \cup Q_{k-1}$ is well coblocked, so there is a 3-separation R_4 equivalent to $Q_2 \cup \cdots \cup Q_{k-1}$ such that $R_4 = P_{i+1} \cup P_{i+2} \cup \cdots \cup P_m \cup (P_1 \cap R_4)$. Again we may replace $(Q_1, Q_2, \ldots, Q_k)$ by an equivalent flower so that $R_4 = Q_2 \cup Q_3 \cup \cdots \cup Q_{k-1}$. It is easily seen that the only petal affected by such equivalence moves is Q_{k-1}, so that we have both $R_3 = Q_2 \cup Q_3$ and $R_4 = Q_2 \cup Q_3 \cup \cdots \cup Q_{k-1}$. It now follows that $Q_4 \cup Q_5 \cup \cdots \cup Q_{k-1} \subseteq P_1$.

As $Q_4 \cup Q_5 \cup \cdots \cup Q_{k-1} \subseteq P_1$, and P_1 is fully closed in $M \backslash f$, it follows from Lemma 3.16 that P_1 contains all but one petal of $(Q_1 \cup Q_2, Q_3, \ldots, Q_k)$. Assume that Q_3 is not a subset of P_1, then $Q_3 \supseteq P_2 \cup P_3 \cup \cdots \cup P_m$ contradicting the fact that $\check{P}_2 \subseteq P_1 \cup P_2 \cup \cdots \cup P_i$. Thus Q_3, and similarly Q_k, is contained in P_1.

Summing up we have $Q_3 \cup Q_4 \cup \cdots \cup Q_k \subseteq P_1$. Moreover, by choosing $(Q_1, Q_2, \ldots, Q_k)$ to satisfy the constraints imposed by R_3 and R_4, we have $P_1 \cup P_2 \cup \cdots \cup P_i \subseteq Q_1$ and $P_{i+1} \cup P_{i+2} \cup \cdots \cup P_k \subseteq Q_2$. Let $Z_1 = Q_1 \cap P_1$ and $Z_2 = Q_2 \cap P_1$. As Q_1 is coblocked by f and $\lambda_M(P_{i+1} \cup P_{i+2} \cup \cdots \cup P_m) = 2$, we see that $Q_1 \neq P_{i+1} \cup P_{i+2} \cup \cdots \cup P_m$. Thus $Z_1 \neq \emptyset$. Also $\lambda_{M\backslash f}(P_1) = 2$, $\lambda_{M\backslash f}(Q_1) = 3$, $\lambda_{M\backslash f}(P_1 \cup Q_1) = 2$. So by uncrossing we have $\lambda_{M\backslash f}(Z_1) \leqslant 3$. As $f \in \mathrm{cl}(Q_2)$, $\lambda_M(Z_1) \leqslant 3$ and similarly $\lambda_M(Z_2) \leqslant 3$. It is now a matter of routine bookkeeping to verify that $(P_1, P_2, \ldots, P_m)$ and $(Q_1, Q_2, \ldots, Q_k)$ form a feral display for f.

Consider the second case. Assume that f is 2-spanned by $\mathbf{F}_d$. Note that this means that $m = k$. The bloom $\mathbf{F}_d$ is both 1-blocked and 2-spanned and $\mathbf{F}_c$ is 2-spanned. By Lemma 4.26, $\mathbf{F}_c$ is also 1-coblocked. Thus we have a flower $(P_1, P_2, \ldots, P_k)$ displayed by $\mathbf{F}_d$ such that $P_1 \cup P_2 = \hat{P}_1 \cup \hat{P}_2$, the element f blocks P_1 and no other petal, and $f \in \mathrm{cl}(P_1 \cup P_2)$. We also have a flower $(Q_1, Q_2, \ldots, Q_k)$ displayed by $\mathbf{F}_c$ such that $Q_1 \cup Q_2 = \hat{Q}_1 \cup \hat{Q}_2$, the element f coblocks Q_1 and no other petal, and $f \in \mathrm{cl}^*(Q_1 \cup Q_2)$. Let $P = P_3 \cup P_4 \cup \cdots \cup P_k$ and $Q = Q_3 \cup Q_4 \cup \cdots \cup Q_k$. Note that $\lambda_M(P) = \lambda_M(Q) = \lambda_M(P_2) = \lambda_M(Q_2) = 2$.

By Lemma 3.16, either Q or $Q_1 \cup Q_2$ is contained in a petal of $\mathbf{F}_d$. But $f \in \mathrm{cl}_M(Q_1 \cup Q_2)$ and no petal of $\mathbf{F}_d$ has that property. Hence $Q \subseteq \hat{P}_j$ for some $j \in \{1, 2, \ldots, k\}$. By Lemma 4.28, $(Q_2 \cup Q_3, Q_4 \cup \cdots \cup Q_k \cup Q_1)$ is well coblocked by f. Thus, by Lemma 4.28, either $Q_2 \cup Q_3 \subseteq P_1 \cup P_2$, or $Q_4 \cup Q_5 \cup \cdots \cup Q_k \cup Q_1 \subseteq P_1 \cup P_2$.

From the above information we draw two conclusions. First we conclude that either Q_3 or Q_4 is contained in $P_1 \cup P_2$. Thus, as $Q_3 \cup Q_4 \subseteq Q$, and $Q \subseteq \hat{P}_j$ for some $j \in \{1, 2, \ldots, k\}$, we have $Q \subseteq \hat{P}_s$ for some $s \in \{1, 2\}$. The second conclusion is that $P \subseteq \hat{Q}_t$ for some $t \in \{1, 2\}$. Up to duality, there are three cases: $(s, t) = (1, 2)$, $(s, t) = (1, 1)$, and $(s, t) = (2, 2)$.

4.32.1. *The lemma holds if* $(s, t) = (1, 2)$*.*

Subproof. In this case $Q \subseteq \hat{P}_1$ and $P \subseteq \hat{Q}_2$. We may assume that $P_1 = \hat{P}_1$ and $Q_1 = \hat{Q}_1$. As Q_2 is not coblocked, $\lambda_{M\backslash f}(Q_2) = 2$.

We now show that we may assume that $Q_2 \cap P_2 = \emptyset$. By Lemma 3.33 either $Q_1 \cap P_2$ consists of loose elements of P_2 contained in $\mathrm{fcl}_{M\backslash f}(P_1)$ or $P_2 \cap Q_2$ consists of loose elements of P_2 contained in $\mathrm{fcl}_{M\backslash f}(P_3)$. The former case contradicts the fact that P_1 is fully closed. Hence the latter case holds and, by moving to a flower equivalent to $(P_1, P_2, \ldots, P_k)$, we may assume that $Q_2 \cap P_2 = \emptyset$, so that $P_2 \subseteq Q_1$.

Next we show that we may assume that $P_1 \cap Q_2 = \emptyset$. Either $Q_2 \cap P_1$ or $P_1 - Q_2$ is a set of loose elements of P_1 in $M\backslash f$. The latter case implies that $Q \subseteq \mathrm{fcl}_{M\backslash f}(Q_2)$, so that $Q \subseteq \mathrm{fcl}_M(Q_2 \cup \{f\})$

contradicting the fact that $(Q_1 \cup Q_2 \cup \{f\}, Q_3, \ldots, Q_k)$ is a swirl-like flower of order $k-1$ in M. Hence $Q_2 \cap P_1$ is a set of loose elements of P_1, so that $(P_1, P_2, \ldots, P_k) \cong (P_1 - Q_2, P_2, \ldots, P_k \cup (Q_2 - P_1))$. By Lemma 4.26, some petal of this flower is blocked by f. Certainly P_2 is not blocked. Moreover, $Q_1 \subseteq (P_1 - Q_1) \cup P_2$ and $f \in \mathrm{cl}_M(Q_1)$, so no petal in $\{P_3, P_4, \ldots, P_{k-1}, P_k \cup (Q_2 - P_1)\}$ is blocked by f. Therefore $P_1 - Q_2$ is blocked by f. We may now relabel the above flower to $(P_1, P_2, \ldots, P_k)$. In this flower P_1 is blocked and $P_1 \cap Q_2 = \emptyset$.

Let $Z_1 = Q_1 \cap P_1$. We have established the following: $Q_2 = P_3 \cup P_4 \cup \cdots \cup P_k$, $P_2 \subseteq Q_1$, and $P_1 = Q_3 \cup \cdots \cup Q_k \cup (Q_1 \cap P_1)$. Let $(P'_1, P'_2, \ldots, P'_k) = (P_1, P_k, P_{k-1}, \ldots, P_2)$. Then we have $Q_2 = P'_2 \cup P'_3 \cup \cdots \cup P'_{k-1}$, $Q_1 = P'_k \cup Z_1$, and $P'_1 = Q_3 \cup Q_4 \cup \cdots \cup Q_k \cup Z_1$. It is now a matter of routine bookkeeping that $(P'_1, P'_2, \ldots, P'_k)$ and $(Q_1, Q_2, \ldots, Q_k)$ form a feral display for f. $\square$

4.32.2. *The lemma holds if* $(s,t) = (1,1)$.

Subproof. In this case $Q \subseteq \hat{P}_1$ and $P \subseteq \hat{Q}_1$. Clearly we may assume that $P_1 = \hat{P}_1$ and $Q_1 = \hat{Q}_1$. Consider Q_2. Note that $Q_2 \cap P_2 \neq \emptyset$, otherwise $Q \cup Q_2 \subseteq P_1$ so that $f \notin \mathrm{cl}(Q \cup Q_2)$ implying that Q_1 is not coblocked. As $\lambda_{M\backslash f}(Q_2) = 2$, and $\mathbf{F}_d$ is a maximal bloom of $M\backslash f$, either $P_1 \cong P_1 \cap Q_2$ or $P_2 \cong P_2 \cup Q_2$. The former case contradicts the fact that P_1 is fully closed. Thus $P_2 \cong P_2 \cup Q_2$. Thus, by moving to the equivalent flower, $(P_1 - Q_2, P_2 \cup Q_2, P_3, \ldots, P_k)$ we have $Q_2 \subseteq P_2$. It may be that $P_1 - Q_2$ is no longer blocked. But then $P_2 \cup Q_2$ is blocked and we are in the case covered by 4.32.1. Therefore we may assume that $P_1 - Q_2$ is blocked by f and, after an appropriate relabelling, we may assume that $Q_2 \subseteq P_2$.

We now have $Q_2 \subseteq P_2 \subseteq Q_2 \cup Q_1$. As $\mathbf{F}_c$ is a maximal bloom of M/f and $\lambda_{M/f}(P_2) = 2$, either $P_2 - Q_2$ is a set of loose elements of Q_2, or $Q_1 - P_1$ is a set of loose elements of Q_1. The latter case implies that $P = P_3 \cup P_4 \cup \cdots \cup P_k$ is contained in $\mathrm{fcl}_M(P_1 \cup P_2 \cup \{f\})$, contradicting the fact that $(P_1 \cup P_2 \cup \{f\}, P_3, \ldots, P_k)$ is a swirl-like flower of order $k-1$ in M. Hence $P_2 - Q_2$ is a set of loose elements of P_2 and, by moving to an equivalent flower we may assume that $P_2 = Q_2$.

Summing up, we have the following: $Q_2 = P_2$, $Q_3 \cup Q_4 \cup \cdots \cup Q_k \subseteq P_1$, and $P_3 \cup P_4 \cup \cdots \cup P_k \subseteq Q_1$. Let $Z_1 = P_1 \cap Q_1$, $Z_2 = P_1 \cap Q_2$. Then $Z_2 = \emptyset$. Moreover, it is easily checked that $Z_1 \neq \emptyset$ and $\lambda_M(Z_1) \leqslant 3$. Thus $(P_1, P_2, \ldots, P_k)$ and $(Q_1, Q_2, \ldots, Q_k)$ form a feral display for f. $\square$

4.32.3. *The lemma holds if* $(s,t) = (2,2)$.

Subproof. In this case $Q \subseteq \hat{P}_2$ and $P \subseteq \hat{Q}_2$. If f does not block $(\hat{P}_1 \cup \hat{P}_2) - \hat{P}_2$, then f blocks $\hat{P}_2$ and we are in a case that is equivalent to that of 4.32.1. So we may assume that f blocks $(\hat{P}_1 \cup \hat{P}_2) - \hat{P}_2$, and similarly, f coblocks $(\hat{Q}_1 \cup \hat{Q}_2) - \hat{Q}_2$. Thus there are flowers $(P_1, P_2, \ldots, P_k)$ and $(Q_1, Q_2, \ldots, Q_k)$ displayed by $\mathbf{F}_d$ and $\mathbf{F}_c$ respectively such that f blocks P_1, f coblocks Q_1, $P_1 \cup P_2 = \hat{P}_1 \cup \hat{P}_2$, $Q_1 \cup Q_2 = \hat{Q}_1 \cup \hat{Q}_2$, $P \subseteq Q_2$, and $Q \subseteq P_2$.

As f does not block Q_2, we may argue, just as in the previous case, that, by moving to an equivalent flower, we may assume that $Q_2 = P$. We now have $Q \subseteq P_2 \subseteq Q \cup Q_1$ and $\lambda_{M/f}(P_2) = 2$. Therefore in M/f, either $Q \cong P_2$, or $P_2 \cong P_2 \cup Q_1$, that is, $P_2 \cong P_2 \cup P_1$. The former case implies that there is a flower displayed by $\mathbf{F}_c$ such that no petal is coblocked by f, so that case does not occur. Consider the latter case. In this case, P_1 is a set of loose elements of Q_1, so we may move to an equivalent flower where $Q_1 \subseteq P_2$. We now have a flower $(Q'_1, Q'_2, \ldots, Q'_k)$, where $Q'_1 \subseteq P_2$, so that Q'_1 is not coblocked by f. But this means that f blocks $\hat{Q}_2$ and we are again in the case covered by 4.32.1. $\square$

All cases have been covered and the lemma follows. $\square$

Theorem 4.25 is an immediate consequence of Lemmas 4.32 and 4.31.

5. 3-Trees and *k*-coherence

Flowers provide a way of representing certain 3-separations in a matroid. It was shown in [20] that, by using a certain type of tree, one can simultaneously display a representative of each equivalence class of non-sequential 3-separations of a 3-connected matroid M. In this section we describe these trees and their interaction with k-coherence.

Let π be a partition of a finite set E, where some members of π may be empty, and let T be a tree such that every member of π labels a vertex of T. We say that T is a *π-labelled tree*; labelled vertices are called *bag vertices* and members of π are called *bags*.

Let G be a subgraph of T with components $G_1, G_2, \dots, G_m$. Let X_i be the union of those bags that label vertices of G_i. Then *the subsets of E displayed by G* are $X_1, X_2, \dots, X_m$. In particular, if $V(G) = V(T)$, then $\{X_1, X_2, \dots, X_m\}$ is the *partition of E displayed by G*. Let e be an edge of T. The *partition of E displayed by e* is the partition displayed by $T\backslash e$. If $e = v_1v_2$ for vertices v_1 and v_2, then (Y_1, Y_2) is the (*ordered*) *partition of $E(M)$ displayed by v_1v_2* if Y_1 is the union of the bags in the component of $T\backslash v_1v_2$ containing v_1. Let v be a vertex of T. The *partition of E displayed by v* is the partition displayed by $T - v$. The edges incident with v correspond to the components of $T - v$, and hence to the members of the partition displayed by v. Note that, if v is not a bag vertex, then the partition displayed by v is a partition of E, while, if v is a bag vertex, then the partition is a partition of $E - B$ where B is the bag labelling v. In what follows, if a cyclic ordering $(e_1, e_2, \dots, e_n)$ is imposed on the edges incident with v, this cyclic ordering is taken to represent the corresponding cyclic ordering on the members of the partition displayed by v.

Let M be a 3-connected matroid with ground set E. An *almost partial 3-tree T* for M is a π-labelled tree, where π is a partition of E such that:

(i) For each edge e of T, the partition (X, Y) of E displayed by e is 3-separating, and, if e is incident with two bag vertices, then (X, Y) is a non-sequential 3-separation.
(ii) Every non-bag vertex v is labelled either D or A; if v is labelled D, then there is a cyclic ordering on the edges incident with v.
(iii) If a vertex v is labelled A, then the partition of E displayed by v is a tight maximal anemone of order at least 3.
(iv) If a vertex v is labelled D, then the partition of E displayed by v, with the cyclic order induced by the cyclic ordering on the edges incident with v, is a tight maximal daisy of order at least 3.

By conditions (iii) and (iv), a vertex v labelled D or A corresponds to a flower of M. The 3-separations displayed by this flower are the 3-separations *displayed by v*. A vertex of a partial 3-tree is referred to as a *daisy vertex* or an *anemone vertex* if it is labelled D or A, respectively. A vertex labelled either D or A is a *flower vertex*. A 3-separation is *displayed by an almost partial 3-tree T* if it is displayed by some edge or some flower vertex of T.

A 3-separation (R, G) of M *conforms* with an almost partial 3-tree T if either (R, G) is equivalent to a 3-separation that is displayed by a flower vertex or an edge of T, or (R, G) is equivalent to a 3-separation (R', G') with the property that either R' or G' is contained in a bag of T.

An almost partial 3-tree for M is a *partial 3-tree* if every non-sequential 3-separation of M conforms with T. We now define a quasi-order on the set of partial 3-trees for M. Let T_1 and T_2 be two partial 3-trees for M. Then $T_1 \preccurlyeq T_2$ if all of the non-sequential 3-separations displayed by T_1 are displayed by T_2. If $T_1 \preccurlyeq T_2$ and $T_2 \preccurlyeq T_1$, then T_1 is *equivalent* to T_2. A partial 3-tree is *maximal* if it is maximal with respect to this quasi-order.

Note that while flower vertices need to be labelled D or A, we may suppress these labels when they are clear from context. The following theorem is the main result of [20, Theorem 9.1].

Theorem 4.33. *Let M be a 3-connected matroid with $|E(M)| \geqslant 9$, and let T be a maximal partial 3-tree for M. Then every non-sequential 3-separation of M is equivalent to a 3-separation displayed by T.*

Maximal partial 3-trees are by no means unique. Consider the following situation. Let (P_1, P_2, P_3) be a maximal flower of order three in a 3-connected matroid M. Then it may be the case that P_2 is sequential, in which case $(P_1, P_2 \cup P_3)$ and $(P_1 \cup P_2, P_3)$ are inequivalent non-sequential 3-separations. Assume that these are the only inequivalent non-sequential 3-separations of M. Then given such a flower, one can obtain distinct maximal partial 3-trees for M as follows. Let T_1 be a tree consisting of a path with vertices (v_1, v_2, v_3) such that v_i labels the bag P_1. Let T_2 be a star with a flower vertex of degree 3 and leaf vertices v_1, v_2 and v_3 labelling the bags P_1, P_2 and P_3. Then both T_1 and T_2 are maximal partial 3-trees for M. Indeed the situation can be further complicated by splitting the elements of P_2 into smaller bags along a path. To get a more canonical structure we follow [21] and say that a maximal partial 3-tree for M is a 3-*tree* if

(I) for every tight maximal flower of M of order three, there is an equivalent flower that is displayed by a vertex of T; and
(II) if a vertex is incident with two edges e and f that display equivalent 3-separations, then the other ends of e and f are flower vertices, v has degree two, and v labels a nonempty bag.

The next theorem summarises results from [21].

Theorem 4.34. *Let M be a 3-connected matroid. Then M has a 3-tree T. Moreover, if $\mathbf{P}$ is a flower of M of order at least three, then T has a flower vertex v that displays a flower equivalent to $\mathbf{P}$.*

Let T be a 3-tree for M. A bag is a *leaf bag* if it labels a leaf of T. We omit the easy proof of the next lemma.

Lemma 4.35. *Let M be a 3-connected matroid. If L is a leaf bag of a 3-tree for M, and $x \in \mathrm{cl}^{(*)}(L)$, then $L \cup \{x\}$ is a leaf bag of a 3-tree for M. Thus $\mathrm{fcl}(L)$ is a leaf bag for a 3-tree for M.*

Note that converse of Lemma 4.35 does not hold in that it is not always the case that a 3-separating set equivalent to one in a leaf bag can be displayed in a 3-tree for M.

A subset L of $E(M)$ is a *peripheral set* of M if it is a leaf bag for some 3-tree T for M. By Lemma 4.35 the full closure of a peripheral set is a peripheral set. An element x of M is *peripheral* if x is an element of a peripheral set. The theme in what follows is that peripheral elements of k-coherent matroids are well behaved. Indeed this is true in a broader setting. The next theorem is [24, Theorem 4.2].

Theorem 4.36. *Let M be a 3-connected matroid other than a wheel or a whirl. Suppose $|E(M)| \geqslant 9$ and let S be a peripheral set of M. Then $\mathrm{fcl}(S)$ contains an element e such that either $M\backslash e$ or M/e is 3-connected and e does not expose any 3-separations.*

This gives the following as an immediate corollary.

Corollary 4.37. *Let M be a k-coherent matroid other than a wheel or a whirl and let S be a peripheral set of M. Then $\mathrm{fcl}(S)$ contains an element e such that either $M\backslash e$ or M/e is k-coherent.*

The remainder of this section examines life in peripheral sets in more detail. We next show that elements of k-wild triangles are not peripheral.

Lemma 4.38. *Let W be a k-wild triangle of the k-coherent matroid M. Then no element of W is peripheral.*

Proof. Say $W = \{a, b, c\}$ and let T be a 3-tree for M. By Theorem 4.16, W is either a standard or costandard k-wild triangle. In either case the triangle $\{a, b, c\}$ has a k-wild display $(A_1, A_2, \dots, A_{k-2}, B_1, B_2, \dots, B_{k-2}, C_1, C_2, \dots, C_{k-2})$. Let $A = A_1 \cup A_2 \cup \dots \cup A_{k-2}$, $B = B_1 \cup B_2 \cup \dots \cup B_{k-2}$ and $C = C_1 \cup C_2 \cup \dots \cup C_{k-2}$. Then, $\mathbf{A} = (A_1, A_2, \dots, A_{k-2}, B \cup C \cup W)$, $\mathbf{B} = (B_1, B_2, \dots, B_{k-2}, A \cup C \cup W)$ and $\mathbf{C} = (C_1, C_2, \dots, C_{k-2}, A \cup B \cup W)$ are tight maximal flowers of M of order at least four. Assume that a is peripheral. Then we may assume that $a \in L$ for some leaf bag of T. By Theorem 4.34, there are flower vertices v_A, v_B and v_C of T that display flowers equivalent to $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. By Lemma 4.24,

$\mathrm{fcl}(A_i) \cap W = \emptyset$ for all $i \in \{1, 2, \ldots, k-2\}$, and symmetric conclusions hold for the analogous petals of **B** and **C**. It follows that, by moving to equivalent flowers, we may assume that it is precisely **A**, **B** and **C** that are displayed by the vertices v_A, v_B and v_C. We conclude that $L \subseteq W$. As $|L| \geqslant 2$, and W is a triangle, Lemma 4.35 implies that we may assume that $L = \{a, b, c\}$. As L is sequential, the vertex adjacent to the vertex of T labelled by L is a flower vertex v, and as such, displays a tight flower of order at least three. Indeed, one readily checks that the partition displayed by v is (A, B, C, W). But then, by Lemma 4.11, $M \backslash a$ is k-coherent, contradicting the fact that W is k-wild. $\square$

Next we show that feral elements are not peripheral. Recall that a set $A \subseteq E(M)$ is *cohesive* if $E(M) - A$ is fully closed. We denote by $\mathrm{coh}(X)$ the set $X - \mathrm{fcl}(E(M) - X)$. Evidently $\mathrm{coh}(X)$ is the unique maximal cohesive set contained in X. Note that the 3-separating set X is non-sequential if and only if $\mathrm{coh}(X) \neq \emptyset$.

Lemma 4.39. *If f is a feral element of the k-coherent matroid M, then f is not peripheral.*

Proof. Let f be a feral element of M. Then up to duality there are partitions $(P_1, P_2, \ldots, P_m)$ and $(Q_1, Q_2, \ldots, Q_k)$ of $E(M)$ that form a feral display for f. In other words, for this pair of partitions, there is an $i \in \{2, 3, \ldots, m-1\}$ such that properties (i)–(x) of a feral display hold. By possibly reversing the order of indices of $(P_1, P_2, \ldots, P_m)$, we may assume that $i \geqslant 3$.

Assume that f is peripheral. Then there is a 3-tree T for M with a leaf bag L such that $f \in L$.

4.39.1. *L is non-sequential and $f \in \mathrm{coh}(L)$.*

Subproof. Assume that L is sequential. If $|L| \geqslant 4$, then $M \backslash f$ is k-coherent by Corollary 4.6. Certainly f is not in a triangle or triad. Thus $|L| = 2$. As L is sequential, it follows from the definition of 3-tree that L is a petal of a tight flower of order at least three in M. Only swirl-like and spike-like flowers can have 2-element tight petals, so the flower must be swirl-like or spike-like. By Corollary 4.10 and the fact that f is not in a triangle or a triad we again deduce that $M \backslash f$ or M/f is k-coherent. Thus L is non-sequential.

Assume that $f \notin \mathrm{coh}(L)$. By considering the full closure of $E(M) - L$, we see that there is a non-sequential 3-separating set $L' \subseteq L$ such that f is in either the guts or coguts of L' meaning that either $M \backslash f$ or M/f is not 3-connected. Hence $f \in \mathrm{coh}(L)$. $\square$

By property (vi) of feral display, $(Q_1 \cup Q_2 \cup \{f\}, Q_3, \ldots, Q_k)$ is a swirl-like flower of M. By Theorem 4.34, there is a vertex v of T that displays an equivalent flower. If $f \in \mathrm{fcl}(Q_i)$ for some $i \in \{3, 4, \ldots, k\}$, then we contradict the fact that $f \in \mathrm{coh}(L)$. Thus $\mathrm{coh}(L) \subseteq Q_1 \cup Q_2 \cup \{f\}$. By property (vii) of feral display, $(P_2, P_3, \ldots, P_i, P_{i+1} \cup \cdots \cup P_m \cup P_1 \cup \{f\})$ is a swirl-like flower of order i in M. Arguing as above we deduce that $\mathrm{coh}(L) \subseteq P_{i+1} \cup \cdots \cup P_m \cup Z_1 \cup Z_2$.

Let $K = \mathrm{coh}(L) - \{f\}$. Then $\lambda_{M \backslash f}(K) = 2$ and $f \in \mathrm{cl}_M(K)$. We now consider the location of K relative to the flower $(P_1, P_2, \ldots, P_m)$ of $M \backslash f$. If $K \subseteq \mathrm{fcl}_{M \backslash f}(P_i)$ for some $i \in \{1, 2, \ldots, m\}$, then M is k-fractured. Hence K is equivalent to a 3-separation displayed by $(P_1, P_2, \ldots, P_m)$. As $Q_3 \cup Q_4 \cup \cdots \cup Q_k \subseteq P_1$, we see that K is equivalent to the union of a consecutive subset of petals in $(P_{i+1}, P_{i+2}, \ldots, P_m)$. If there is more than one petal in this set, we contradict the fact that L is a peripheral set of M as in this case the flower $(P_1 \cup \cdots \cup P_i \cup \{f\}, P_{i+1}, P_{i+2}, \ldots, P_m)$ of M has order at least three and needs to be displayed by T. On the other hand, we have already shown that there cannot be only one petal in the set. The lemma follows from this contradiction. $\square$

By Corollary 4.37 there is always an element of the full closure of a peripheral set that can be removed to preserve k-coherence. The next lemma strengthens that outcome in the non-sequential case.

Lemma 4.40. *Let L be a fully-closed peripheral set of the k-coherent matroid M such that either $L = E(M)$ or L is a non-sequential 3-separating set. Let x be an element of $\mathrm{coh}(L)$.*

(i) *If M is not a wheel or a whirl, and x is in a triangle T, then* $\mathrm{fcl}(T) \subseteq L$ *and there is an element* $z \in \mathrm{fcl}(T)$ *such that either $M\backslash z$ or M/z is k-coherent.*
(ii) *If x is not in a triangle or a triad, then either $M\backslash x$ or M/x is k-coherent.*

Proof. Say that x is in a triangle T. Then $|T \cap \mathrm{coh}(L)| \geqslant 2$ and $\mathrm{fcl}(T) \subseteq L$. Part (i) now follows straightforwardly from Corollary 4.6, Lemma 4.38 and the fact that M is not a wheel or a whirl.

Assume that x is not in a triangle or a triad. Say M/x is not 3-connected. Then there is a 3-separation $(A \cup \{x\}, B)$ of M with $x \in \mathrm{cl}(A), \mathrm{cl}(B)$. Assume that $(A \cup \{x\}, B)$ is sequential. Then we may assume that $A \cup \{x\}$ is sequential. If $|A| = 2$, then x is in a triangle. Hence $|A \cup \{x\}| \geqslant 4$. By Bixby's Lemma and the fact that x is in no triads, $M\backslash x$ is 3-connected. Then, by Corollary 4.6, $M\backslash x$ is k-coherent. It follows that $(A \cup \{x\}, B)$ is non-sequential.

If $L = E(M)$, then a 3-tree for M consists of a single vertex and M has no non-sequential 3-separations. Thus M/x, and similarly $M\backslash x$ is 3-connected. By Lemma 4.39, one of $M\backslash x$ or M/x is k-coherent as required.

We may now assume that $\lambda(L) = 2$. Let $K = E(M) - \mathrm{coh}(L)$. Then K is fully closed and $(K, \mathrm{coh}(L))$ is a non-sequential 3-separation. If M/x is not 3-connected, then x is in the guts of a non-sequential 3-separation $(A \cup \{x\}, B)$. But $x \in \mathrm{fcl}(B)$ and $x \notin \mathrm{fcl}(K)$ so $(A \cup \{x\}, B)$ is not equivalent to $(K, \mathrm{coh}(L))$. A 3-separation equivalent to $(A \cup \{x\}, B)$ must be displayed in a 3-tree for M. As $\mathrm{coh}(L)$ is a peripheral set, and K is fully closed, it is easily seen that we may assume that $B \subseteq K$. But now $x \in \mathrm{cl}(B)$ so $x \in \mathrm{cl}(K)$ contradicting the fact that K is fully closed. Therefore both M/x and $M\backslash x$ are 3-connected. Again by Lemma 4.39, we see that one of M/x or $M\backslash x$ is k-coherent. □

6. Extending a k-coherent matroid

Let x be an element of the matroid M such that $M\backslash x$ is 3-connected. If M is not 3-connected, then it is easily seen that x is either a loop, a coloop, or is in a parallel pair in M. The situation when $M\backslash x$ is k-coherent, but M is not is a little more complicated. This section gives some straightforward lemmas describing the structures that arise in this and related situations.

We omit the routine proof of the next lemma.

Lemma 4.41. *Let M be a* 3*-connected matroid and let $(l_1, l_2, \dots, l_n)$ be a maximal fan of loose elements between a pair of petals in a swirl-like flower of M of order at least* 3*. Assume that l_1 is a guts element.*

(i) *$M\backslash l_1$ is k-coherent if and only if M is.*
(ii) *If i is odd and $i > 1$, then $M\backslash l_i/l_{i-1}$ is* 3*-connected. Moreover $M\backslash l_i/l_{i-1}$ is k-coherent if and only if M is.*

Let M be a 3-connected matroid with a maximal k-fracture $\mathbf{P}$. Then M is *uniquely* k-fractured *by* $\mathbf{P}$ if every k-fracture $\mathbf{Q}$ of M has the property that $\mathbf{Q} \preccurlyeq \mathbf{P}$. Note that if M is uniquely fractured by $\mathbf{P}$ and $\mathbf{P}$ has order k, then every k-fracture of M is equivalent to $\mathbf{P}$.

Recall the definition of quasi-flower. Note that if a quasi-flower has exactly one 1-element petal, then this petal is contained in either the closure or coclosure of either of its adjacent petals, so it is quite properly regarded as a loose petal.

Lemma 4.42. *Let M be a* 3*-connected matroid with an element e such that $M\backslash e$ is k-coherent. If M is not k-coherent, then the following hold.*

(i) *M is uniquely k-fractured by a maximal swirl-like flower* $\mathbf{P}$ *of order k.*
(ii) *If $(P_1, P_2, \dots, P_k)$ is a k-fracture of M with $e \in P_1$, then $P_1 - \{e\}$ is a loose petal in the swirl-like quasi-flower $(P_1\backslash\{e\}, P_2, \dots, P_k)$ of M. This quasi-flower has order $k - 1$.*

Proof. Up to labels M has a k-fracture $(P_1, P_2, \dots, P_k)$, where $e \in P_1$. But $(P_1 - \{e\}, P_2, \dots, P_k)$ cannot be a k-fracture of $M\backslash e$. The latter part of the lemma follows easily from this observation. Say that $(Q_1, Q_2, \dots, Q_k)$ is another k-fracture of M. By Lemma 3.16, up to labels, equivalence and symmetry, we may assume that $e \in Q_1$, that Q_1 is fully closed and that Q_1 contains all but one petal of $(P_1, P_2, \dots, P_k)$. By the above, $Q_1 - \{e\}$ is a set of loose elements of the flower $((Q_1 \cup Q_2) - \{e\}, Q_3, \dots, Q_k)$ of $M \backslash e$. But it is now easily seen that this is not possible. Thus M is uniquely fractured by $\mathbf{P}$ and (i) holds. □

Viewed from the perspective of moving from $M \backslash e$ to M, we have

Lemma 4.43. *Assume that $M \backslash e$ is k-coherent and that M is 3-connected and k-fractured. If $\mathbf{P}$ is a flower of $M \backslash e$ of order at least three, then, $e \in \mathrm{cl}(\hat{P})$ for some petal P of $\mathbf{P}$.*

Proof. Let $(P_1, P_2, \dots, P_k)$ be a k-fracture of M, where $e \in P_1$. By Lemma 4.42, $((P_1 \cup P_2) - \{e\}, P_3, \dots, P_k)$ is a maximal flower in $M \backslash e$ and it follows that $e \in \mathrm{cl}((P_1 \cup P_2) - \{e\})$ and the lemma holds in this case.

Say that $\mathbf{Q} = (Q_1, Q_2, \dots, Q_m)$ is another flower in M of order at least three. By Lemma 3.16, we may assume up to labels in $\mathbf{Q}$ that either $P_k \subseteq \hat{Q}_1$ or $P_2 \subseteq \hat{Q}_1$. Assume without loss of generality that the latter case holds. Then $\hat{P}_2 \subseteq \hat{Q}_1$ and $e \in \mathrm{cl}(\hat{P}_2)$ so that $e \in \mathrm{cl}(\hat{Q}_1)$ as required. □

We omit the routine proof of the next lemma.

Lemma 4.44. *Assume that $M \backslash e$ is k-coherent and that M is 3-connected and k-fractured. Then there is a swirl-like quasi-flower $(P_1, L, P_2, \dots, P_{k-1})$ of $M \backslash e$, where L is a nonempty loose petal, such that the following hold.*

(i) *$(P_1, L \cup \{e\}, P_2, \dots, P_{k-1})$ is a k-fracture of M.*
(ii) *$e \in \mathrm{cl}(P_1 \cup L)$ and $e \in \mathrm{cl}(P_2 \cup L)$.*
(iii) *If $|L| > 1$, then $e \in \mathrm{cl}(L)$.*

The next lemma is unsurprising. If we block every k-fracture we would expect to become k-coherent.

Lemma 4.45. *Assume that M and $M \backslash e$ are 3-connected and k-fractured. Then there is a k-fracture $(P_1, P_2, \dots, P_n)$ of $M \backslash e$ such that $e \in \mathrm{cl}_M(\hat{P}_i)$ for some $i \in \{1, 2, \dots, n\}$.*

Proof. Let $(Q_1 \cup \{e\}, Q_2, \dots, Q_n)$ be a maximal k-fracture of M. If the lemma fails, it must be the case that $n = k$ and that Q_1 is a set of loose elements of the maximal flower $(Q_1 \cup Q_2, Q_3, \dots, Q_k)$ of $M \backslash e$. Note that $e \in \mathrm{cl}_M(\hat{Q}_2)$ and $e \in \mathrm{cl}_M(\hat{Q}_k)$. As $M \backslash e$ is k-fractured, it has a maximal k-fracture $(P_1, P_2, \dots, P_n)$. By Lemma 3.16, there is an $i \in \{1, 2, \dots, n\}$ such that either $Q_2 \subseteq \hat{P}_i$ or $Q_k \subseteq \hat{P}_i$. In either case it follows that $e \in \mathrm{cl}_M(\hat{P}_i)$ as required. □

Note that Lemma 4.45 would fail if we were to insist in the statement that the k-fracture $(P_1, P_2, \dots, P_n)$ was *maximal*. As an easy corollary of Lemma 4.45, we have

Corollary 4.46. *Assume that M and $M \backslash e$ are 3-connected and that, $M \backslash e$ has a unique maximal k-fracture $(P_1, P_2, \dots, P_k)$. Then M is k-fractured if and only if $e \in \mathrm{cl}(\hat{P}_i)$ for some $i \in \{1, 2, \dots, k\}$.*

Lemma 4.47. *Assume that M is 3-connected and that $(\{e, f\}, P_2, \dots, P_k)$ is a maximal flower that uniquely k-fractures M, where $\{e, f\}$ is fully closed. Then $M \backslash e$ is k-coherent.*

Proof. Assume that $M \backslash e$ is not k-coherent. Then, by Lemma 4.45, there is a k-fracture $(Q_1, Q_2, \dots, Q_n)$ of $M \backslash e$ such that $e \in \mathrm{cl}(Q_1)$. But then $(Q_1 \cup \{e\}, Q_2, \dots, Q_n)$ is a k-fracture of M. As M has a unique k-fracture and $\{e, f\}$ is fully closed, $Q_1 \cup \{e\} = \{e, f\}$ so that $Q_1 = \{f\}$ contradicting the assumption that Q_1 is a petal of a flower of $M \backslash e$. □

We omit the easy proof of the next lemma.

Lemma 4.48. *Let (A, B) be a 3-separation of the k-coherent matroid M, where $a \in A$, and $|A| \geqslant 4$. If $M \backslash a$ is 3-connected and k-fractured, then there is a k-fracture $(P_1, P_2, \ldots, P_n)$ of $M \backslash a$ such that, for some $i \in \{2, 3, \ldots, n-1\}$, $A = P_1 \cup P_2 \cup \cdots \cup P_i$, and $(A, P_{i+1}, P_{i+2}, \ldots, P_n)$ is a swirl-like flower of M.*

Note that the flower $(A, P_{i+1}, P_{i+2}, \ldots, P_n)$ of Lemma 4.48 may be trivial in the sense that is has only two petals.

Chapter 5. *k*-Skeletons

It was noted in the introduction that the underlying cause of inequivalent representations of a matroid is that elements may have "freedom". Loosely speaking k-skeletons are k-coherent matroids whose elements have maximal freedom. The next section of this chapter makes these notions precise. The remaining sections of the chapter examine freedom in matroids and the connection with k-coherence in more detail. The last two sections describe certain structures that turn out to be of importance. Life would have been easier if we could have avoided worrying about these structures, but, sadly, this seems not to be the case.

1. Clones, fixed elements, and *k*-skeletons

Let M be a matroid. Elements e and f of M are *clones* if swapping the labels of e and f is an automorphism of M. A *clonal set* of M is a set of elements every pair of which are clones. A *clonal class* of M is a maximal clonal set. The elements z, z' are *independent clones* if they are clones and the set $\{z, z'\}$ is independent. An element z of M is *fixed* in M if there is no single-element extension of M by an element z' in which z and z' are independent clones. Dually, an element z in M is *cofixed* in M if it is fixed in M^*. Note that if z already has a clone, say x, and $\{x, z\}$ is independent, then z is not fixed since we can add a new element z' freely on the line through x and z.

Let $k \geqslant 5$ be an integer, and M be a k-coherent matroid. Then M is a *k-skeleton* if the following hold.

(i) M is not a wheel or a whirl of rank at least 3.
(ii) If x is fixed in M, then $M \backslash x$ is not k-coherent.
(iii) If x is cofixed in M then M/x is not k-coherent.

As with k-coherence, in any unexplained context, when referring to a k-skeleton, we always assume that k is an integer greater than four. Note that condition (i) is required in the definition simply because wheels and whirls vacuously satisfy (ii) and (iii).

It is shown in Lemma 12.3 that the number of inequivalent representations of a k-coherent matroid over a finite field $\mathbb{F}$ is bounded above by the maximum of the number of inequivalent $\mathbb{F}$-representations of its k-skeleton minors. The importance of k-skeletons in studying inequivalent representations of k-coherent matroids is due to this fact.

2. Freedom and cofreedom

In this section we develop further material related to freedom in matroids. Most of the results here are either straightforward or are proved in [12,13].

A flat F of the matroid M is *cyclic* if, for each element $e \in F$, there is a circuit C such that $e \in C \subseteq F$. It is easily seen that F is a cyclic flat of M if and only if $E(M) - F$ is a cyclic flat of M^*. The next result is straightforward.

Proposition 5.1. *Elements e and f of a matroid M are clones if and only if e and f are contained in the same set of cyclic flats.*

Let e and f be elements of the matroid M. Then e is *freer* than f, denoted $e \succcurlyeq f$, if every cyclic flat containing e also contains f. The *freedom* of an element e of M is the maximum size of an independent clonal class containing e amongst all matroids containing M as a restriction, that is, amongst all extensions of M. We denote the freedom of e in M by $\mathrm{fr}_M(e)$, or $\mathrm{fr}(e)$ if the matroid M is clear. This maximum does not exist if and only if e is a coloop of M in which case $\mathrm{fr}(e)$ is infinity. A loop has freedom 0 and an element is fixed if and only if it has freedom at most 1.

The notion of freedom in matroids was introduced by Duke [6]. His definition was different from that given above, but it is shown in [13, Lemma 2.8] that Duke's definition is equivalent to ours. The next two lemmas are also proved in [13].

Lemma 5.2. *Let a and b be elements of the matroid M such that $a \succcurlyeq b$. Then $\mathrm{fr}(a) \geqslant \mathrm{fr}(b)$. Moreover, either a and b are clones or $\mathrm{fr}(a) > \mathrm{fr}(b)$.*

Lemma 5.3. *Let a and b be elements of the matroid M. Then $a \succcurlyeq b$ if and only if the following holds: for all $X \subseteq E(M) - \{a, b\}$, if $a \in \mathrm{cl}(X)$, then $b \in \mathrm{cl}(X)$.*

The next lemma is elementary.

Lemma 5.4. *Let a and b be elements of the matroid M, and let N be a minor of M whose ground set contains a and b. If a is freer than b in M, then a is freer than b in N.*

Lemma 5.5. *Let a and b be elements of the matroid M. Then:*

(i) $\mathrm{fr}_{M\backslash a}(b) \geqslant \mathrm{fr}_M(b)$.
(ii) $\mathrm{fr}_{M/a}(b) \geqslant \mathrm{fr}_M(b) - 1$, *and, if* $\mathrm{fr}_{M/a}(b) = \mathrm{fr}_M(b) - 1$, *then* $a \preccurlyeq b$ *in* M.

Proof. The lemma is trivial if $\mathrm{fr}(b) = 0$. Assume that b has freedom $k \geqslant 1$ and let N be a matroid that extends M in which b belongs to an independent clonal set B of size k. It is easily seen that such an extension exists with the property that $a \notin B$. As B is an independent clonal set in $N\backslash a$, part (i) holds.

Consider N/a. In this matroid B is a clonal set and contains an independent subset containing b of size $k - 1$, so $\mathrm{fr}_{M/a}(b) \geqslant \mathrm{fr}_M(b) - 1$. Assume that $\mathrm{fr}_{M/a}(b) < \mathrm{fr}_M(b)$. Then B is not independent in N/a. Thus $a \in \mathrm{cl}_N(B)$. But every cyclic flat of N containing b also contains B, so every cyclic flat of N containing b contains a. Hence b is freer than a in N, and by Lemma 5.4 $b \succcurlyeq a$ in M. □

The *cofreedom* of an element e of M, denoted $\mathrm{fr}^*(e)$, is the freedom of e in M^*. Note that $e \succcurlyeq f$ in M^* if and only if $f \succcurlyeq e$ in M. This is a consequence of the fact that the cyclic flats of M^* are the complements of the cyclic flats of M. The following lemma is the dual of Lemma 5.5.

Lemma 5.6. *Let a and b be elements of the matroid M. Then:*

(i) $\mathrm{fr}^*_{M/a}(b) \geqslant \mathrm{fr}^*_M(b)$.
(ii) $\mathrm{fr}^*_{M\backslash a}(b) \geqslant \mathrm{fr}^*_M(b) - 1$, *and, if* $\mathrm{fr}^*_{M\backslash a}(b) = \mathrm{fr}^*_M(b) - 1$, *then* $a \succcurlyeq b$ *in* M.

The next two results are immediate corollaries of Lemmas 5.5 and 5.6. We apply them frequently.

Corollary 5.7. *Let a and b be elements of the matroid M.*

(i) *If b is fixed in M/a, but not in M, then $a \preccurlyeq b$ in M and $\mathrm{fr}_M(b) = 2$. Moreover, either a and b are clones in M, or a is fixed in M.*
(ii) *If b is cofixed in $M\backslash a$ but not in M, then $a \succcurlyeq b$ in M and $\mathrm{fr}^*_M(b) = 2$. Moreover, either a and b are clones in M, or a is cofixed in M.*

Corollary 5.8. *Let a and b be elements of the matroid M.*

(i) *If b is fixed in M/a and a is cofixed in M, then b is fixed in M.*
(ii) *If b is cofixed in $M\backslash a$ and a is fixed in M, then b is cofixed in M.*

Elements a and b of M are *comparable* if either $a \succcurlyeq b$ or $b \succcurlyeq a$; otherwise they are *incomparable*.

Corollary 5.9. *Let a and b be incomparable elements of the matroid M.*

(i) *If b is fixed in $M\backslash a$ or M/a, then b is fixed in M.*
(ii) *If b is cofixed in $M\backslash a$ or M/a, then b is cofixed in M.*

The next lemma is [13, Lemma 2.13].

Lemma 5.10. *Let a, b and e be elements of the matroid M such that a and b are clones and have freedom 2 in $M\backslash e$. If a and b are not clones in M, then either a or b is fixed in M.*

At times local connectivity provides a useful way of bounding the freedom of an element in a matroid.

Lemma 5.11. *Let X and Y be disjoint sets of elements of a matroid M and suppose that $a \in E(M) - (X \cup Y)$. If $a \in \mathrm{cl}(X), \mathrm{cl}(Y)$, then $\mathrm{fr}(a) \leqslant \sqcap(X, Y)$.*

Proof. Let N be an extension of M in which a belongs to an independent clonal set A of size $\mathrm{fr}(a)$. By Lemma 5.3, $A \subseteq \mathrm{cl}(X)$ and $A \subseteq \mathrm{cl}(Y)$. Moreover, $r(\mathrm{cl}_N(X) \cap \mathrm{cl}_N(Y)) \leqslant r_N(X) + r_N(Y) - r_N(\mathrm{cl}_N(X) \cup \mathrm{cl}_N(Y)) = r_M(X) + r_M(Y) - r_M(X \cup Y) = \sqcap_M(X, Y)$. We now have $\mathrm{fr}_M(a) = |A| \leqslant r(\mathrm{cl}_N(X) \cap \mathrm{cl}_N(Y))$, and the lemma follows. □

The next result follows easily from Lemma 5.11.

Corollary 5.12. *Let A, B be disjoint subsets of a matroid M.*

(i) *If $x \in \mathrm{cl}(A), \mathrm{cl}(B)$, and A, B are skew in M/x, then x is fixed in M.*
(ii) *If $x \in \mathrm{cl}^*(A), \mathrm{cl}^*(B)$, and A, B are coskew in $M\backslash x$, then x is cofixed in M.*

A useful consequence of Corollary 5.12(ii) is

Lemma 5.13. *Let a be an element of the 3-connected matroid M. If a blocks non-adjacent petals of a swirl-like flower of $M\backslash a$, then a is cofixed in M.*

There are connections between freedom and connectivity. Loosely speaking, the freer an element b of M is, the less likely it is that connectivity will be damaged when b is contracted from M. If contracting b does damage connectivity, and $a \preccurlyeq b$, then contracting a should also damage connectivity. The next lemma has easy generalisations. We focus on the case that is useful for us.

Lemma 5.14. *Let b be an element of the 3-connected matroid M. Assume that M/b is not 3-connected. Then the following hold.*

(i) $\mathrm{fr}(b) \leqslant 2$.
(ii) *If $a \in E(M)$, and $a \preccurlyeq b$, then M/a is not 3-connected.*

Proof. Let (P, Q) be a 2-separation of M/b. As M is 3-connected, b coblocks this 2-separation so that $b \in \mathrm{cl}_M(P)$ and $b \in \mathrm{cl}_M(Q)$. But $\sqcap(P, Q) = 2$, so by Lemma 5.11, $\mathrm{fr}_M(b) \leqslant 2$, and (i) holds. Say $a \preccurlyeq b$. Assume that $a \in P$. Observe that $a \in \mathrm{cl}((P - \{a\}) \cup \{b\})$. As $a \preccurlyeq b$, we have $a \in \mathrm{cl}(Q)$. Hence $((P - \{a\}) \cup \{b\}, Q)$ is a 2-separation of M/a, and (ii) holds. □

We write $a \prec b$ if $a \preccurlyeq b$ in M, but a and b are not clones in M. By Lemma 5.2, $a \prec b$ if $a \preccurlyeq b$ and $\mathrm{fr}(a) < \mathrm{fr}(b)$. The next result is a routine corollary of this fact and Lemma 5.14. We omit the proof.

Corollary 5.15. *Let a and b be elements of the 3-connected matroid M such that $a \prec b$.*

(i) *If b is in a triangle, then there is a triangle containing a and b, and a is fixed in M.*
(ii) *If $\mathrm{si}(M/b)$ is not 3-connected, then a is fixed in M and $\mathrm{si}(M/a)$ is not 3-connected.*

As elements in triangles have freedom at most 2, we do not need much extra information to fix them.

Lemma 5.16. *If a is in a triangle T of the matroid M and there is a cyclic flat of M that contains a but not T, then a is fixed in M.*

Proof. Say F is a cyclic flat of M containing a but not T. Observe that $a \in \mathrm{cl}(F - \{a\})$, $a \in \mathrm{cl}(T - \{a\})$, and $\sqcap(F - \{a\}, T - \{a\}) = 1$, so that the lemma holds by Lemma 5.11. □

The next lemma sees a single application in Chapter 8.

Lemma 5.17. *Let $\{a, b, c\}$ be a clonal triangle of the matroid M and let e be an element of $E(M) - \{a, b, c\}$.*

(i) *If e is fixed in M, then e is fixed in $M \backslash a$.*
(ii) *If e is cofixed in M, then e is cofixed in $M \backslash a$.*

Proof. Consider (i). Let $N = M \backslash a$ and assume that e is not fixed in $M \backslash a$. Let N' be a matroid obtained by independently cloning e by e'. Let M' be a matroid obtained from N' by freely placing the point a in the span of $\{b, c\}$. Consider $M' \backslash e'$. We have the following: $M \backslash a = M' \backslash e', a$, the set $\{a, b, c\}$ is a triangle in $M' \backslash e'$, and a is freely placed on the line $\{a, b, c\}$. It follows straightforwardly that $M' \backslash e' = M$.

Assume that e is fixed in M. Then e is fixed in M'. It follows that there is a cyclic flat F of M' containing e but not e'. If F does not contain a, then we contradict the fact that e and e' are clones in N'. Hence F contains a. But as a is freely placed in $\{a, b, c\}$, we see that a is freer than both b and c in M'. Therefore b and c are contained in F. Let C be a circuit such that $e \in C$ and $C \subseteq F$. If $a \in C$, then we may apply circuit elimination to C and $\{a, b, c\}$ to obtain a circuit C' containing e that does not contain a. Thus we may assume that C does not contain a. But then $\mathrm{cl}_{N'}(C)$ is a cyclic flat of N' that contains e but not e' contradicting the fact that $\{e, e'\}$ is a clonal pair in N'.

Part (ii) follows by a similar argument, but this time by coindependently cocloning e by e' in $M \backslash a$. □

As always the connection with flowers is important to us.

Lemma 5.18. *Let M be a 3-connected matroid and let $\{a, b\}$ be a 2-element tight petal of a swirl-like or spike-like flower with at least four petals. If a is not fixed in M, then the following hold.*

(i) $\mathrm{fr}(a) = 2$.
(ii) $b \preccurlyeq a$.
(iii) *If M' is obtained by independently cloning a by a', then $\{a, a', b\}$ is a triangle in M'.*

Proof. Say that flower is $(P_1, \{a, b\}, P_3, \ldots, P_m)$. Let M' be a matroid obtained by independently cloning a by a'. As $\{a, b\}$ is a tight petal, $a \in \mathrm{cl}(P_1 \cup \{b\})$ so that $a' \in \mathrm{cl}(P_1 \cup \{b\})$, that is, $a' \in \mathrm{cl}(P_1 \cup \{a, b\})$ and, similarly, $a' \in \mathrm{cl}(P_3 \cup \{a, b\})$. Thus, by Lemma 3.31, $\{a, a', b\}$ is a triangle in M'. Hence (iii) holds. Parts (i) and (ii) follow easily. □

Recall that the loose elements in a swirl-like flower can be partitioned into guts and coguts elements in a canonical way. The next lemma is perhaps a terminological surprise, in that it is quite easy for a loose element to be fixed.

Lemma 5.19. *Let M be a 3-connected matroid and l be a loose element of a swirl-like or spike-like flower $\mathbf{F}$ of M of order at least 3. Then the following hold.*

(i) *If $\mathbf{F}$ is spike-like and l is the tip, then l is fixed in M.*
(ii) *If $\mathbf{F}$ is spike-like and l is the cotip, then l is cofixed in M.*
(iii) *If $\mathbf{F}$ is swirl-like and l is a loose guts element, then l is fixed in M.*
(iv) *If $\mathbf{F}$ is swirl-like and l is a loose coguts element, then l is cofixed in M.*

Proof. Assume that either (i) or (iii) holds. Then it is clear that M has a swirl-like or spike-like quasi-flower $(P_1, \{l\}, P_2, P_3)$ where $l \in \mathrm{cl}(P_1)$, $l \in \mathrm{cl}(P_2)$, and $|P_3| \geqslant 2$. In this case $\sqcap(P_1, P_2) = 1$ and the lemma follows by Lemma 5.11. □

3. Freedom and *k*-coherence

In this section we collect some lemmas that deal with the relationship between the relative freedom of elements in a matroid and k-coherence.

Lemma 5.20. *Let a and b be elements of the k-coherent matroid M, where $a \preccurlyeq b$. If $M\backslash a$ is not k-coherent, then b is cofixed in $M\backslash a$.*

Proof. Assume that $M\backslash a$ is not k-coherent. Say that $M\backslash a$ is not 3-connected. Then $M\backslash a$ has a 2-separation (X, Y). Assume that $b \notin \mathrm{cl}^*_{M\backslash a}(X - \{b\})$. Then $b \in \mathrm{cl}_{M\backslash a}(Y - \{b\})$, that is, $b \in \mathrm{cl}_M(Y - \{b\})$. As $a \preccurlyeq b$, we have $a \in \mathrm{cl}_M(Y)$, contradicting the fact that a coblocks (X, Y). Therefore $b \in \mathrm{cl}^*_{M\backslash a}(X - \{b\})$ and $b \in \mathrm{cl}^*_{M\backslash a}(Y - \{b\})$ so that b is cofixed in $M\backslash a$ by the dual of Lemma 5.11.

Assume that $M\backslash a$ is 3-connected. Let $(P_1, P_2, P_3, \ldots, P_k)$ be a k-fracture of $M\backslash a$, where $b \in P_2$. If b is a coloop of neither $M|(P_1 \cup P_2)$ nor $M|(P_2 \cup P_3)$, then, by Lemma 5.3, $a \in \mathrm{cl}_M(P_1 \cup P_2)$ and $a \in \mathrm{cl}_M(P_2 \cup P_3)$, so that, by Lemma 3.31, $a \in \mathrm{cl}_M(P_2)$, contradicting the fact that M is k-coherent. Hence we may assume that b is a coloop of $M|(P_1 \cup P_2)$. But this means that b is in the coguts of $(P_1 \cup P_2, P_3 \cup \cdots \cup P_k)$. In this case, by Lemma 3.10, b is a loose coguts element of P_2. By Lemma 5.19 b is cofixed in $M\backslash a$. □

Corollary 5.21. *Let a and b be elements of the k-coherent matroid M, where $a \preccurlyeq b$.*

(i) *If $\mathrm{fr}(a) \geqslant 3$, then M/b is k-coherent.*
(ii) *If $\mathrm{fr}^*(b) \geqslant 3$, then $M\backslash a$ is k-coherent.*

Proof. Consider (ii). Say that $\mathrm{fr}^*(b) \geqslant 3$. Then, by Lemma 5.6, $\mathrm{fr}^*_{M\backslash a}(b) \geqslant 2$ so that b is not cofixed in $M\backslash a$. So by Lemma 5.20 $M\backslash a$ is k-coherent. Part (i) is the dual of (ii). □

The next lemma is easily seen to hold for arbitrary flowers, but we only need it in the swirl-like case.

Lemma 5.22. *Let $\mathbf{F}$ be a swirl-like flower of the 3-connected matroid M of order at least 4. If $a, b \in E(M)$, and $a \preccurlyeq b$, then there is a flower equivalent to $\mathbf{F}$ in which a and b are in the same petal.*

Proof. Say that $(P_1, P_2, \dots, P_n)$ is a tight swirl-like flower equivalent to $\mathbf{F}$ that is chosen so that $a \in P_1$, $b \in P_i$, and, amongst all equivalent flowers, i is minimal. Assume that $i > 1$. By the minimality assumption b is not a loose coguts element between P_{i-1} and P_i. Thus, $b \in \mathrm{cl}((P_i \cup P_{i+1}) - \{b\})$. By Lemma 5.3, $a \in \mathrm{cl}(P_i \cup P_{i+1})$. By Lemma 3.9, either $a \in \mathrm{cl}(P_i)$ or $a \in \mathrm{cl}(P_{i+1})$. The former case implies a contradiction. Thus we may assume that $a \in \mathrm{cl}(P_{i+1})$. Now $a \in P_{i+2}$ and $n \geqslant 4$, so that $i > 2 \bmod n$, and a and b are not in adjacent petals. However putting a in P_{i+1} gives an equivalent flower in which a and b are in adjacent petals and we have contradicted the minimality of the choice of $\mathbf{F}$ and i. $\square$

Knowing that elements of a k-coherent matroid are comparable gives us valuable information that enables us to deduce that certain minors are k-coherent. This point is exemplified in the next lemma. Let $(P_1, P_2, \dots, P_n)$ be a swirl-like flower of a matroid. Recall that, when we say that x is a coguts element of (P_i, P_{i+1}) we mean that x is a rim element of the fan of loose elements between P_i and P_{i+1}.

Lemma 5.23. *Let x and y be elements of the k-coherent matroid M where $x \preccurlyeq y$. Assume that $M\backslash x$ is 3-connected and k-fractured. Then*

(i) *M/y is k-coherent, and*
(ii) *either x and y are clones in M or y is cofixed in M.*

Proof. Let $\mathbf{F} = (P_1 \cup \{y\}, P_2, \dots, P_n)$ be a maximal k-fracture of $M\backslash x$.

5.23.1. *Up to labels y is in the coguts of $(P_1 \cup \{y\}, P_2)$. Moreover $n = k$.*

Subproof. Assume the sublemma does not hold. Then we may assume that y is not in the coguts of $(P_n, P_1 \cup \{y\})$ nor in the coguts of $(P_1 \cup \{y\}, P_2)$. Thus $y \in \mathrm{cl}(P_n \cup P_1)$ and $y \in \mathrm{cl}(P_1 \cup P_2)$. Then, as $x \preccurlyeq y$, we have, by Lemma 5.3, $x \in \mathrm{cl}(P_n \cup P_1)$ and $x \in \mathrm{cl}(P_1 \cup P_2)$ so that $x \in \mathrm{cl}(P_n \cup (P_1 \cup \{y\}))$ and $x \in \mathrm{cl}((P_1 \cup \{y\}) \cup P_2)$. But then, by Lemma 3.31, $x \in \mathrm{cl}(P_1 \cup \{y\})$ implying that M is k-fractured. Thus we may assume that y is in the coguts of $(P_1 \cup \{y\}, P_2)$.

Now $y \in \mathrm{cl}(P_1 \cup P_2)$ so that $x \in \mathrm{cl}(P_1 \cup P_2)$ and hence $(P_1 \cup P_2 \cup \{x, y\}, P_3, \dots, P_n)$ is a tight swirl-like flower in M. As M is k-coherent $n = k$. $\square$

As y is in the coguts of $(P_1 \cup \{y\}, P_2)$ we have $y \in \mathrm{cl}^*_{M\backslash x}(P_1)$ and $y \in \mathrm{cl}^*_{M\backslash x}(P_2)$. Also $y \in \mathrm{cl}(P_1 \cup P_2)$ and, as $y \succcurlyeq x$, we have $x \in \mathrm{cl}(P_1 \cup P_2)$.

5.23.2. *M/y and $M\backslash x/y$ are 3-connected.*

Subproof. Note that $M\backslash x, y$ is not 3-connected up to series pairs (for example, $(P_n \cup P_1, P_2 \cup P_3 \cup \dots \cup P_{n-1})$ is a 2-separation of $M\backslash x, y$). So, by Bixby's Lemma, $M\backslash x/y$ is 3-connected up to parallel pairs. Hence M/y is 3-connected up to parallel classes. To prove the sublemma it suffices to show that M/y has no non-trivial parallel classes. Assume otherwise. Then y is in a triangle of M. As y is freer than x, there is an element t such that $\{x, y, t\}$ is a triangle of M. As $\{x, y\} \subseteq \mathrm{cl}(P_1 \cup P_2)$, we have $t \in \mathrm{cl}(P_1 \cup P_2)$. By taking an equivalent flower if necessary, we may assume that $P_1 \cup P_2 \cup \{y\}$ is closed in $M\backslash x$. Up to symmetry we may assume that $t \in P_1$. But now $x \in \mathrm{cl}(P_1 \cup \{y\})$ and $(P_1 \cup \{x, y\}, P_2, \dots, P_n)$ is a k-fracture of M, contradicting the fact that M is k-coherent. $\square$

The element y is in the coguts of $(P_1 \cup \{y\}, P_2)$ in $M\backslash x$. As $M\backslash x/y$ is 3-connected, y is an initial element of the fan of loose elements between $P_1 \cup \{y\}$ and P_2. It now follows from the dual of Lemma 4.41 that

5.23.3. *$(P_1, P_2, \dots, P_k)$ is a swirl-like flower of order k in $M\backslash x/y$ so that $M\backslash x/y$ is k-fractured.*

We now consider the reverse and show that all k-fractures of $M\backslash x/y$ have the same form. Assume that $\mathbf{Q}$ is a k-fracture of $M\backslash x/y$.

5.23.4. *$\mathbf{Q}$ has order k, and there is a swirl-like flower $(Q_1, Q_2, \ldots, Q_k) \cong \mathbf{Q}$ such that $(Q_1 \cup \{y\}, Q_2, \ldots, Q_k)$ is a k-fracture of $M\backslash x$ and x is in the coguts of (Q_1, Q_2).*

Subproof. By Lemma 3.16, there is a petal Q of $\mathbf{Q}$ such that either $P_1 \subseteq \hat{Q}$ or $P_2 \subseteq \hat{Q}$. But $y \in \mathrm{cl}^*_{M\backslash x}(P_1)$ and $y \in \mathrm{cl}^*_{M\backslash x}(P_2)$, so $y \in \mathrm{cl}^*_{M\backslash x}(\hat{Q})$. Now choose a flower $(Q_1, Q_2, \ldots, Q_m)$ equivalent to $\mathbf{Q}$ such that $Q_1 = \hat{Q}$. Then $(Q_1 \cup \{y\}, Q_2, \ldots, Q_m)$ is a maximal swirl-like flower of $M\backslash x$. By 5.23.1, $m = k$ and we may assume that y is in the coguts of $(Q_1 \cup \{y\}, Q_2)$. $\square$

Assume that part (i) of the lemma fails. Then, as M/y is 3-connected, it must be the case that M/y is k-fractured. As $M\backslash x/y$ is k-fractured, it follows from Lemma 4.45 that there is a tight swirl-like flower $(R_1, R_2, \ldots, R_k)$ of $M\backslash x/y$ such that, for some $i \in \{1, 2, \ldots, k\}$, we have $x \in \mathrm{cl}_{M/y}(R_i)$. By 5.23.4, we may assume that $y \in \mathrm{cl}_M(R_1 \cup R_2)$, and, as $y \succcurlyeq x$, we have $x \in \mathrm{cl}_M(R_1 \cup R_2)$, so that $x \in \mathrm{cl}_{M/y}(R_1 \cup R_2)$. Thus we may assume that $i \in \{1, 2\}$ and, up to labels, we have $x \in \mathrm{cl}_{M/y}(R_1)$. But then, $x \in \mathrm{cl}_M(R_1 \cup \{y\})$ so that $y \in \mathrm{cl}_M(R_1 \cup \{x\})$, and $(R \cup \{x, y\}, R_2, \ldots, R_k)$ is a k-fracture of M, contradicting the fact that M is k-coherent. Thus (i) holds.

Consider part (ii). As y is in the coguts of P_1 and P_2 in $M\backslash x$, by Lemma 5.20, y is cofixed in $M\backslash x$. As $x \preccurlyeq y$, the element x is not cofixed in M. It follows from the dual of Corollary 5.7(ii), that either x and y are clones or y is cofixed in M. $\square$

Lemma 5.23 has some useful consequences. The condition that $\{a, b, c\}$ is not in a 4-element fan required by the next lemma is necessary. Let (p, q, r, s) be a 4-element fan of a 3-connected matroid M. Then it is quite possible for q and r to be comparable—indeed they can even be clones. Yet neither $M\backslash q$ nor M/q is 3-connected.

Lemma 5.24. *Let $\{a, b, c\}$ be a triangle of the 3-connected matroid M that is not contained in a 4-element fan. If a is not fixed, then $M\backslash b$ is 3-connected.*

Proof. Assume that $M\backslash b$ is not 3-connected. Let (A, C) be a 2-separation of $M\backslash b$. As M is 3-connected we cannot have $\{a, c\} \subseteq A$ or $\{a, c\} \subseteq C$, so we may assume that $a \in A$ and $c \in C$. As $\{a, b, c\}$ is not contained in a 4-element fan, a does not belong to a triad so that $|A| \geqslant 3$. If $a \notin \mathrm{cl}(A - \{a\})$, then $a \in \mathrm{cl}^*_{M\backslash b}(C)$, so that $(A - \{a\}, C \cup \{a\})$ is also a 2-separation of $M\backslash b$. But this implies that $(A - \{a\}, C \cup \{a, b\})$ is a 2-separation of M. Thus $a \in \mathrm{cl}(A - \{a\})$. As M is 3-connected, $b \notin \mathrm{cl}(A - \{a\})$. Hence, by Lemma 5.16, a is fixed in M. $\square$

Lemma 5.24 extends to k-coherence.

Corollary 5.25. *Let $\{a, b, c\}$ be a triangle of the k-coherent matroid M that is not contained in a 4-element fan. If a is not fixed in M, then $M\backslash b$ is k-coherent.*

Proof. By Lemma 5.24 $M\backslash b$ is 3-connected. By Lemma 5.16, $b \preccurlyeq a$ and, as a is in a triangle, M/a is not 3-connected and hence not k-coherent. Thus, by Lemma 5.23, $M\backslash b$ is k-coherent. $\square$

Corollary 5.26. *Let x and y be clones of the k-coherent matroid M. If $\{x, y\}$ is not contained in a 4-element fan, then either $M\backslash x$ or M/x is k-coherent.*

Proof. Say that x is in a triangle. Then there is a triangle containing both x and y. By Corollary 5.25, $M\backslash x$ is k-coherent. Dually, if x is in a triad, then M/x is k-coherent.

Assume that x is not in a triangle or a triad. Then, by Bixby's Lemma, either $M\backslash x$ or M/x is 3-connected. Assume the former. If $M\backslash x$ is not k-coherent, then, by Lemma 5.23, M/y is k-coherent. But M/y is isomorphic to M/x and the corollary follows. $\square$

4. Elementary properties of k-skeletons

In this section we give some basic properties of k-skeletons. The first is a more-or-less immediate consequence of the definition.

Lemma 5.27. *The matroid M is a k-skeleton if and only if M^* is.*

A pleasant property of k-skeletons is that they do not have large fans.

Lemma 5.28. *If M is a k-skeleton, then M has no 4-element fans.*

Proof. Say that M is a skeleton and that M has a 4-element fan. Then there is a fan $F = (f_1, f_2, f_3, f_4)$ such that f_1 is an initial element of a maximal fan of M. As M is not a wheel or a whirl, F is exactly 3-separating in M. Up to duality we may assume that $M\backslash f_1$ is 3-connected, so that, by Corollary 4.6, $M\backslash f_1$ is k-coherent. Now $\{f_1, f_2, f_3\}$ is a triangle of M so that $f_1 \in \mathrm{cl}(\{f_2, f_3\})$. Also $f_1 \in \mathrm{cl}(E(M) - F)$, and no other element of $\{f_1, f_2, f_3\}$ is in the closure of this set, so, by Lemma 5.16, f_1 is fixed in M, contradicting the definition of a k-skeleton. □

We can also be quite specific about elements in triangles or triads in k-skeletons.

Lemma 5.29. *Let T be a triangle of the k-skeleton M. Then the following hold.*

(i) *Either T is a clonal triple, or T is a standard or costandard k-wild triangle.*
(ii) *$M\backslash t$ is 3-connected for all $t \in T$.*

Proof. Say $T = \{t_1, t_2, t_3\}$. Assume that T is not a standard or costandard k-wild triangle. By Lemma 5.28, T is not contained in a 4-element fan, so by Theorem 4.16, there is an element $t \in T$ such that $M\backslash t$ is k-coherent. Assume that $t = t_1$. By the definition of a k-skeleton, t_1 is not fixed, so, by Corollary 5.25, both $M\backslash t_2$ and $M\backslash t_3$ are k-coherent. Thus neither t_1, t_2 nor t_3 is fixed. It now follows easily that $\{t_1, t_2, t_3\}$ is a clonal triple. Thus (i) holds.

If T is a clonal triple then it is clear that $M\backslash t$ is 3-connected for all $t \in T$. On the other hand, if T is a standard or costandard k-wild triangle, then it follows from the definition of these structures that $M\backslash t$ is 3-connected for all $t \in T$. Hence (ii) holds. □

We had to make a special case of wheels and whirls and exclude them from the definition of k-skeleton. The reason for this is that wheels and whirls trivially satisfy properties (ii) and (iii) of a k-skeleton as they do not have an element x such that either $M\backslash x$ or M/x is k-coherent. There exists a danger that we may fall into this case in minors hence complicating analyses in proofs. The next lemma shows that this will not be a problem for us.

Lemma 5.30. *Let e be an element of the k-skeleton M of rank at least 3. Then $M\backslash e$ is not a wheel or a whirl.*

Proof. Assume that $M\backslash e$ is a wheel or a whirl. If T is a triangle of $M\backslash e$, then at least two elements of T are fixed in $M\backslash e$ and hence fixed in M. By Lemma 5.28, M has no 4-element fans. This means that e must block every triad of $M\backslash e$. It follows easily from this that if (A, B) is a 3-separation of M, then either A or B is a triangle. Thus M has no non-sequential 3-separations so that a 3-tree for M consists of a single vertex. This means that all elements of M are peripheral. By Lemma 4.38 M has no k-wild triangles. By Lemma 5.29, T is a clonal triple contradicting the fact that T contains fixed elements. □

Quads in k-skeletons are also well behaved.

Lemma 5.31. *Let M be a k-skeleton and let D be a quad of M. If $d \in D$, then d is in neither a triangle nor a triad and both $M\backslash d$ and M/d are k-coherent.*

Proof. Consider a 3-tree T for M. This displays a 3-separation equivalent to D, and it follows routinely that the members of D are peripheral elements of T. By Lemma 4.38, no element of D is in a k-wild triangle or triad. Consider $d \in D$. Assume that d is in a triangle T. Then, as D is both a circuit and a cocircuit, T must have exactly one element t that is not in D. It is easily seen that Lemma 5.16 applies so that t is fixed in M. However, by Lemma 5.29, T is a clonal triple so that no element of T is fixed. This contradiction shows that d is not in a triangle, and dually, d is not in a triad. By Lemma 4.7 both $M\backslash d$ and M/d are k-coherent. □

A flower is *canonical* if it has no loose elements.

Lemma 5.32. *If* **F** *is a tight swirl-like flower of a k-skeleton with at least three petals then* **F** *is canonical.*

Proof. Say **F** is not canonical. Then, by taking an appropriate concatenation, we see that M has a tight swirl-like flower (P_1, P_2, P_3) with loose elements between P_1 and P_2. Let l be the initial element of the fan of loose elements between P_1 and P_2. Up to equivalence of flowers and duality, we may assume that $l \in P_1$ and $l \in \mathrm{cl}(P_2)$. By Lemma 3.13, $M\backslash l$ is 3-connected. As (P_1, P_2, P_3) is tight, $|P_1| > 2$. If $|P_2| = |P_3| = 2$, then $P_2 \cup P_3$ is a quad and l is in a triangle with P_2, contradicting Lemma 5.31. Thus the hypotheses of Lemma 4.13(iii) hold, and, by that lemma, we deduce that l does not expose any 3-separations in $M\backslash l$. Hence $M\backslash l$ is k-coherent. But $l \in \mathrm{cl}(P_2)$ and $l \in \mathrm{cl}(P_1 - \{l\})$. Moreover, $\sqcap(P_1 - \{l\}, P_2) = 1$, so, by Lemma 5.11, l is fixed in M and we have contradicted the definition of k-skeleton. □

Ideally we would like to find an element e in a k-skeleton such that either $M\backslash e$ or M/e is a k-skeleton. This would give us a neat chain theorem for skeletons. Unfortunately, life is not that simple in the world of k-skeletons. In the remaining sections of this chapter we prove theorems that identify situations when it is possible to remove elements to keep a k-skeleton and identify certain annoying structures that make life difficult for us.

5. Comparable elements in *k*-skeletons

Knowing that an element is comparable with another is certainly helpful. This section is devoted to proving the following theorem.

Theorem 5.33. *Let a and b be elements of the k-skeleton M, where $a \preccurlyeq b$. If either*

(i) $a \prec b$, *or*
(ii) $a \cong b$, *and* $\mathrm{fr}^*(a) \geqslant 3$,

then $M\backslash a$ is a k-skeleton.

Proof. Assume that either (i) or (ii) holds. We first show that

5.33.1. *$M\backslash a$ is k-coherent, and, if $a \prec b$, then both $M\backslash a$ and M/b are k-coherent.*

Subproof. If condition (ii) holds this follows from Corollary 5.21. Assume that $a \prec b$. Assume that neither $M\backslash a$ nor M/b is k-coherent. If $M\backslash a$ is 3-connected and k-fractured, then by Lemma 5.23, M/b is k-coherent and b is cofixed in M, contrary to assumption. Thus $M\backslash a$ is not 3-connected, and similarly M/b is not 3-connected.

Assume that b is in a triangle. Then, as $b \succ a$, there is a triangle T containing both b and a. By Lemma 5.29, $M\backslash a$ is 3-connected, contradicting the fact that $M\backslash a$ is not 3-connected. Hence b is not in a triangle. Thus $\mathrm{si}(M/b)$ is not 3-connected. As $a \prec b$ it follows from Corollary 5.15, that $\mathrm{si}(M/a)$ is not 3-connected. But then $\mathrm{co}(M\backslash a)$ is 3-connected, and, as a is not in a triad, we have contradicted the assumption that $M\backslash a$ is not 3-connected.

Thus either $M\backslash a$ or M/b is k-coherent. Assume that M/b is k-coherent. If $M\backslash a$ is not 3-connected, then, by the dual of Corollary 5.15, b is cofixed in M contradicting the fact that M is a skeleton. Thus $M\backslash a$ is 3-connected. If $M\backslash a$ is not k-coherent, then by Lemma 5.23, we again obtain the contradiction that b is cofixed in M. Thus $M\backslash a$ is indeed k-coherent. A dual argument shows that, if $M\backslash a$ is k-coherent, then M/b is k-coherent. Thus, both $M\backslash a$ and M/b are k-coherent. $\square$

5.33.2. *b is not cofixed in $M\backslash a$.*

Proof. As M/b is k-coherent, and M is a k-skeleton, b is not cofixed in M. If condition (ii) of this theorem holds, then $\mathrm{fr}^*_M(b) \geqslant 3$, so $\mathrm{fr}^*_{M\backslash a}(b) \geqslant 2$ by Lemma 5.6. Therefore b is not cofixed in $M\backslash a$. If condition (i) holds, then M/b is k-coherent by 5.33.1, so b is not cofixed in M. If b is cofixed in $M\backslash a$, then Lemma 5.6 implies that $b \preccurlyeq a$, a contradiction. $\square$

We now work towards showing that $M\backslash a$ is a k-skeleton.

5.33.3. *If z is fixed in $M\backslash a$, then $M\backslash a, z$ is not k-coherent.*

Proof. Assume that z is fixed in $M\backslash a$ and that $M\backslash a, z$ is k-coherent. Then z is fixed in M and, as M is a k-skeleton, $M\backslash z$ is not k-coherent. As $M\backslash a, z$ is k-coherent, $M\backslash z$ is 3-connected. Hence $M\backslash z$ is k-fractured. Altogether we have M, $M\backslash a$ and $M\backslash a, z$ are k-coherent and $M\backslash z$ is k-fractured. As z is fixed in M, $z \neq b$. Let $(P_1, P_2, \ldots, P_k)$ be a k-fracture of $M\backslash z$. By Lemma 5.22, we may assume that both a and b are in P_1.

Consider $M\backslash z, a$. By Lemma 4.42(i), $P_1 - \{a\}$ is a set of loose elements of the swirl-like quasi-flower $(P_1 - \{a\}, P_2, P_3, \ldots, P_k)$ of $M\backslash z, a$. By Lemma 5.19, elements of $P_1 - \{a\}$ are either fixed or cofixed in $M\backslash z, a$. But b is not fixed in M and hence not fixed in $M\backslash z, a$. Therefore b is cofixed in $M\backslash z, a$. Now z is fixed in $M\backslash a$, so it is not the case that $z \succcurlyeq a$ in M. Therefore, by Corollary 5.7(ii), b is cofixed in $M\backslash a$, contradicting 5.33.2. $\square$

5.33.4. *If z is cofixed in $M\backslash a$, then $M\backslash a/z$ is not k-coherent.*

Proof. Assume that z is cofixed in $M\backslash a$ and that $M\backslash a/z$ is k-coherent. By 5.33.2, $z \neq b$. Assume that z is not cofixed in M. Then, by Corollary 5.7(ii), $z \preccurlyeq a$ in M, and hence $z \preccurlyeq b$ in M so that $z \preccurlyeq b$ in $M\backslash a$. But then, b is cofixed in $M\backslash a$ contradicting 5.33.2. Hence z is cofixed in M.

We now show that M/z is 3-connected. Assume not. As $M/z\backslash a$ is k-coherent and hence 3-connected, there is an element p of M such that $\{a, z, p\}$ is a triangle. As z is cofixed in M, $z \not\preccurlyeq b$. Hence there is a cyclic flat of M containing b but not z. As $a \preccurlyeq b$, this cyclic flat contains a. But now, by Lemma 5.16, a is fixed in M, and $M\backslash a$ is k-coherent, contradicting the fact that M is a skeleton. Thus M/z is 3-connected.

Since z is cofixed in M and M/z is 3-connected, it must be the case that M/z is k-fractured. By Lemma 5.22, there is a k-flower $(P_1, P_2, \ldots, P_k)$ of M/z such that $\{a, b\} \subseteq P_1$. But $M/z\backslash a$ is k-coherent, so by Lemma 4.42(i), the elements of $P_1 - \{a\}$ are all fixed or cofixed in $M/z\backslash a$, in particular, b is either fixed or cofixed in $M/z\backslash a$. As $a \preccurlyeq b$ in M/z, and M/z is 3-connected, b is not fixed in $M/z\backslash a$. Thus b is cofixed in $M/z\backslash a$. But now, b is cofixed in $M\backslash a$, and we have contradicted 5.33.2. $\square$

By 5.33.1, 5.33.3 and 5.33.4, $M\backslash a$ is a k-skeleton. $\square$

Note that, in the case that (i) holds in Theorem 5.33, that is that $a \prec b$, then it follows by duality that M/b is also a k-skeleton.

As an easy consequence of Theorem 5.33 we have

Corollary 5.34. *Let a and b be comparable elements in the k-skeleton M. If neither $M\backslash a$ nor M/a is a k-skeleton, then $\{a, b\}$ is a clonal class, and* $\mathrm{fr}(a) = \mathrm{fr}^*(a) = 2$.

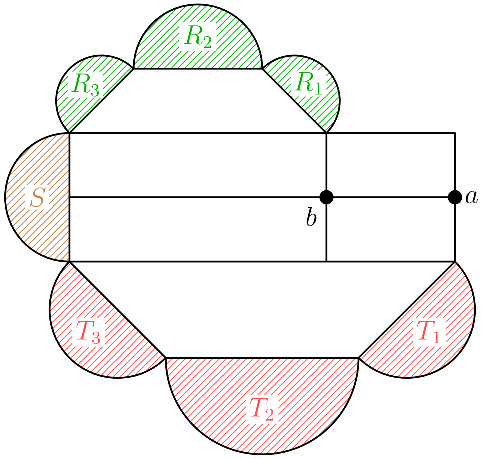

Fig. 5.1. A bogan couple.

6. Bogan couples

In this section we examine one of the problematic structures that turns out to be of importance to us. Recall that a flower is *canonical* if it is the only member of its equivalence class. If f is an element of the k-skeleton M such that $M\backslash f$ is 3-connected but not k-coherent, then $M\backslash f$ is certainly not a k-skeleton. But one might hope that the fact that M is a k-skeleton would impose structure on a k-fracture in $M\backslash f$. In particular it would be useful to know that such a k-fracture is canonical. In this section we show that this is almost always the case unless f is a very particular type of feral element.

Let a, b be elements of the k-coherent matroid M. Then a *bogan display* for $\{a, b\}$ is a partition $(R, S, T, \{a, b\})$ of $E(M)$ together with partitions $(R_1, R_2, \dots, R_{k-2})$, $(S_1, S_2, \dots, S_r)$ and $(T_1, T_2, \dots, T_{k-2})$ of R, S and T respectively such that the following hold.

(i) $(R_1 \cup \{b\}, R_2, \dots, R_{k-2}, S_1, S_2, \dots, S_r, T)$ is a maximal k-fracture of $M\backslash a$, and b is in the coguts of $R_1 \cup \{b\}$ and T.
(ii) $(R, S \cup \{b\}, T_1, T_2, \dots, T_{k-2})$ is a maximal k-fracture of M/a.
(iii) $(R, S_1, S_2, \dots, S_r, T_1, T_2, \dots, T_{k-3}, T_{k-2} \cup \{a\})$ is a maximal k-fracture of $M\backslash b$, and a is in the coguts of $T_{k-2} \cup \{a\}$ and R.
(iv) $(R_1, R_2, \dots, R_{k-2}, S \cup \{a\}, T)$ is a maximal k-fracture of M/b.
(v) $(R_1, R_2, \dots, R_{k-2}, S \cup T \cup \{a, b\})$, $(S_1, S_2, \dots, S_r, R \cup T \cup \{a, b\})$ and $(T_1, T_2, \dots, T_{k-2}, R \cup S \cup \{a, b\})$ are maximal swirl-like flowers in M.
(vi) a and b are cofixed in M.

The pair $\{a, b\}$ is a *bogan couple* if it has a bogan display (see Fig. 5.1). The property of being a bogan couple is not self-dual. If $\{a, b\}$ is a bogan couple in M^*, then $\{a, b\}$ is a *cobogan couple*.

Theorem 5.35. *Let a be an element of the k-skeleton M such that $M\backslash a$ is k-fractured. If b is a loose element of a k-fracture* **P** *of $M\backslash a$, then b is a loose coguts element of* **P** *and either*

(i) *$a \succcurlyeq b$, or*
(ii) *$\{a, b\}$ is a bogan couple of M.*

Proof. Assume that $\mathbf{P} = (P_1 \cup \{b\}, P_2, \dots, P_l)$ is a maximal k-fracture of $M\backslash a$ and that b is in the guts or the coguts of $(P_1 \cup \{b\}, P_2)$.

5.35.1. *b is in the coguts of $(P_1 \cup \{b\}, P_2)$.*

Subproof. Assume that b is in the guts of $(P_1 \cup \{b\}, P_2)$. Then, by Lemma 5.19, b is fixed in $M\backslash a$, so that b is fixed in M. By Lemma 3.13, $M\backslash a, b$ is 3-connected up to series pairs. Consider $M\backslash b$. Assume that $M\backslash b$ is not 3-connected. Then it is either the case that $M\backslash b$ has a series pair, or there is a series pair $\{s_1, s_2\}$ of $M\backslash a, b$ such that $\{s_1, s_2, a\}$ is a triangle of $M\backslash b$. In the latter case $\{s_1, s_2, a, b\}$ is a sequential 3-separating set of M. But $M\backslash a$ is 3-connected, so, by Lemma 4.5, $M\backslash a$ is k-coherent, contradicting the assumption that this matroid is k-fractured.

We now eliminate the former case. Assume that $M\backslash b$ has a series pair. Then b is in a triad $\{p, b, q\}$ of M and $\mathrm{co}(M\backslash b)$ is 3-connected. (Otherwise M has a 4-element fan, contradicting Lemma 5.28.) As b is fixed in M, this triad is not in a clonal triple, so, by Lemma 5.29, it is k-wild. But $\mathrm{co}(M\backslash b)$ is 3-connected, so by Lemma 4.22(iii), it is not standard. We need to show that it is not costandard. By Lemma 3.26 (p, b, q) is a consecutive set of loose elements in the flower $(P_1 \cup \{b\}, P_2, \dots, P_l)$ of $M\backslash a$. But then, $\mathrm{si}(M\backslash a/p, q)$ is 3-connected, as p and q are loose coguts elements of this flower. Thus $\mathrm{si}(M/p, q)$ is 3-connected, contradicting Lemma 4.23.

Therefore $M\backslash b$ is 3-connected. As b is fixed in M, the matroid $M\backslash b$ is k-fractured. By considering a k-fracture of $M\backslash b$, we see that there is a 3-separation $(R \cup \{a\}, G)$ of $M\backslash b$ such that $a \in \mathrm{cl}(R)$, and $(R \cup \{a\}, G)$ is well blocked by b. Now $(R \cup \{b\}, G)$ and $(R, G \cup \{b\})$ are unsplit 4-separating sets of $M\backslash a$. If (R, G) does not cross P_1, then b does not block (R, G), as $b \in \mathrm{cl}(P_1)$. Thus (R, G) crosses P_1, and, similarly, (R, G) crosses P_2.

Now Lemma 3.34 applies and we may assume that, for some $T \in \{R, G\}$, we have $T \cup \{b\} \subseteq P_1 \cup P_2 \cup \{b\}$. By considering the flower $(P_1 \cup \{b\}, P_2, \dots, P_l)$, Lemma 3.36 implies that $\lambda_{M\backslash a}((P_1 \cup \{b\}) \cap T) = 2$ and by considering the flower $(P_1, P_2 \cup \{b\}, \dots, P_l)$, Lemma 3.36 implies that $\lambda_{M\backslash a}(P_1 \cap T) = 2$. Hence $b \in \mathrm{cl}^{(*)}(P_1 \cap T)$. But $b \in \mathrm{cl}(P_2)$, so $b \notin \mathrm{cl}^*(P_1 \cap T)$. Therefore $b \in \mathrm{cl}(P_1 \cap T)$, contradicting the assumption that b blocks (R, G). $\square$

Assume that $a \not\succcurlyeq b$. We need to prove that $\{a, b\}$ is a bogan couple. We proceed by accumulating properties of the pair $\{a, b\}$.

5.35.2. *$M/b\backslash a$ is 3-connected. The element b is cofixed in M and M/b is 3-connected and k-fractured.*

Subproof. As b is a coguts element of $\mathbf{P}$, the element b is cofixed in $M\backslash a$. As $a \not\succcurlyeq b$, it follows from Corollary 5.9 that b is cofixed in M. It follows from 5.35.1 that b is the unique loose element between P_1 and P_2. So, by Lemma 3.13, $M\backslash a/b$ is 3-connected. Thus M/b is 3-connected unless there is a triangle of M containing $\{a, b\}$.

Assume that T is such a triangle. As b is cofixed in M, the triangle T is not a clonal triple and hence is k-wild. As b is in the coguts of $(P_1 \cup \{b\}, P_2)$, the matroid $\mathrm{co}(M\backslash a, b)$ is not 3-connected. It follows from the definition of k-wild triangle that, if T is standard, then $\mathrm{co}(M\backslash a, b)$ is 3-connected. Moreover, by basic properties of the $\Delta - Y$ operation, the same conclusion holds if T is costandard.

It follows from the above contradiction that M/b is 3-connected. As b is cofixed in M, the matroid M/b is k-fractured. $\square$

5.35.3. *We may assume that, for some $i_1, i_2 \in \{1, 2, \dots, l\}$, with $2 \leqslant i_1 \leqslant i_2 \leqslant l$ the partition $(P_1, \dots, P_{i_1 - 1}, P_{i_1} \cup \dots \cup P_{i_2} \cup \{a\}, P_{i_2+1}, \dots, P_l)$ is a maximal k-fracture of M/b.*

Subproof. Let $(Q_1 \cup \{a\}, Q_2, \dots, Q_m)$ be a maximal k-fracture of M/b. By Lemma 4.13, $(P_1, P_2, \dots, P_l)$ is a maximal flower in $M\backslash a/b$.

Consider the partition $(Q_1, Q_2, \dots, Q_m)$ of $E(M\backslash a/b)$. This will be a flower $\mathbf{Q}$ in $M\backslash a/b$ unless $|Q_1| = 1$, in which case it is a quasi-flower. If $\mathbf{Q} \preccurlyeq (P_1, P_2, \dots, P_l)$, then the sublemma clearly holds. Assume otherwise. Let $\mathbf{Q}'$ be a maximal flower that refines $\mathbf{Q}$. Then, by Lemma 3.16, we may assume that there is a pair i, j', such that $E(M\backslash a/b) = P_i \cup Q'_{j'}$. It follows immediately that we may assume that there is a pair i, j such that $E(M\backslash a/b) = P_i \cup Q_j$.

Assume that $j = 1$. Then either $P_1 \subseteq Q_1$ or $P_2 \subseteq Q_1$. We lose no generality in assuming that $P_1 \subseteq Q_1$, so that $P_1 \cup \{a\} \subseteq Q_1 \cup \{a\}$. But $b \in \mathrm{cl}^*_{M\backslash a}(P_1)$. Hence $b \in \mathrm{cl}^*_M(Q_1 \cup \{a\})$ and it follows that $(Q_1 \cup \{a, b\}, Q_2, \dots, Q_l)$ is a k-fracture of M, contradicting the fact that M is k-coherent.

Hence we may assume that $j \neq 1$. Then either $Q_1 \cup Q_2 \subseteq P_i$ or $Q_1 \cup Q_m \subseteq P_i$ and we may assume that the former case holds. As $M/b\backslash a$ is 3-connected, a is not a coguts element of Q_1, so that $a \in \mathrm{cl}_{M/b}(Q_1 \cup Q_2)$, that is, $a \in \mathrm{cl}_{M/b}(P_i)$. Hence $(P_1, P_2, \dots, P_i \cup \{a\}, \dots, P_l)$ fractures M/b and the sublemma holds with $i_1 = i_2 = i$. □

Assume that i_1 and i_2 are chosen so that 5.35.3 is satisfied and $i_2 - i_1$ is maximum.

5.35.4.

(i) *$i_1 \geqslant 3$, $i_2 \leqslant l$.*
(ii) *Either $i_1 > 3$ or $i_2 < l$.*
(iii) *a blocks both $P_1 \cup \{b\}$ and $P_2 \cup \{b\}$.*
(iv) *a blocks either $P_3 \cup P_4 \cup \dots \cup P_{l-1}$ or $P_4 \cup P_5 \cup \dots \cup P_l$.*
(v) *a does not block P_1 or P_2.*

Subproof. Say $i_1 = 2$. Then $a \in \mathrm{cl}_M(P_2 \cup P_3 \cup \dots \cup P_{i_2} \cup \{b\})$, so that $(P_1 \cup P_2 \cup \dots \cup P_{i_2} \cup \{a, b\}, P_{i_2+1}, \dots, P_l)$ is a k-fracture of M. Hence $i_1 \geqslant 3$ and similarly $i_2 \leqslant l$ so that (i) holds. Part (ii) follows from the fact that $(P_1, P_2, \dots, P_{i_1-1}, P_{i_1} \cup \dots \cup P_{i_2} \cup \{a\}, P_{i_2+1}, \dots, P_l)$ is a k-fracture of M/b.

Assume that a does not block $P_1 \cup \{b\}$. Certainly $a \notin \mathrm{cl}(P_1 \cup \{b\})$, so $a \in \mathrm{cl}(E(M) - (P_1 \cup \{a, b\}))$. Then $b \in \mathrm{cl}^*(P_1)$ and $(P_1 \cup \{b\}, P_2, \dots, P_{i_1} \cup \dots \cup P_{i_2} \cup \{a\}, \dots, P_l)$ is a k-fracture of M. Thus a blocks $P_1 \cup \{b\}$ and similarly a blocks $P_2 \cup \{b\}$ and (iii) holds.

Consider (iv). Assume that a blocks neither $P_3 \cup P_4 \cup \dots \cup P_{l-1}$ nor $P_4 \cup P_5 \cup \dots \cup P_l$. As a does not block $P_3 \cup P_4 \cup \dots \cup P_{l-1}$ and a does block $P_1 \cup \{b\}$, we see that $a \notin \mathrm{cl}(P_3 \cup P_4 \cup \dots \cup P_{l-1})$. Hence $a \in \mathrm{cl}(P_l \cup P_1 \cup \{b\} \cup P_2)$. Similarly $a \in \mathrm{cl}(P_1 \cup \{b\} \cup P_2 \cup P_3)$ and it follows from Lemma 3.31 that $a \in \mathrm{cl}(P_1 \cup \{b\} \cup P_2)$. By (ii) and symmetry we may assume that $i_2 < l$. As $a \in \mathrm{cl}(P_{i_1} \cup P_{i_1+1} \cup \dots \cup P_{i_2} \cup \{b\})$, we have $a \in \mathrm{cl}(\{b\} \cup P_2 \cup P_3 \cup \dots \cup P_{i_2})$ so that by Lemma 3.31 $a \in \mathrm{cl}(\{b\} \cup P_2)$, contradicting the fact that a blocks this set. Thus (iv) holds.

Consider (v). As $i_1 \geqslant 3$ and $i_2 \leqslant l$, the set $\mathrm{cl}(P_{i_1} \cup P_{i_1+1} \cup \dots \cup P_{i_2} \cup \{b\})$ avoids P_1 and P_2. As $a \in \mathrm{cl}(P_{i_1} \cup P_{i_1+1} \cup \dots \cup P_{i_2} \cup \{b\})$, neither P_1 nor P_2 is blocked by a. □

5.35.5. *a is cofixed in M.*

Subproof. By 5.35.4(iv), we may assume that a blocks $P_3 \cup P_4 \cup \dots \cup P_{l-1}$. By 5.35.4(iii), a blocks $P_1 \cup \{b\}$. By 5.35.4(iv), $(P_1 \cup \{b\}, P_2, P_3 \cup P_4 \cup \dots \cup P_{l-1}, P_l)$ is a swirl-like flower of $M\backslash a$. It now follows from Lemma 5.13 that a is cofixed in M. □

5.35.6. *M/a is 3-connected.*

Subproof. Assume that a is in a triangle. As $M/b\backslash a$ is 3-connected, this triangle must contain b, contradicting the fact that M/b is 3-connected. Thus a is not in a triangle. Assume that M/a is not 3-connected. As a is not in a triangle, $\mathrm{si}(M/a)$ is not 3-connected, so there is a 3-separation $(R \cup \{a\}, B)$ of M such that (R, B) is a 3-separation of $M\backslash a$ and $a \in \mathrm{cl}(R), \mathrm{cl}(B)$. By Lemma 3.32, (R, B) conforms with the maximal flower $\mathbf{P}$ of $M\backslash a$. If either R or B is contained in a petal of a flower equivalent to $\mathbf{P}$, then, as $a \in \mathrm{cl}(R), \mathrm{cl}(B)$ we obtain the contradiction that M is k-fractured. Thus (R, B) is displayed by $\mathbf{P}$. But then, either a does not block $P_1 \cup \{b\}$ or $P_2 \cup \{b\}$, contradicting 5.35.4(iii). □

As M/a is 3-connected, and a is cofixed in M we see that M/a is k-fractured. Thus a is a feral element of M. Let $\mathbf{Q} = (Q_1, \dots, Q_m)$ be a maximal k-fracture of M/a. We may assume that there is a labelling of $\mathbf{P}$ and $\mathbf{Q}$ that forms a feral display in either M or M^*. In the proof of the next claim we assume that the reader is familiar with terminology from Section 4 associated with feral elements.

5.35.7.

(i) *$m = k$, $i_1 = k$, and $i_2 = l$.*
(ii) *Up to labels $Q_2 = P_2 \cup P_3 \cup \cdots \cup P_{k-1}$, $Q_1 = P_k \cup P_{k+1} \cup \cdots \cup P_l \cup \{b\}$, and $\{Q_3, Q_4, \ldots, Q_m\}$ is a partition of P_1.*
(iii) *$(P_2, P_3, \ldots, P_{k-1}, E(M) - Q_2)$, $(P_k, P_{k+1}, \ldots, P_l, E(M) - (Q_1 - \{b\}))$, and $(Q_3, Q_4, \ldots, Q_k, E(M) - P_1)$ are displayed flowers in M.*

Subproof. Consider the k-fracture $\mathbf{P}$ of $M\backslash a$. By Lemma 4.26(iv), a is either 1-blocking for $\mathbf{P}$ or is 2-spanned by $\mathbf{P}$.

Our first, somewhat painful, task is to show that a is 1-blocking for $\mathbf{P}$. Assume otherwise. Then a is 2-spanned by $\mathbf{P}$. As a is 2-spanned but not 1-blocking, a blocks exactly two adjacent petals of a flower equivalent to $\mathbf{P}$. We also have $l = k$ so that $P_l = P_k$. As a is not 1-blocking for $\mathbf{P}$ we are in the case where we have a feral display for a in M^*. We know that a blocks $P_1 \cup \{b\}$ and, by 5.35.4(v), a does not block P_2. Thus it must be the case that a blocks P_k. Moreover, as a is 2-spanned, $a \in \mathrm{cl}(P_k \cup P_1 \cup \{b\})$. We have committed to a labelling of $\mathbf{P}$, but we are still free to label $\mathbf{Q}$. Given this, we can say that $(Q_1, Q_2, \ldots, Q_m)$ and $(P_k, P_1 \cup \{b\}, P_2, \ldots, P_{k-1})$ form a feral display in M^*. In other words, $(Q_1, Q_2, \ldots, Q_m)$ and $(P_k, P_1 \cup \{b\}, P_2, \ldots, P_{k-1})$ play the roles played by $(P_1, P_2, \ldots, P_m)$ and $(Q_1, Q_2, \ldots, Q_k)$ in the definition of feral display. With this in mind, we see from the properties of a feral display that $P_2 \cup P_3 \cup \cdots \cup P_{k-1} \subseteq Q_1$, and, for some $j \in \{2, 3, \ldots, m-1\}$, we have $Q_{j+1} \cup Q_{j+2} \cup \cdots \cup Q_m \subseteq P_k$ and $Q_2 \cup Q_3 \cup \cdots \cup Q_j \subseteq P_1 \cup \{b\}$.

We next show that $i_1 = i_2 = k$. As $P_{i_1} \cup P_{i_1+1} \cup \cdots \cup P_{i_2} \cup \{a\}$ is 3-separating in M/b but not in M, we have $b \in \mathrm{cl}_M(P_{i_1} \cup P_{i_1+1} \cup \cdots \cup P_{i_2} \cup \{a\})$. But $a \notin \mathrm{cl}_M(P_{i_1} \cup P_{i_1+1} \cup \cdots \cup P_{i_2})$, as otherwise M is k-fractured, so we also have $a \in \mathrm{cl}_M(P_{i_1} \cup P_{i_1+1} \cup \cdots \cup P_{i_2} \cup \{b\})$. It follows from this and the fact that a blocks P_k, that $i_2 = k$. Now $a \in \mathrm{cl}_{M/b}(P_1 \cup P_k)$ and $a \in \mathrm{cl}_{M/b}(P_{i_1} \cup P_{i_1+1} \cup \cdots \cup P_{i_2} \cup P_k)$. So, by Lemma 3.31, $a \in \mathrm{cl}_{M/b}(P_k)$. It now follows from 5.35.3 that $i_1 = k$. Thus we do have $i_1 = i_2 = k$ and we can also conclude that $b \in \mathrm{cl}_M(P_k \cup \{a\})$.

Assume that $b \in Q_1$. By 5.35.4(iii), a blocks $P_2 \cup \{b\}$, so $a \in \mathrm{cl}^*_M(P_2 \cup \{b\})$. But $Q_1 \supseteq P_2 \cup \{b\}$, so $a \in \mathrm{cl}^*_M(Q_1)$. This implies the contradiction that $(Q_1 \cup \{a\}, Q_2, \ldots, Q_m)$ is a k-fracture of M. Thus $b \notin Q_1$, so that $b \in Q_2 \cup Q_3 \cup \cdots \cup Q_j$. Assume that $|Q_2 \cup \cdots \cup Q_j| \geqslant 3$. By property (vii) of a feral display, $Q_2 \cup Q_3 \cup \cdots \cup Q_j$ is 3-separating in M and hence $M\backslash a$. It is now easily seen that b is in the coguts of $Q_2 \cup Q_3 \cup \cdots \cup Q_j$. Thus $b \notin \mathrm{cl}_M(E(M) - (Q_2 \cup Q_3 \cup \cdots \cup Q_j))$, contradicting the fact that $b \in \mathrm{cl}_M(P_k \cup \{a\})$.

We are left with the annoying case where $|Q_2 \cup Q_3 \cup \cdots \cup Q_j| = 2$, so that $j = 2$, $m = k$ and $|Q_2| = 2$. Say $Q_2 = \{b, z\}$. Recall that there is a set Z_2 such that $P_k = Q_3 \cup Q_4 \cup \cdots \cup Q_k \cup Z_2$. We know that $b \in \mathrm{cl}_M(P_k \cup \{a\})$. Thus $\lambda_M(P_k \cup \{a, b\}) \leqslant \lambda_M(P_k \cup \{a\}) = 3$. We now show that $\lambda_M(P_k \cup \{a, b, z\}) \geqslant 3$. Assume otherwise. Then $\lambda_{M\backslash a}(P_k \cup \{b, z\}) = 2$. Consider the flower $(P_1 \cup \{b\}, P_2, \ldots, P_k)$ of $M\backslash a$. The petal $P_1 \cup \{b\}$ is tight and b is loose between P_1 and P_2, so that, by Lemma 3.9(ii), $z \notin \mathrm{fcl}_{M\backslash a}(P_k)$. But now it is clear that $P_k \cup \{b, z\}$ does not have the form of Lemma 3.33. Hence $\lambda_M(P_k \cup \{a, b, z\}) \geqslant 3$. As a coblocks $Q_2 \cup Q_3 \cup \cdots \cup Q_k$, we also deduce that $\lambda_M(\{a, b, z\} \cup Q_3 \cup Q_4 \cup \cdots \cup Q_k) = 3$. We have shown that $\lambda_M(P_k \cup \{a, b\}) \leqslant 3$ and $\lambda_M(\{a, b, z\} \cup Q_3 \cup Q_4 \cup \cdots \cup Q_k) = 3$. The union of these sets is $P_k \cup \{a, b, z\}$, and $\lambda_M(P_k \cup \{a, b, z\}) \geqslant 3$. But the intersection of these sets is $\{a, b\} \cup Q_3 \cup Q_4 \cup \cdots \cup Q_k$. Hence $\lambda_M(\{a, b\} \cup Q_3 \cup Q_4 \cup \cdots \cup Q_k) \leqslant 3$. Now $a \notin \mathrm{cl}^{(*)}(Q_3 \cup Q_4 \cup \cdots \cup Q_k)$, as otherwise one of $M\backslash a$ of M/a is not 3-connected. Hence $\lambda_M(\{a\} \cup Q_3 \cup Q_4 \cup \cdots \cup Q_k) = 3$. Thus $b \in \mathrm{cl}^{(*)}_M(Q_3 \cup Q_4 \cup \cdots \cup Q_k \cup \{a\})$. But $b \in \mathrm{cl}_M(P_1 \cup P_2)$ as b is a loose coguts element between P_1 and P_2 in $\mathbf{P}$. Therefore $b \notin \mathrm{cl}^*_M(Q_3 \cup Q_4 \cup \cdots \cup Q_k \cup \{a\})$. Hence $b \in \mathrm{cl}_M(Q_3 \cup Q_4 \cup \cdots \cup Q_k \cup \{a\})$ so that $b \in \mathrm{cl}_{M/a}(Q_3 \cup Q_4 \cup \cdots \cup Q_k)$ and it follows that $\{z, b\}$ is a loose petal of $\mathbf{Q}$ contradicting the fact that it is not.

The above contradiction shows at last that a is indeed 1-blocking for $\mathbf{P}$ and we consider the structure that arises in this case now. By the symmetry between P_1 and P_2, we may assume that a blocks $P_1 \cup \{b\}$ and no other petal of $\mathbf{P}$. Indeed, we may assume that $m = k$ and that $(P_1 \cup \{b\}, P_2, \ldots, P_l)$ and $(Q_1, Q_2, \ldots, Q_k)$ form a feral display. By the properties of a feral display $P_1 \cup \{b\}$ properly contains $Q_3 \cup Q_4 \cup \cdots \cup Q_k$, and for some $i \in \{2, 3, \ldots, l-1\}$, we have $P_2 \cup P_3 \cup \cdots \cup P_i \subseteq Q_2$, and $P_{i+1} \cup P_{i+2} \cup \cdots \cup P_k \subseteq Q_1$. By 5.35.4(v), P_1 is not blocked by a. Hence P_1 is 3-separating in M/a and it follows easily that P_1 is displayed by $\mathbf{Q}$. But $P_1 \cup \{b\}$ properly contains $Q_3 \cup Q_4 \cup \cdots \cup Q_k$.

Hence $P_1 = Q_3 \cup Q_4 \cup \cdots \cup Q_k$. Also by the properties of a feral display, $\lambda_M(P_2 \cup P_3 \cup \cdots \cup P_i) = 2$ and $\lambda_M(P_{i+1} \cup P_{i+2} \cup \cdots \cup P_l) = 2$. Note that, using the notation of the definition of feral display we have $Z_1 \cup Z_2 = \{b\}$.

Either $b \in Q_1$ or $b \in Q_2$. Assume for a contradiction that $b \in Q_2$, so that $(Q_2, Q_1) = (\{b\} \cup P_2 \cup P_3 \cup \cdots \cup P_i, P_{i+1} \cup P_{i+2} \cup \cdots \cup P_l)$. By a property of feral elements a coblocks either Q_1 or Q_2. But $Q_1 = P_{i+1} \cup P_{i+2} \cup \cdots \cup P_l$ and it was observed above that this set is 3-separating in M. Hence a coblocks Q_2. Therefore $a \in \mathrm{cl}_M(Q_3 \cup Q_4 \cup \cdots \cup Q_k \cup Q_1)$. But $a \notin \mathrm{cl}_M(P_{i+1} \cup P_{i+2} \cup \cdots \cup P_l) = \mathrm{cl}_M(Q_1)$. As a blocks $P_1 \cup \{b\}$, we see that $a \notin \mathrm{cl}_M(P_1)$, and this set contains $Q_3 \cup Q_4 \cup \cdots \cup Q_l$ so that $a \notin \mathrm{cl}_M(Q_3 \cup Q_4 \cup \cdots \cup Q_k)$. Hence, by Lemma 2.10

$$\begin{aligned}\sqcap_{M/a}(Q_1, Q_3 \cup Q_4 \cup \cdots \cup Q_k) &= \sqcap_M(Q_1, Q_3 \cup Q_4 \cup \cdots \cup Q_k) + 1 \\ &\geqslant \sqcap_M(P_l, P_1) + 1 \\ &= 2,\end{aligned}$$

contradicting the fact that $(Q_1, Q_2, Q_3 \cup Q_4 \cup \cdots \cup Q_k)$ is a swirl-like flower of M/a.

It follows that $b \in Q_1$. Thus $Q_2 = P_2 \cup P_3 \cup \cdots \cup P_i$, $Q_1 = P_{i+1} \cup P_{i+2} \cup \cdots \cup P_l \cup \{b\}$, and $P_1 = Q_3 \cup Q_4 \cup \cdots \cup Q_k$. Recall that $(P_1, P_2, \ldots, P_l)$ is a maximal k-fracture of $M/b\backslash a$.

We next show that $i_1 = i + 1$. We first show that $i_1 \geqslant i + 1$. Assume otherwise, so that $i_1 < i + 1$. We have $a \in \mathrm{cl}_{M/b}(P_{i_1} \cup P_{i_1+1} \cup \cdots \cup P_{i_2})$. Also, by a property of feral elements, a coblocks either Q_2 or Q_1. But $Q_2 = P_2 \cup P_2 \cup \cdots \cup P_i$ and this set is 3-separating in M. Hence a coblocks Q_1, so $a \in \mathrm{cl}_{M/b}(P_{i+1} \cup P_{i+2} \cup \cdots \cup P_l)$. Now, by Lemma 3.31, $a \in \mathrm{cl}_{M/b}(P_{i+1} \cup P_{i+2} \cup \cdots \cup P_{i_2})$. Therefore $(P_1, P_2, \ldots, P_{i_1}, \ldots, P_{i+1} \cup P_{i+2} \cup \cdots \cup P_{i_2} \cup \{a\}, \ldots, P_l)$ is a flower of M/b that properly refines $(P_1, P_2, \ldots, P_{i_1} \cup \cdots \cup P_{i_2} \cup \{a\}, \ldots, P_l)$. The former flower is clearly tight contradicting the fact established in 5.35.3 that the latter flower is maximal. Thus $i_1 \geqslant i + 1$.

Assume that $i_1 > i+1$. As a coblocks Q_1, we see that $a \in \mathrm{cl}_M(P_1 \cup P_2 \cup \cdots \cup P_i)$. Thus $a \in \mathrm{cl}_{M/b}(P_1 \cup P_2 \cup \cdots \cup P_i)$. Also $a \in \mathrm{cl}_{M/b}(P_{i_1} \cup P_{i_1+1} \cup \cdots \cup P_l \cup P_1)$. Hence, by Lemma 3.31, $a \in \mathrm{cl}_{M/b}(P_1)$ so that $a \in \mathrm{cl}_M(P_1 \cup \{b\})$ contradicting the fact that a blocks this set. Thus $i_1 = i + 1$.

We now prove that $i_2 = l$. Assume otherwise, so that $i_2 < l$. We have $a \in \mathrm{cl}_{M/b}(P_{i_1} \cup P_{i_1+1} \cup \cdots \cup P_{i_2})$ so that $a \in \mathrm{cl}_{M/b}(P_2 \cup P_3 \cup \cdots \cup P_{i_2})$. Hence $a \in \mathrm{cl}_M(\{b\} \cup P_2 \cup P_3 \cup \cdots \cup P_{i_2})$. Hence $\lambda_M(P_l \cup P_1) = 2$, so that $\lambda_{M/a}(P_l \cup P_1) = 2$. But $P_1 = Q_3 \cup Q_4 \cup \cdots \cup Q_k$ and $P_l \subseteq Q_1$. There are several ways to see that neither P_l nor $Q_1 - P_l$ is a set of loose elements of $\mathbf{Q}$. One way is to observe that both $|P_l| \geqslant 2$ and $|Q_1 - P_l| \geqslant 2$, so that, if either of these sets is loose, one of the elements is a loose coguts element and we have contradicted the dual of 5.35.1. It follows that $P_l \cup P_1$ does not conform with $\mathbf{Q}$ and we have contradicted the fact that this is a maximal flower in M/a. Hence $i_2 = l$.

We have now established that (i) and (ii) hold. Consider (iii). The partition $(P_2, P_3, \ldots, P_l, P_1 \cup \{b\})$ is a flower in $M\backslash a$, and $a \in \mathrm{cl}(Q_1 \cup \{b\})$ so that $(P_2, P_3, \ldots, P_{k-2}, E(M) - Q_2)$ is a flower in M and the first claim of (iii) holds. As a coblocks Q_1, we see that $a \in \mathrm{cl}_M(E(M) - Q_1)$. Hence a does not block $P_{k-1} \cup \cdots \cup P_l$ so that the second claim of (iii) holds. As a blocks P_1, we have $a \in \mathrm{cl}^*(Q_1 \cup Q_2)$. Thus a does not coblock $Q_1 \cup Q_2$ and the third claim of (iii) holds. $\square$

To simplify notation a little, set $Q_1' = P_k \cup P_{k+1} \cup \cdots \cup P_l$, so that $Q_1' = Q_1 - \{b\}$. Summarising some of the information gained so far, we see that the partition $\mathbf{P} = (P_1 \cup \{b\}, P_2, \ldots, P_l)$ is a maximal k-fracture of $M\backslash a$, the partition $\mathbf{Q} = (Q_1' \cup \{b\}, Q_2, Q_3, \ldots, Q_k)$ is a maximal k-fracture of M/a, and the partition $(P_2, P_3, \ldots, P_{k-1}, Q_1' \cup \{a\}, P_1)$ is a maximal k-fracture of M/b. We next prove

5.35.8. *The matroid $M\backslash b$ is 3-connected and $\mathbf{O} = (Q_2, P_k, P_{k+1}, \ldots, P_l, Q_k, Q_{k-1}, \ldots, Q_3 \cup \{a\})$ is a maximal k-fracture of $M\backslash b$. Moreover a is in the coguts of $Q_3 \cup \{a\}$ and Q_2 in $\mathbf{O}$.*

Subproof. We omit the routine verification that $M\backslash b$ is 3-connected. We first show that $\mathbf{O}$ is a flower in $M\backslash b$. To do this, by [20, Lemma 4.11(i)], it suffices to show the union of each pair of consecutive members of the partition $\mathbf{O}$ in the linear order is 3-separating in $M\backslash b$. In other words we need to check all consecutive pairs, but we do not have to check that $(Q_3 \cup \{a\}) \cup Q_2$ is 3-separating.

Consider $Q_2 \cup P_k$. As b is in the coguts of P_1 and P_2 in $M\backslash a$, we have $\lambda_{M\backslash a,b}(P_2 \cup \cdots \cup P_k) = 1$, that is, $\lambda_{M\backslash a,b}(Q_2 \cup Q_3 \cup P_k) = 1$, so that $\lambda_{M\backslash b}(Q_2 \cup P_k) \leqslant 2$ and, as $M\backslash b$ is 3-connected, equality holds here. If $i \in \{k, k+1, \ldots, l-1\}$, then it follows from 5.35.7(iii) that $\lambda_{M\backslash b}(P_i \cup P_{i+1}) = 2$.

Consider $P_l \cup Q_k$. We have $\lambda_{M/a}(Q_1 \cup Q_k) = 2$, so that $\lambda_{M/a\backslash b}(Q_1' \cup Q_k) = 2$. Also $b \in \mathrm{cl}^*_M(Q_2 \cup \{a\})$, so $a \in \mathrm{cl}^*_M(Q_2 \cup \{b\})$, that is, $a \in \mathrm{cl}^*_{M\backslash b}(Q_2)$. Hence $\lambda_{M\backslash b}(Q_1' \cup Q_k) = 2$. We also know that $\lambda_{M\backslash a,b}(P_l \cup P_1) = 1$, so that $\lambda_{M\backslash b}(P_l \cup P_1) = 2$. As $(P_l \cup P_1) \cap (Q_1' \cup Q_k) = P_l \cup Q_k$, an uncrossing argument gives $\lambda_{M\backslash b}(P_l \cup Q_k) = 2$ as required. By 5.35.7(iii), if $i \in \{k, k-1, \ldots, 4\}$, then $\lambda_{M\backslash b}(Q_i \cup Q_{i-1}) = 2$.

It remains to prove that $\lambda_{M\backslash b}(Q_4 \cup Q_3 \cup \{a\}) = 2$. We will use an uncrossing argument in $M\backslash a, b$ to show that $\lambda_{M\backslash a,b}(Q_4 \cup Q_3) \leqslant 1$. We first show that $\lambda_{M\backslash a,b}(Q_2 \cup Q_3 \cup Q_4) \leqslant 2$. We know that $\lambda_{M/a\backslash b}(Q_2 \cup Q_3 \cup Q_4) = 2$, as $b \in \mathrm{cl}_{M/a}(Q_1')$. By 5.35.4(iii), a blocks $P_2 \cup \{b\}$, so $a \notin \mathrm{cl}_M(Q_5 \cup Q_6 \cup \cdots \cup Q_k \cup Q_1')$. Hence a does not coblock the 3-separation $(Q_2 \cup Q_3 \cup Q_4, Q_5 \cup Q_6 \cup \cdots \cup Q_k \cup Q_1')$ of $M/a\backslash b$. It follows from this fact that the above 3-separation is induced in $M\backslash b$ and hence $\lambda_{M\backslash a,b}(Q_2 \cup Q_3 \cup Q_4) \leqslant 2$. As b is in the coguts of P_1 and P_2 in $M\backslash a$, we see that $\lambda_{M\backslash a,b}(P_1) = 1$, in other words, $\lambda_{M\backslash a,b}(Q_3 \cup Q_4 \cup \cdots \cup Q_k) = 1$. Also, $b \in \mathrm{cl}_{M\backslash a}(P_1 \cup P_2)$, and $M\backslash a$ is 3-connected, so $\lambda_{M\backslash a,b}(Q_1') = \lambda_{M\backslash a}(Q_1') = 2$; that is, $\lambda_{M\backslash a,b}(Q_2 \cup Q_3 \cup \cdots \cup Q_k) = 2$. Altogether we have $\lambda_{M\backslash a,b}(Q_2 \cup Q_3 \cup Q_4) \leqslant 2$; $\lambda_{M\backslash a,b}(Q_3 \cup Q_4 \cup \cdots \cup Q_k) = 1$; and $\lambda_{M\backslash a,b}(Q_2 \cup Q_3 \cup \cdots \cup Q_k) = 2$. Uncrossing these separations gives us the desired outcome that $\lambda_{M\backslash a,b}(Q_3 \cup Q_4) \leqslant 1$. It follows from this fact, and the fact that $M\backslash b$ is 3-connected, that $\lambda_{M\backslash b}(Q_3 \cup Q_4 \cup \{a\}) = 2$, as required.

We have proved that **O** is a flower in $M\backslash b$. As **P** is a swirl-like flower in $M\backslash a$, we have $\sqcap(Q_1', Q_2) = \sqcap(P_2 \cup P_3 \cup \cdots \cup P_{k-1}, P_k \cup P_{k+1} \cup \cdots \cup P_l) = 1$. Thus **O** is a swirl-like or spike-like. As $a \notin \mathrm{cl}(P_1)$, and $a \in \mathrm{cl}(Q_1 \cup Q_2)$, we see that $\sqcap_{M/a}(Q_3, Q_k) = \sqcap_M(Q_3, Q_k)$. But $\sqcap_{M/a}(Q_3, Q_k) = 0$ and it follows that **O** is swirl-like. □

Relabel $(P_2, P_3, \ldots, P_{k-1})$ by $(R_1, R_2, \ldots, R_{k-2})$, relabel $(P_k, P_{k+1}, \ldots, P_l)$ by $(S_1, S_2, \ldots, S_r)$, and relabel $(Q_3, Q_4, \ldots, Q_k)$ by $(T_1, T_2, \ldots, T_{k-2})$. It is now easily checked that these partitions form a bogan display for $\{a, b\}$, so that $\{a, b\}$ is indeed a bogan couple. □

7. Gangs of three

We know that if x is an element of the skeleton M that is comparable with some other element of M, then either $M\backslash x$ or M/x is a k-skeleton unless x belongs to a clonal pair. What if x is not comparable to another element and M/x is k-coherent? In this situation it will almost always be the case that M/x is a k-skeleton. But there is one exceptional structure for which this is not true and we describe this structure now. Recall that a 3-connected matroid M is *uniquely* k-fractured it there is a bloom **F** of M such that every k-fracture of M is displayed by this bloom.

Let M be a k-coherent matroid and $\{r, s, t\} \subseteq E(M)$. Then $\{r, s, t\}$ is a *gang of three* in M if there is a partition $(R, S, T, Z, \{r, s, t\})$ of $E(M)$, and partitions $(R_2, R_3, \ldots, R_{k-1})$, $(S_2, S_3, \ldots, S_{k-1})$ and $(T_2, T_3, \ldots, T_{k-1})$, of R, S and T respectively such that the following hold.

(i) M/r, M/s and M/t are k-coherent.
(ii) $M\backslash r$, $M\backslash s$ and $M\backslash t$ are 3-connected.
(iii) $(R_2, R_3, \ldots, R_{k-1}, E(M) - R)$, $(S_2, S_3, \ldots, S_{k-1}, E(M) - S)$ and $(T_2, T_3, \ldots, T_{k-1}, E(M) - T)$ are tight swirl-like flowers in M.
(iv) $(\{s, t\}, R_2, R_3, \ldots, R_{k-1}, S \cup T \cup Z)$, $(\{r, t\}, S_2, S_3, \ldots, S_{k-1}, R \cup T \cup Z)$ and $(\{r, s\}, T_2, T_3, \ldots, T_{k-1}, R \cup S \cup Z)$ are canonical maximal k-fractures of $M\backslash r$, $M\backslash s$ and $M\backslash t$ respectively. Moreover these k-fractures are unique.
(v) r, s and t are fixed in M.

Fig. 5.2 illustrates a gang of three with notation as in the definition. Note that gangs of three are not self-dual. A gang of three in M^* is a *cogang of three*. The goal of this section is to prove

Theorem 5.36. *Let x be an element of the k-skeleton M such that M/x is k-coherent and x is not comparable with any other element of M. Then either M/x is a k-skeleton, or x is a member of a gang of three in M.*

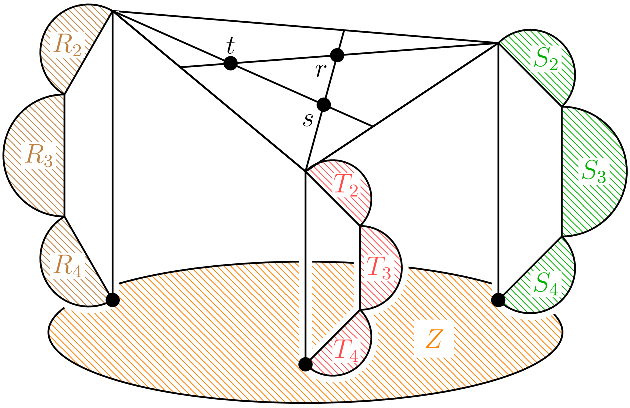

Fig. 5.2. A gang of three.

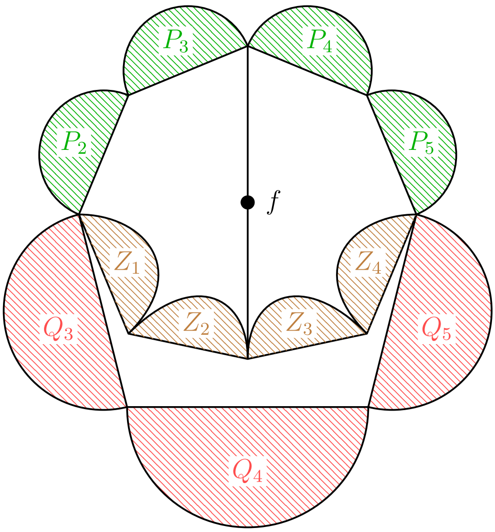

Fig. 5.3. $M\backslash f$ is not uniquely fractured.

We first prove some preliminary lemmas. One might expect that, if f is feral, then $M\backslash f$ would be uniquely k-fractured. Perhaps surprisingly this is not always true; a feral element for which $M\backslash f$ is not uniquely k-fractured is illustrated in Fig. 5.3. But it is true for members of bogan couples.

Lemma 5.37. *If M is k-coherent and a is a member of a bogan couple in M, then $M\backslash a$ and M/a are uniquely k-fractured.*

Proof. Say that $\{a, b\}$ is a bogan couple of M. Then it has an associated bogan display. In what follows we use the notation for a bogan display given in the definition. We know that $(R_1, R_2, \dots, R_{k-2}, S, T \cup \{b\})$ is a k-fracture of $M\backslash a$. Say $(P_1, P_2, \dots, P_l)$ is another k-fracture. By Lemma 3.16, we may assume that $P_1 \cup P_2 \cup \dots \cup P_{l-1}$ is contained in the full closure of a petal of $(R_1, R_2, \dots, R_{k-2}, S, T \cup \{b\})$. This petal must be blocked by a, otherwise M is k-fractured. By property (v) of bogan displays the only possibilities are that $P_1 \cup P_2 \cup \dots \cup P_{l-1} \subseteq R_1 \cup \{b\}$ or that $P_1 \cup P_2 \cup \dots \cup P_{l-1} \subseteq T \cup \{b\}$. But, in either case we see that $b \in \mathrm{cl}^*(P_l)$ so that we may assume that $P_1 \cup P_2 \cup \dots \cup P_{l-1}$ is contained in T or R_1. As neither T nor R_1 is blocked by a we again get the contradiction that M is k-fractured. Thus $M\backslash a$ is uniquely k-fractured. The argument for M/a is similar and is omitted. $\square$

The next lemma shows that, while an element can be loose in more than one flower of a matroid, this can only happen in a very restricted way. The first part of the lemma is true for any type of flowers, but would take slightly longer to prove. Recall that, if Q_i is a petal of a swirl-like flower $\mathbf{P}$, then Q_i^+ denotes the ordered sequences of loose elements between Q_i and Q_{i+1} and Q_i^- denotes the ordered sequence of loose elements between Q_{i-1} and Q_i.

Lemma 5.38. *Let $\mathbf{P}$ and $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_n)$ be inequivalent tight maximal swirl-like flowers of the 3-connected matroid M of order at least three. Assume that the petal P of $\mathbf{P}$ is contained in $\hat{Q}_i$ for some $i \in \{1, 2, \ldots, n\}$. Then P contains at most one element z of Q_i^+. In the case that P contains such an element z, then z is the first element of Q_i^+.*

Proof. By Lemma 3.16, $\hat{Q}_i$ contains all but one petal of $\mathbf{P}$. It follows that P is not displayed by $\mathbf{Q}$. The lemma now follows from Lemma 3.33. $\square$

The following lemma is an easy consequence of the structure of k-wild triangles.

Lemma 5.39. *Let T be a k-wild triangle of the k-coherent matroid M, let z_1, z_2 be elements of $E(M) - T$. Assume that $N \in \{M\backslash z_1, z_2, M\backslash z_1/z_2\}$ and that N is 3-connected. Then T is not a set of loose elements of a swirl-like flower of N.*

The next two lemmas provide the bulk of the proof of Theorem 5.36. Lemma 5.40 below is the more straightforward. Note that we use the *dual* of Lemma 5.40 for Theorem 5.36.

Lemma 5.40. *Let x be an element of the k-skeleton M such that x is not comparable with any other element and $M\backslash x$ is k-coherent. Then there is no element $y \in E(M\backslash x)$ such that y is fixed in $M\backslash x$ and $M\backslash x, y$ is k-coherent.*

Proof. Assume that y is fixed in $M\backslash x$ and that $M\backslash x, y$ is k-coherent. Then $M\backslash y$ is 3-connected. The element y is fixed in M so that $M\backslash y$ is k-fractured. Let $\mathbf{F} = (P_1, P_2, \ldots, P_m)$ be a maximal k-fracture of $M\backslash y$, where $x \in P_1$. By Lemma 4.42, $m = k$, and $(P_1 - \{x\}, P_2, \ldots, P_k)$ is a swirl-like quasi-flower of $M\backslash x, y$ and P_1 is a set of loose elements of this quasi-flower.

5.40.1. *P_1 contains no triangles. Moreover x belongs to no triangles.*

Subproof. Assume that T is a triangle in P_1. Say $x \notin T$. By Lemma 5.19, T contains an element that is fixed in $M\backslash x, y$ and consequently fixed in M. But, by Lemma 5.39, T is not a k-wild triangle of M. But then, by Lemma 5.29, T is a clonal triple in M, and does not contain an element that is fixed in M. On the other hand, if $x \in T$, then, as x is not fixed in M, we see, again by Lemma 5.29, that the triangle T is a clonal triple, contradicting the fact that x is not comparable with any other element of M. $\square$

By 5.40.1, and the fact that $P_1 - \{x\}$ is a fan in $M\backslash x, y$, we have $|P_1| \leqslant 4$.

5.40.2. $|P_1| \neq 2$.

Subproof. Assume that $|P_1| = 2$, say $P_1 = \{x, z\}$. As x is not fixed in M, there is a matroid M' obtained by independently cloning x by x'. By Lemma 5.18, $\{x, x', z\}$ is a triangle in $M'\backslash y$ and hence in M'. This proves that $z \preccurlyeq x$ in M, contradicting the fact that x is not comparable with any other element of M. $\square$

5.40.3. $|P_1| \neq 3$.

Subproof. Say $|P_1| = 3$. Then, as P_1 is 3-separating in $M\backslash y$ and $M\backslash x, y$ is 3-connected, P_1 is a triangle of $M\backslash y$ and hence of M, contradicting 5.40.1. □

5.40.4. $|P_1| \neq 4$.

Subproof. Assume that $|P_1| = 4$. By 5.40.1 $P_1 - \{x\}$ is a triad of $M\backslash x, y$. Say $P_1 - \{x\} = \{a, b, c\}$, where (a, b, c) is a fan between P_k and P_2 in $M\backslash x, y$. By Lemma 5.19, a and c are cofixed in $M\backslash x, y$. As y is fixed in $M\backslash x$, we see, by Corollary 5.8(ii), that a and c are cofixed in $M\backslash x$. But x is not comparable with any element of M, so, again by Corollary 5.7(ii), we conclude that a and c are cofixed in M.

Say $p \in \{a, c\}$. By Lemma 4.41, $M\backslash x, y/p$ is k-coherent. By 5.40.1, x is not in a triangle of $M\backslash y$, so $M\backslash y/p$ is 3-connected. Now, by Lemma 4.42, either $M\backslash y/p$ is uniquely fractured by $(P_1 - \{p\}, P_2, \dots, P_k)$, or $M\backslash y/p$ is k-coherent and $P_1 - \{p\}$ is a loose petal of $(P_1 - \{p\}, P_2, \dots, P_k)$. It is conceivable that p is in a triangle T of M. Assume that this is the case. As p is cofixed in M, the triangle T cannot be a clonal triple, so, by Lemma 5.29, T must be k-wild. Readers who find it obvious that T cannot be k-wild should skip the remainder of this paragraph. Note that T meets P_1 in $\{p\}$ as otherwise P_1 contains a triangle. Observe that $p \in \mathrm{cl}_M(P_1 - \{p\})$; that is, there is a 3-element subset Z of $E(M) - T$ such that $p \in \mathrm{cl}_M(Z)$. It is easily checked from properties of a k-wild display that this is only possible if T is a standard k-wild triangle. In this case $\mathrm{si}(M/p)$ is not 3-connected by Lemma 4.22(iii). But $M\backslash y/p$ is 3-connected, so $\mathrm{si}(M/p)$ is 3-connected. Hence p is not in a triangle.

It follows that M/p is 3-connected. As p is cofixed in M, M/p is not k-coherent. It now follows from either Lemma 4.42 (in the case that $M\backslash y/p$ is k-coherent) or Corollary 4.46 (in the case that $M\backslash y/p$ is uniquely k-fractured) that there is an i and j in $\{1, 2, \dots, k\}$ such that $y \in \mathrm{cl}_{M/a}(\hat{P}_i)$ and $y \in \mathrm{cl}_{M/c}(\hat{P}_j)$. Hence $y \in \mathrm{cl}_M(\{a\} \cup \hat{P}_i)$ and $y \in \mathrm{cl}_M(\{c\} \cup \hat{P}_j)$. If $i = 1$, then $y \in \mathrm{cl}_M(P_1)$ and M is k-fractured. Hence $i, j \in \{2, 3, \dots, k\}$.

We now show that $2 \in \{i, j\}$. Assume otherwise. Let $P'_3 = P_3 \cup P_4 \cup \dots \cup P_k$. Then $y \in \mathrm{cl}(\{a\} \cup P'_3)$, and $y \in \mathrm{cl}(\{c\} \cup P'_3)$. If $y \in \mathrm{cl}(P'_3)$, then, by Lemma 3.31, $y \in \mathrm{cl}(P_i)$, contradicting the fact that M is k-coherent. Hence $c \in \mathrm{cl}(P'_3 \cup \{y\})$ and $a \in \mathrm{cl}(P'_3 \cup \{y\})$, so that $c \in \mathrm{cl}(P'_3 \cup \{a\})$. Now consider the swirl-like flower $(\{a, b, c\}, P_2, \dots, P_k)$ of $M\backslash x, y$. Here (a, b, c) is a fan between P_k and P_2 in $M\backslash x, y$, and c is a rim element of this fan. This implies that $b \in \mathrm{cl}_{M\backslash x,y}(P_2 \cup \{c\})$, so that $c \in \mathrm{cl}_M(P_2 \cup \{b\})$. Hence $c \notin \mathrm{cl}^*_M(\{a\} \cup P'_3)$. From this contradiction it follows that $2 \in \{i, j\}$, and, similarly, $k \in \{i, j\}$. But now, $y \in \mathrm{cl}(P_1 \cup P_2)$ and $y \in \mathrm{cl}(P_1 \cup P_k)$, so, by Lemma 3.31, $y \in \mathrm{cl}(P_1)$ implying that M is k-fractured. □

The lemma now follows from 5.40.2, 5.40.3 and 5.40.4. □

We now come to the more substantial lemma.

Lemma 5.41. *Let x be an element of the k-skeleton M such that x is not comparable with any other element of M and M/x is k-coherent. If there is an element $y \in E(M/x)$ such that y is fixed in M/x and $M/x\backslash y$ is k-coherent, then there is exactly one more such element z and $\{x, y, z\}$ is a gang of three in M.*

Proof. Assume that M/x is k-coherent, that y is fixed in M/x, and that $M/x\backslash y$ is k-coherent.

5.41.1. *y is fixed in M, and $M\backslash y$ is 3-connected and k-fractured.*

Subproof. As x is not comparable with any other element of M, we see, by Corollary 5.7(i), that y is fixed in M. Thus $M\backslash y$ is not k-coherent. If $M\backslash y$ is not 3-connected, then, as $M\backslash y/x$ is 3-connected, y is in a triad with x. But x is not cofixed in M, so, by Corollary 5.25 the triad is not k-wild, and x is not comparable with any other element of M, so the triad cannot be a clonal triple. It now follows by Lemma 5.29 that y is not in a triad so that $M\backslash y$ is 3-connected and k-fractured. □

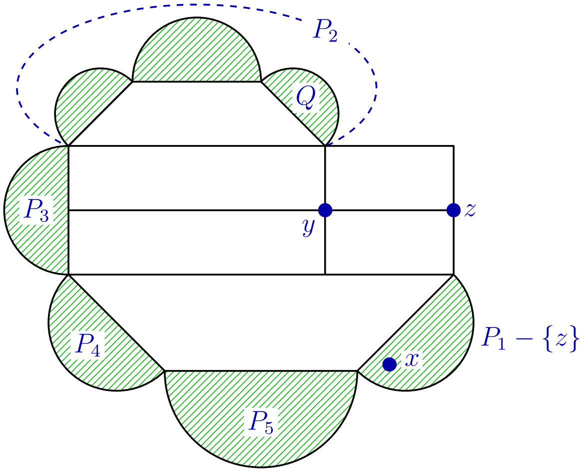

Fig. 5.4. The labelling for the bogan couple of 5.41.2.

Let $\mathbf{P} = (P_1, P_2, \ldots, P_m)$ be a k-fracture of $M\backslash y$ where $x \in P_1$ and P_1 is fully closed. As $M\backslash y/x$ is k-coherent, it follows from Lemma 4.42 that $m = k$ and that $(P_1, P_2, \ldots, P_k)$ uniquely fractures $M\backslash y$. Also, as $M\backslash y/x$ is k-coherent, $P_1 - \{x\}$ is a fan of loose elements in the quasi-flower $(P_1 - \{x\}, P_2, \ldots, P_k)$ of $M\backslash y/x$.

5.41.2.

(i) $\mathbf{P}$ *has no loose elements in* P_1.
(ii) *The fan* $P_1 - \{x\}$ *in* $M\backslash y/x$ *begins and ends with spoke elements.*
(iii) P_1 *has an even number of elements.*

Subproof. If P_1 has a loose element z, then, by Theorem 5.35, either $y \succcurlyeq z$ or $\{y, z\}$ is a bogan couple. The former case contradicts 5.41.1. Hence $\{y, z\}$ is a bogan couple. Again, by Theorem 5.35, z is a loose coguts element of P_1. It follows from the properties of a bogan display that y does not block $P_1 - \{z\}$. As M/x is k-coherent, $z \neq x$.

Consider a bogan display for $\{z, y\}$. By the properties of a bogan display, either $P_1 - \{z\}$ or P_2 contains $k - 2$ petals of a k-fracture for $M\backslash z$. As $P_1 - \{x\}$ is a fan in $M\backslash y/x$, the latter case must hold. An illustration of the bogan display that we now have is given for the case $k = 5$ in Fig. 5.4. By considering the k-fracture of $M\backslash z$ displayed by the bogan display, and the properties of bogan displays as given in the definition, we observe that there is a subset Q of P_2 such that $y \in \mathrm{cl}(P_4 \cup P_5 \cup \cdots \cup P_k \cup P_1 \cup Q)$ and $z \notin \mathrm{cl}(P_4 \cup P_5 \cup \cdots \cup P_k \cup P_1 \cup Q)$. Therefore $y \in \mathrm{cl}_{M/x}(P_4 \cup P_5 \cup \cdots \cup P_k \cup (P_1 - \{x\}) \cup Q)$ and $z \notin \mathrm{cl}_{M/x}(P_4 \cup \cdots \cup P_k \cup (P_1 - \{x\}) \cup Q)$, so $y \not\succcurlyeq z$ in M/x. Moreover z is clearly cofixed in $M/x\backslash y$, and now, by Corollary 5.7, z is cofixed in M/x.

By Lemma 5.37, $(P_1 - \{z\}, P_2, P_3 \cup \{y\}, \ldots, P_k)$ uniquely fractures M/z. Also $P_1 - \{z, x\}$ is a set of loose elements in the quasi-flower $(P_1 - \{x, z\}, P_2, P_3 \cup \{y\}, \ldots, P_k)$ of $M/z, x$. Hence $M/z, x$ is k-coherent. As z is cofixed in M/x, we have contradicted the dual of Lemma 5.40. It follows that P_1 has no loose elements and (i) holds.

Consider (ii). We know that $P_1 - \{x\}$ is a fan between P_k and P_2 in $M\backslash y/x$. If an end of this fan, say z, is a rim element, then $z \in \mathrm{cl}^*_{M\backslash y/x}(P_i)$ for some $i \in \{k, 2\}$. Hence $z \in \mathrm{cl}^*_{M\backslash y}(P_2)$, contradicting (i). Thus (ii) holds. Part (iii) follows immediately. $\square$

By 5.41.2(i), we may assume that P_2 and P_k are fully closed.

5.41.3.

(i) *P_1 contains no triangles in M.*
(ii) *If T is a triad of $P_1 - \{x\}$ in $M\backslash y/x$, then T is blocked by y.*

Subproof. Assume that (i) fails. Let S be a triangle of M contained in P_1. Evidently $x \notin \mathrm{cl}_M(S)$, so S is a triangle of $M\backslash y/x$. By Lemma 5.18 S contains an element s that is fixed in $M\backslash y/x$. Then s is fixed in M/x and, by Corollary 5.9, s is fixed in M. Thus S is not a clonal triple in M. It is straightforwardly verified that S is a peripheral set in M. Hence S is neither a standard nor costandard k-wild triangle. We have thus contradicted the assumption that M is a k-skeleton.

The proof of (ii) is similarly routine and is omitted. □

Consider the fan $P_1 - \{x\}$ between P_k and P_2 in $M\backslash y/x$. Denote this fan by $(z_k, f_1, f_2, \ldots, f_l, z_2)$. If $|P_2| = 2$, then $z_k = z_2$ and $\{f_1, f_2, \ldots, f_l\} = \emptyset$. Otherwise, by 5.41.2(iii), l is odd. By 5.41.2(ii), $z_k \in \mathrm{cl}_{M\backslash y/x}(P_k)$ and $z_2 \in \mathrm{cl}_{M\backslash y/x}(P_2)$.

5.41.4. *$z_k \notin \mathrm{cl}_{M\backslash y}(P_k)$ and $z_2 \notin \mathrm{cl}_{M\backslash y}(P_2)$.*

Subproof. If the sublemma fails, then either z_k or z_2 is a loose element of P_1 contradicting 5.41.2(i). □

5.41.5. *Both z_k and z_2 are fixed in M.*

Subproof. Say $i \in \{k, 2\}$. As z_i is a spoke element of a fan between P_k and P_2 in $M\backslash y/x$, we see, by Lemma 5.19(iii), that z_i is fixed in $M\backslash y/x$. Hence z_i is fixed in M/x. By hypothesis, x is not comparable with any other element of M, so by Corollary 5.9, z_i is fixed in M. □

Our next task is to reduce to the case that $|P_1| = 2$. Until further notice assume that $|P_1| > 2$.

5.41.6. *$\{z_k, z_2, x\}$ is not a triad of $M\backslash y$.*

Subproof. Assume that $\{z_k, z_2, x\}$ is a triad of $M\backslash y$. We have $z_2 \in \mathrm{cl}_{M\backslash y/x}(P_2)$ and by 5.41.4, $z_2 \notin \mathrm{cl}_{M\backslash y}(P_2)$, so $x \in \mathrm{cl}_{M\backslash y}(P_2 \cup \{z_2\})$. Say $z_k \in \mathrm{cl}_{M\backslash y}(P_2 \cup \{z_2\})$. Then, as $\sqcap(P_2, \{z_k, z_2, x\}) \leqslant \sqcap(P_2, P_1) = 1$, an easy rank calculation shows that $\{z_k, z_2, x\}$ is a triangle in M, contradicting 5.41.3(i). Thus $z_k \notin \mathrm{cl}_{M\backslash y}(P_2 \cup \{z_2\})$ and also $z_2 \notin \mathrm{cl}_{M\backslash y}(P_2 \cup \{z_k\})$.

Next we show that z_2 and z_k are cofixed in M. As $x \in \mathrm{cl}_{M\backslash y}(P_2 \cup \{z_2\})$, but $z_k \notin \mathrm{cl}_{M\backslash y}(P_2 \cup \{z_2\})$, there is a cyclic flat of $M\backslash y$ that contains x but not z_k. Hence there is a cyclic flat of $(M\backslash y)^*$ that contains z_k but not x. But $\{z_k, z_2, x\}$ is a triangle of $(M\backslash y)^*$, so by Lemma 5.16, z_k is fixed in $(M\backslash y)^*$. Thus z_k, and similarly z_2, are cofixed in $M\backslash y$. Since y is fixed in M, Corollary 5.7 implies that z_k and z_2 are cofixed in M.

We now show that M/z_k is 3-connected. We know that z_k and z_2 are spoke ends of a fan between P_k and P_2 in $M\backslash y/x$. Moreover P_1 is fully closed in $M\backslash y$ so that $P_1 - \{x\}$ is fully closed in $M\backslash y/x$. Hence the fan is maximal so that $M\backslash y/x\backslash z_k, z_2$ is 3-connected. As $\{x, z_k, z_2\}$ is a triad of $M\backslash y$, we see that $M\backslash y/x\backslash z_k, z_2 = M\backslash y/z_k\backslash x, z_2$. Hence $M\backslash y/z_k\backslash x, y_1, z_2$ is 3-connected and M/z_k is 3-connected up to parallel classes. If M/z_k is not 3-connected, then z_k is in a triangle T of M. By 5.41.3(i), and the fact that P_1 is fully closed in $M\backslash y$, the triangle T contains y. But y is fixed in M, so T is k-wild. As $\mathrm{si}(M/z_k)$ is 3-connected, T is not standard. Assume that T is costandard. By Lemma 4.23, $\mathrm{co}(M\backslash y, z_k)$ is not 3-connected. But $M\backslash y, z_k/x$ is 3-connected, so $\mathrm{co}(M\backslash y, z_k)$ is 3-connected and T is not costandard. Hence M/z_k, and similarly M/z_2, is 3-connected. Since z_k and z_2 are cofixed in M, and M/z_k and M/z_2 are 3-connected, we deduce that both M/z_k and M/z_2 are k-fractured.

Consider the flower $(P_1 - \{z_k\}, P_2, \ldots, P_k)$ of $M\backslash y/z_k$. We now show that $P_1 - \{z_k\}$ is not a loose petal of this flower. Assume that it is a loose petal. Then there is an element $r \in P_1 - \{z_k\}$ such that $r \in \mathrm{cl}^{(*)}_{M\backslash y/z_k}(P_2)$. Assume that $r \in \mathrm{cl}_{M\backslash y/z_k}(P_2)$. Then $r \in \mathrm{cl}_{M\backslash y}(P_2 \cup \{z_k\})$. As $\{z_k, z_2, x\}$ is a triad of $M\backslash y$, we see that $r \in \{z_2, x\}$. If $r = x$, then $z_k \in \mathrm{cl}_{M\backslash y/x}(P_2)$, so $r \neq x$. Assume that $r = z_2$. Then $z_2 \in \mathrm{cl}_{M\backslash y}(P_2 \cup \{z_k\})$, and hence $z_k \in \mathrm{cl}_{M\backslash y}(P_2 \cup \{z_2\})$. But also, $x \in \mathrm{cl}_{M\backslash y}(P_2 \cup \{z_2\})$ and we have already seen that this situation does not occur. Thus $r \notin \mathrm{cl}_{M\backslash y/z_k}(P_2)$. Then $r \in \mathrm{cl}^*_{M\backslash y/z_k}(P_2)$ so that $r \in \mathrm{cl}^*_{M\backslash y}(P_2)$ contradicting 5.41.2(i). Therefore $P_1 - \{z_k\}$ is not a loose petal of $(P_1 - \{z_k\}, P_2, \dots, P_k)$.

Assume that the flower $(P_1 - \{z_k\}, P_2, \dots, P_k)$ of $M\backslash y/z_k$ is not maximal. In this case it is routinely seen that there is a partition (P', P'') of $P_1 - \{z_k\}$, such that $(P', P'', P_2, \dots, P_k)$ is a tight swirl-like flower in $M\backslash y/z_k$. As $z_k \in \mathrm{cl}_{M\backslash y/x}(P_k)$ and $z_k \notin \mathrm{cl}_{M\backslash y}(P_k)$, we know that $x \in \mathrm{cl}_{M\backslash y/z_k}(P_k)$. Say $z_2 \in P'$. Then, as $\sqcap_{M\backslash y/z_k}(P_k \cup P', P_2) = 0$, we see that $z_2 \notin \mathrm{cl}_{M\backslash y/z_k,x}(P_2)$, contradicting the fact that $z_2 \in \mathrm{cl}_{M\backslash y/x}(P_2)$. Say $z_2 \in P''$. Then, as $\sqcap(P_k \cup \{x\}, P'' \cup P_2) = 0$, and $z_2 \notin \mathrm{cl}_{M\backslash y/z_k}(P_2)$, we also have $z_2 \notin \mathrm{cl}_{M\backslash y/z_k,x}(P_2)$, and again we contradict the fact that $z_2 \in \mathrm{cl}_{M\backslash y/x}(P_2)$.

Therefore $(P_1 - \{z_k\}, P_2, \dots, P_k)$ is a maximal k-fracture of $M\backslash y/z_k$ and it is easily seen that this fracture is unique. Similarly $(P_1 - \{z_2\}, P_2, \dots, P_k)$ is the unique maximal k-fracture of $M\backslash y/z_2$. In what follows we discuss related flowers in different matroids. By $\hat{P}_i$ we will always mean $\mathrm{fcl}_{M\backslash y}(P_i)$. Observe that, if $i \in \{2, 3, \dots, k-1\}$, then $\mathrm{fcl}_{M\backslash y/z_k}(P_i) = \hat{P}_i$ and, if $i \in \{3, 4, \dots, k\}$, then $\mathrm{fcl}_{M\backslash y/z_2}(P_i) = \hat{P}_i$.

As M/z_k is uniquely k-fractured, it follows by Corollary 4.46 that $y \in \mathrm{cl}_{M/z_k}(\mathrm{fcl}_{M\backslash y/z_k}(P_i))$ for some $i \in \{2, 3, \dots, k\}$, or $y \in \mathrm{cl}_{M/z_k}(\mathrm{fcl}_{M\backslash y/z_k}(P_1 - \{z_k\}))$. If $y \in \mathrm{cl}_{M/z_k}(\mathrm{fcl}_{M\backslash y/z_k}(P_1 - \{z_k\}))$, then $y \in \mathrm{cl}_M(P_1)$ and it follows that M is k-fractured. Thus the former case occurs and we also have $y \in \mathrm{cl}_{M/z_2}(\mathrm{fcl}_{M\backslash y/z_2}(P_i))$ for some $i \in \{2, 3, \dots, k\}$.

Assume that $y \in \mathrm{cl}_{M/z_k}(\hat{P}_i)$ for some $i \in \{2, 3, \dots, k-1\}$. Let $P'_2 = \hat{P}_2 \cup \hat{P}_3 \cup \dots \cup \hat{P}_{k-1}$. Then $y \in \mathrm{cl}_M(P'_2 \cup \{z_k\})$, and, as $z_k \notin \mathrm{cl}_M(P'_2)$, we have $z_k \in \mathrm{cl}_M(P'_2 \cup \{y\})$. Observe that $\mathrm{cl}_M(P'_2 \cup \{y\}) \cap P_1 = \mathrm{cl}_M(P'_2 \cup \{z_k\}) \cap P_1$. As $z_2 \in \mathrm{cl}_{M/x}(P'_2)$, and $z_2 \notin \mathrm{cl}_M(P'_2)$, we see that, if $z_2 \in \mathrm{cl}_M(P'_2 \cup \{y\})$, then $x \in \mathrm{cl}_M(P'_2 \cup \{y\})$. As $\sqcap(P'_2 \cup \{y\}, P_1) = 2$, we deduce that $\{x, z_2, z_k\}$ is a triangle in M. This contradicts a number of things, amongst which is the fact that M/x is 3-connected.

It follows that $z_2 \in \mathrm{cl}_M(\hat{P}_k \cup \{y\})$. But we may apply the previous argument using z_2 to deduce that $z_k \in \mathrm{cl}_M(\hat{P}_2 \cup \{y\})$. Thus $y \in \mathrm{cl}_M(P_1 \cup \hat{P}_k)$ and $y \in \mathrm{cl}_M(P_1 \cup \hat{P}_2)$, so by Lemma 3.31, $y \in \mathrm{cl}_M(P_1)$ contradicting the fact that M is k-coherent. At last we can conclude that $\{z_k, z_2, x\}$ is not a triad of $M\backslash y$. □

5.41.7. *$M\backslash y, z_k$ and $M\backslash y, z_2$ are 3-connected.*

Subproof. Certainly $M\backslash y/x\backslash z_k$ is 3-connected, so that $M\backslash y, z_k$ is 3-connected up to a series pair containing x. Such a series pair must be $\{x, z_2\}$, as otherwise $z_2 \in \mathrm{cl}_{M\backslash y}(P_2)$ contradicting 5.41.4. But, in this case, $\{x, z_k, z_2\}$ is a triad of $M\backslash y$, contradicting 5.41.6. Hence $M\backslash y, z_k$ has no series pair containing x and is therefore 3-connected. □

Consider the flower $(P_1 - \{z_k\}, P_2, \dots, P_k)$ in $M\backslash y, z_k$.

5.41.8. *The ordered elements of $P_1 - \{z_k\}$ that are loose elements between P_k and $P_1 - \{z_k\}$ in $M\backslash y, z_k$ form an initial segment of $(f_1, \dots, f_l, z_2, x)$.*

Subproof. Say $r \in P_1 - \{z_k\}$ and $r \in \mathrm{cl}^{(*)}_{M\backslash y,z_k}(P_k)$. By 5.41.2(i), $r \notin \mathrm{cl}(P_k)$, so $r \in \mathrm{cl}^*_{M\backslash y,z_k}(P_k)$. As $x \in \mathrm{cl}(P_2 \cup \{z_2\})$, and $P_2 \cup \{z\} \subseteq E(M\backslash y, z_k) - P_k$, we see that $x \notin \mathrm{cl}^*_{M\backslash y,z_k}(P_k)$ so that $r \neq x$. Hence $r \in \mathrm{cl}^*_{M\backslash y,z_k/x}(P_k)$, that is, $r \in \mathrm{cl}^*_{M\backslash y/x}(P_k \cup \{z_k\})$. Thus $r = f_1$.

Assume that $l > 1$ and that there is an element $r \in \mathrm{cl}_{M\backslash y,z_k}(P_k \cup \{f_1\})$. If $r = x$, then $x \in \mathrm{cl}_M(P_k \cup \{f_1\})$, so that $f_1 \in \mathrm{cl}_{M/x}(P_k)$. But $z_2 \in \mathrm{cl}_{M/x}(P_k)$ and we have contradicted the structure of loose elements in swirl-like flowers. Hence $r \neq x$. Now $r \in \mathrm{cl}_{M\backslash y,z_k/x}(P_k \cup \{f_1\})$, so $r = f_2$.

An easy induction now proves that the loose elements of P_k form an initial segment of $(f_1, f_2, \dots, f_l, r, s)$, where (r, s) is a permutation of $\{z_2, x\}$. Assume that the elements $\{f_1, f_2, \dots, f_l\}$ are all loose and that we can continue. Note that there is a circuit C such that $\{x, z_2\} \subseteq C \subseteq P_2 \cup \{x, z_2\}$. Thus $r \notin \mathrm{cl}^*_{M\backslash y,z_k}(P_k \cup \{f_1, f_2, \dots, f_l\})$, so that $r \in \mathrm{cl}_{M\backslash y,z_k}(P_k \cup \{f_1, f_2, \dots, f_l\})$. Say $r = x$, so that $x \in \mathrm{cl}_{M\backslash y,z_k}(P_k \cup \{f_1, f_2, \dots, f_l\})$. But $z_k \in \mathrm{cl}_{M\backslash y/x}(P_k \cup \{f_1, f_2, \dots, f_l\})$ and $z_2 \in \mathrm{cl}_{M\backslash y/x}(P_k \cup$

$\{f_1, f_2, \ldots, f_l, z_k\})$, so $z_2 \in \mathrm{cl}_{M\backslash y/x}(P_k \cup \{f_1, f_2, \ldots, f_l\})$. Hence $z_2 \in \mathrm{cl}_{M\backslash y,z_k}(P_k \cup \{f_1, f_2, \ldots, f_l, x\})$. Thus $\{z_2, x\} \subseteq \mathrm{cl}_{M\backslash y,z_k}(P_k \cup \{f_1, f_2, \ldots, f_l\})$, contradicting the structure of loose elements in the swirl-like flower $(P_1 - \{z_k\}, P_2, \ldots, P_k)$ of $M\backslash y, z_k$. Hence $r = z_2$ and the claim holds. □

Now consider the flower $(P_1 - \{z_2\}, P_2, \ldots, P_k)$ of $M\backslash y, z_2$.

5.41.9. *The ordered subset of elements of $P_1 - \{z_2\}$ that are loose elements between P_k and $P_1 - \{z_2\}$ in $M\backslash y, z_2$ is either empty or the first element of the set is x, in which case x is in the coguts of P_k and $P_1 - \{z_2\}$.*

Subproof. If $r \in P_1 - \{z_2\}$ and $r \in \mathrm{cl}^{(*)}_{M\backslash y,z_2}(P_k)$, then $r \in \mathrm{cl}^*_{M\backslash y,z_2}(P_k)$. Say $r \neq x$. Then $r \in \mathrm{cl}^*_{M\backslash y,z_2/x}(P_k)$ and $r \in \mathrm{cl}^*_{M\backslash y/x}(P_k \cup \{z_2\})$. But $P_1 - \{z\}$ is a fan in $M\backslash y/x$ that begins and ends at the spoke elements z_k and z_2. Thus $(M\backslash y/x)|(P_1 - \{x\})$ is connected. Also $z_2 \in \mathrm{cl}_{M\backslash y/x}(P_2)$. It follows that there is a circuit C of $M\backslash y/x$ such that $r \in C \subseteq P_2 \cup (P_1 - \{x, z_2\})$. Hence $r \notin \mathrm{cl}^*_{M\backslash y/x}(P_k)$. The claim follows from this contradiction. □

5.41.10. *If $P_1 - \{z_2\}$ is a tight petal of $(P_1 - \{z_2\}, P_2, \ldots, P_k)$ in $M\backslash y, z_2$, then the set of elements in $P_1 - \{z_2\}$ that are loose between P_k and $P_1 - \{z_2\}$ is empty.*

Subproof. Assume that the set is nonempty. Then, by 5.41.9, x is in the coguts of $P_1 - \{z_2\}$ and P_k. Now $P_1 - \{x, z_2\}$ is a tight petal of the flower $(P_1 - \{x, z_2\}, P_2, \ldots, P_k)$ in the 3-connected matroid $M\backslash y, z_2/x$, so that $P_1 - \{x\}$ is certainly not a fan of loose elements in the flower $(P_1 - \{x\}, P_2, \ldots, P_k)$ of $M\backslash y/x$. □

As both $M\backslash y, z_k$ and $M\backslash y, z_2$ are 3-connected, both $M\backslash z_k$ and $M\backslash z_2$ are 3-connected. As z_k and z_2 are fixed in M, both $M\backslash z_k$ and $M\backslash z_2$ are k-fractured.

5.41.11. *We may assume that $y \notin \mathrm{cl}(P_1 \cup P_2)$.*

Subproof. If $y \in \mathrm{cl}(P_k \cup P_1)$ and $y \in \mathrm{cl}(P_1 \cup P_2)$, then by Lemma 3.31, $y \in \mathrm{cl}(P_1)$, contradicting the fact that M is k-coherent. Thus, up to symmetry, we may assume that $y \notin \mathrm{cl}(P_1 \cup P_2)$. □

5.41.12. *Say $i \in \{k, 2\}$. Assume that $P_1 - \{z_i\}$ is a fan of loose elements between P_k and P_2 in the flower $(P_1 - \{z_i\}, P_2, \ldots, P_k)$ of $M\backslash y, z_i$, with initial element α. Then $y \in \mathrm{cl}(P_k \cup \{\alpha\})$.*

Subproof. Under the hypothesis of the sublemma it is clear that $M\backslash y, z_i$ is k-coherent. It is easily checked that if $(P_1 - \{z_i\}, P_2, \ldots, P_k)$ induces a k-fracture in $M\backslash z_i$, then it follows that M is k-fractured. As $M\backslash z_i$ is k-fractured, by Lemma 4.44 it must be the case that for some petals $\hat{Q}_1, \hat{Q}_2$ of some other swirl-like flower of $M\backslash y, z_i$ of order $k-1$, we have $y \in \mathrm{cl}(\hat{Q}_1), \mathrm{cl}(\hat{Q}_2)$. By Lemma 3.16, we may assume that $\hat{Q}_1 \subseteq \mathrm{fcl}_{M\backslash y,z_i}(P_j)$ for some $j \in \{2, 3, \ldots, k\}$. The only cases that do not quickly lead to a contradiction to the fact that M is k-coherent are when either $\hat{Q}_1 \subseteq \mathrm{fcl}_{M\backslash y,z_i}(P_k)$ or $\hat{Q}_1 \subseteq \mathrm{fcl}_{M\backslash y,z_i}(P_2)$. The latter case contradicts 5.41.11, so the former case holds. Now by Lemma 5.38 we have $\hat{Q}_1 \subseteq P_k \cup \{\alpha\}$, so that $y \in \mathrm{cl}(P_k \cup \{\alpha\})$. □

5.41.13. *Say $i \in \{k, 2\}$. Assume that $P_1 - \{z_i\}$ is a tight petal of $(P_1 - \{z_i\}, P_2, \ldots, P_k)$ in $M\backslash y, z_i$. Then $y \in \mathrm{cl}_M(\mathrm{fcl}_{M\backslash y,z_i}(P_k))$.*

Subproof. We have $(P_1 - \{z_i\}, P_2, \ldots, P_k)$ uniquely fractures $M\backslash y, z_i$ and $M\backslash z_i$ is k-fractured. By Lemma 4.45, either $y \in \mathrm{cl}(P_1 - \{z_i\})$ or $y \in \mathrm{cl}_M(\mathrm{fcl}_{M\backslash y,z_i}(P_j))$ for some $j \in \{2, 3, \ldots, k\}$. The only case that does not either contradict 5.41.11 or the k-coherence of M is if $y \in \mathrm{cl}_M(\mathrm{fcl}_{M\backslash y,z_i}(P_k))$. □

5.41.14. $x \in \mathrm{cl}(P_k \cup \{y\})$.

Subproof. Say that $P_1 - \{z\}$ is a tight petal of the flower $(P_1 - \{z_2\}, P_2, \dots, P_k)$ in $M\backslash y, z_2$. By 5.41.13 we have $y \in \mathrm{cl}_M(\mathrm{fcl}_{M\backslash y,z_2}(P_k))$, and by 5.41.10, we may assume that $\mathrm{fcl}_{M\backslash y,z_2}(P_k) = P_k$, so $y \in \mathrm{cl}_M(P_k)$, giving the contradiction that M is k-fractured. Thus $P_1 - \{z_2\}$ is a fan between P_k and P_1 in $M\backslash y, z_2$. By 5.41.9 and 5.41.12, $y \in \mathrm{cl}_M(P_k \cup \{x\})$. As $y \notin \mathrm{cl}_M(P_k)$, we have $x \in \mathrm{cl}_M(P_k \cup \{y\})$ as required. $\square$

Assume that $P_1 - \{z_k\}$ is a fan between P_k and P_1 in $(P_1 - \{z_k\}, P_2, \dots, P_k)$. Then by 5.41.12, $y \in \mathrm{cl}(P_k \cup \{f_1\})$ and now by 5.41.14 $x \in \mathrm{cl}(P_k \cup \{f_1\})$. But, by 5.41.8, x is the last element of the fan. Moreover, f_1 is an initial rim element of the fan and it is easily seen that the fan structure implies the contradiction that $x \notin \mathrm{cl}(P_k \cup \{f_1\})$.

Thus we may assume that $P_1 - \{z_k\}$ is a tight petal. By 5.41.8, the loose elements of $P_1 - \{z_k\}$ between P_k and $P_1 - \{z_k\}$ are of the form $(f_1, f_2, \dots, f_i)$ for some $i \leqslant l$. Moreover, $x \notin \mathrm{cl}(P_k \cup \{f_1, f_2, \dots, f_i\})$, otherwise x is also a loose element. By 5.41.13, $y \in \mathrm{cl}(P_k \cup \{f_1, f_2, \dots, f_i\})$, so by 5.41.14 $x \in \mathrm{cl}(P_k \cup \{f_1, f_2, \dots, f_i\})$.

From this final contradiction we at last deduce that $|P_1| = 2$ and we consider this case now. Assume that $P_1 = \{x, z\}$. The task now is to show that $\{x, y, z\}$ is a gang of three.

5.41.15. *x, y and z are fixed in M.*

Subproof. We already know that y is fixed in M. Say $\{a, b\} = \{x, z\}$. Then b is fixed in $M\backslash y/a$ as a is in the guts of a pair of petals of a swirl-like flower in $M\backslash y/a$. Thus b is fixed in M/a. As a and b are not comparable in M, it follows from Corollary 5.9 that b is fixed in M. Therefore both x and z are fixed in M. $\square$

The next claim is clear.

5.41.16. *$M\backslash x$, $M\backslash y$ and $M\backslash z$ are 3-connected and k-fractured.*

Let $\mathbf{O} = (O_1, O_2, \dots, O_l)$ and $\mathbf{Q} = (Q_1, Q_2, \dots, Q_m)$ be k-fractures of $M\backslash x$ and $M\backslash z$ respectively. As $\mathbf{P}$ uniquely fractures $M\backslash y$, it follows by Lemma 4.47 that $M\backslash y, x$ and $M\backslash y, z$ are k-coherent. Moreover $(P_2, P_3, \dots, P_k \cup \{z\})$ and $(P_2, P_3, \dots, P_k \cup \{x\})$ are maximal swirl-like flowers in $M\backslash y, x$ and $M\backslash y, z$ respectively. At this stage we almost have symmetry between x and z except that we do not yet know that M/z is k-coherent.

By Lemma 4.43, $y \in \mathrm{cl}_M(\mathrm{fcl}_{M\backslash y,x}(P_i))$ for some $i \in \{2, 3, \dots, k\}$. It is easily seen that we contradict the k-coherence of M unless, up to labels we have $y \in \mathrm{cl}(P_k \cup \{z\})$, $y \notin \mathrm{cl}(P_k)$, and $y \notin \mathrm{cl}(P_2 \cup \{z\})$. A similar conclusion holds by considering the flower $(P_2, P_3, \dots, P_k \cup \{x\})$, establishing the next claim.

5.41.17. *We may assume that $y \in \mathrm{cl}(P_k \cup \{z\})$, $y \in \mathrm{cl}(P_k \cup \{x\})$, $y \notin \mathrm{cl}(P_k)$, $y \notin \mathrm{cl}(P_2 \cup \{z\})$ and $y \notin \mathrm{cl}(P_2 \cup \{x\})$.*

Consider the k-fracture $\mathbf{O} = (O_1, O_2, \dots, O_l)$ of $M\backslash x$. We may assume that O_1 is fully closed and that $y \in O_1$. Consider the quasi-flower $(O_1 - \{y\}, O_2, \dots, O_l)$ in $M\backslash x, y$. As $M\backslash x, y$ is k-coherent, it follows from Lemma 4.42 that $l = k$ and that $O_1 - \{y\}$ is a loose petal of this quasi-flower. As O_1 is fully closed, $O_1 - \{y\}$ is a maximal fan in $M\backslash x, y$. By Lemma 4.44 $y \in \mathrm{cl}((O_1 - \{y\}) \cup O_2)$ and $y \in \mathrm{cl}(O_k \cup (O_1 - \{y\}))$. By Lemma 3.16 we may assume, up to labels, that $(O_1 - \{y\}) \cup O_2$ is contained in the full closure of some petal of the flower $(P_2, P_3, \dots, P_k \cup \{z\})$ of $M\backslash x, y$. As $y \in \mathrm{cl}((O_1 - \{y\}) \cup O_2)$, this petal must be $P_k \cup \{z\}$.

5.41.18. *$P_2 \subseteq \mathrm{fcl}_{M\backslash x,y}(O_k)$, $z \in \mathrm{cl}^*_{M\backslash x,y}(O_k)$, and $z \in O_1$.*

Subproof. By Lemma 3.16, P_2 is contained in the full closure of a petal O_i of $(O_1 - \{y\}, O_2, \dots, O_k)$ in $M\backslash x, y$, where $i \in \{3, 4, \dots, k\}$. But $z \in \mathrm{cl}^*_{M\backslash x,y}(P_2)$, so, $z \in \mathrm{cl}^*_{M\backslash x,y}(O_i)$. By the fact that $(O_1 - \{y\},$

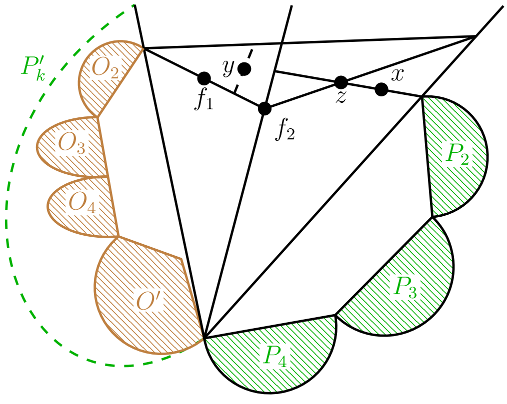

Fig. 5.5. Getting technical.

$O_2, \ldots, O_k)$ is a swirl-like quasi-flower and the fact that $z \in \mathrm{cl}^*_{M\backslash x,y}(O_i)$, we see that $i = k$. Thus $P_2 \subseteq \mathrm{fcl}_{M\backslash x,y}(O_k)$ and $z \in \mathrm{cl}^*_{M\backslash x,y}(O_k)$.

Say $z \in O_k$. Then $P_2 \cup \{z\} \subseteq O_k$. But $P_1 = \{x, z\}$ so that $x \in \mathrm{cl}(P_2 \cup \{z\})$, that is, $x \in \mathrm{cl}(O_k)$, giving the contradiction that M is k-fractured.

Now, if $z \notin O_1 - \{y\}$, then $z \in O_{k-1}$. But, in this case, z is a loose coguts element of the flower $(O_1, O_2, \ldots, O_k)$ in $M\backslash x$, contradicting the fact that $M\backslash x, z$ is 3-connected. Hence $z \in O_1 - \{y\}$. □

The disturbing possibility that $O_1 \neq \{y, z\}$ needs to be eliminated. Fig. 5.5 is an illustration for the proof of 5.41.19. In the diagram, $P'_k = P_k - \{f_1, f_2\}$ and $O' = O_k \cap P_k$.

5.41.19. $O_1 = \{y, z\}$.

Subproof. If the sublemma fails, then $|O_1 - \{y\}| > 1$. In this case, by Lemmas 4.42 and 4.44, $y \in \mathrm{cl}(O_1 - \{y\})$. If $|O_1 - \{y\}| \neq 3$, then O_1 contains a triangle of M. We omit the routine verification that this cannot happen. Hence $|O_1 - \{y\}| = 3$. By 5.41.18, $z \in \mathrm{cl}^*_{M\backslash x,y}(O_k)$, so that there is an ordering (z, f_2, f_1) of $O_1 - \{y\}$ that gives a maximal fan of loose elements between O_k and O_2 in the flower $(O_1 - \{y\}, O_2, \ldots, O_k)$ of $M\backslash x, y$. Note that f_1 is a rim element of this fan, so that, by Lemma 5.19(iv), f_1 is cofixed in $M\backslash x, y$. We next show that f_1 is cofixed in M. Now $x \in \mathrm{cl}(P_2 \cup \{z\})$ and $P_2 \subseteq O_k$. By the fact that (z, f_2, f_1) is a fan of loose elements between O_k and O_2, we see that $f_1 \notin \mathrm{cl}(O_k \cup \{z\})$. Thus $f_1 \notin \mathrm{cl}(P_2 \cup \{z\})$. Hence $x \not\asymp f_1$ in $M\backslash y$. Therefore, by Corollary 5.7, f_1 is cofixed in $M\backslash y$. As y is fixed in M it follows, again by Corollary 5.7, that f_1 is cofixed in M.

The element f_1 is a terminal rim element of a maximal fan between O_k and O_2 in the k-coherent matroid $M\backslash x, y$. Thus, by Lemma 4.41, $M\backslash y, x/f_1$ is k-coherent. But $f_1 \notin \mathrm{cl}_M(P_2 \cup \{x\})$, so $x \notin \mathrm{cl}_{M/f_1}(P_2)$. Therefore,

$$x \in \mathrm{cl}_{M/f_1}\big(P_2 \cup \{z\}\big) - \mathrm{cl}_{M/f_1}(P_2).$$

As $f_1 \in P_k$, by 5.41.17, we have

$$x \in \mathrm{cl}_{M/f_1}\big(\big(P_k - \{f_1\}\big) \cup \{z\}\big) - \mathrm{cl}_{M/f_1}\big(P_k - \{f_1\}\big).$$

It is now easily checked that $(\{x, z\}, P_2, \ldots, P_k - \{f_1\})$ uniquely fractures $M\backslash y/f_1$. We know that $y \in \mathrm{cl}_{M/f_1}((P_k - \{f_1\}) \cup \{x, z\})$. But $y \notin \mathrm{cl}_{M/f_1}(P_k - \{f_1\})$. Assume for a contradiction that $y \in \mathrm{cl}_{M/f_1}(\{x, z\})$. Then $x \in \mathrm{cl}_{M/f_1}(\{y, z\})$ so that $x \in \mathrm{cl}_M(\{y, z, f_1\})$. Recall that **O** is a k-fracture of $M\backslash x$ and that $O_1 = \{y, z, f_1, f_2\}$. It follows that $(O_1 \cup \{x\}, O_2, \ldots, O_k)$ is a k-fracture of M. From this contradiction we deduce that $y \notin \mathrm{cl}_{M/f_1}(\{x, z\})$. Now, by Corollary 4.46, M/f_1 is k-coherent. But f_1 is cofixed in M and we have contradicted the fact that M is a k-skeleton. □

Evidently the same argument works for **Q** so that we have $\mathbf{O} = (\{y, z\}, O_2, \dots, O_k)$ and $\mathbf{Q} = (\{x, y\}, Q_2, \dots, Q_k)$.

5.41.20.

(i) **O**, **P** *and* **Q** *uniquely fracture* $M\backslash x$, $M\backslash y$ *and* $M\backslash z$ *respectively.*
(ii) *Up to labels* $(O_2, O_3, \dots, O_k \cup \{x, y, z\})$, $(P_2, P_3, \dots, P_k \cup \{x, y, z\})$ *and* $(Q_2, Q_3, \dots, Q_k \cup \{x, y, z\})$ *are swirl-like flowers in* M.
(iii) $O_2 \cup O_3 \cup \dots \cup O_{k-1}$, $P_2 \cup P_3 \cup \dots \cup P_{k-1}$ *and* $Q_2 \cup Q_3 \cup \dots \cup Q_{k-1}$ *are mutually disjoint.*

Subproof. We already know that **P** uniquely fractures $M\backslash y$, so that $M\backslash y, x$ is k-coherent. Consider **O**. If **O** did not uniquely fracture $M\backslash x$, then $M\backslash x, y$ would not be k-coherent by Lemma 4.42. Thus **O**, and similarly **Q** uniquely fracture $M\backslash x$ and $M\backslash z$ respectively.

Consider (ii). As $y \in \mathrm{cl}(P_k \cup \{x, z\})$, we see that $(P_2, P_3, \dots, P_k \cup \{x, y, z\})$ is a swirl-like flower in M. Similar arguments to those that establish that $y \in \mathrm{cl}(P_k \cup \{x, z\})$ prove that, up to labels, $x \in \mathrm{cl}(O_k \cup \{y, z\})$ and $z \in \mathrm{cl}(Q_k \cup \{x, y\})$. Thus (ii) holds.

Consider (iii). Consider the inequivalent flowers $(O_2, O_3, \dots, O_k \cup \{z\})$ and $(P_2, P_3, \dots, P_k \cup \{z\})$ in $M\backslash x, y$. We have already observed that, up to labels, $O_2 \subseteq P_k$. Note that $P_k \cup \{z\}$ is fully closed in $M\backslash x, y$. Thus, by Lemma 3.16, P_k contains all but one member of $(O_2, O_3, \dots, O_k)$. As z is in the coguts of O_2 and $O_k \cup \{z\}$ in $M\backslash x, y$ we see that $z \in \mathrm{cl}(O_2 \cup O_k)$ so that $y \in \mathrm{cl}(O_2 \cup O_k)$. But $y \notin \mathrm{cl}(P_k)$, so $O_2 \cup O_k \nsubseteq P_k$. Hence $O_k \nsubseteq P_k$. Thus $O_2 \cup \dots \cup O_{k-1} \subseteq P_k$, so that $O_2 \cup O_3 \cup \dots \cup O_{k-1}$ and $P_2 \cup P_3 \cup \dots \cup P_{k-1}$ are disjoint. The rest of (iii) follows from similar arguments. □

Note that we now have symmetry between y and z.

5.41.21. *M/y and M/z are k-coherent.*

Subproof. By the symmetry between y and z is suffices to prove that M/z is k-coherent. Consider $M\backslash x/z$. Clearly $(P_2, P_3, \dots, P_k \cup \{y\})$ is a swirl-like flower in this matroid. But $x \in \mathrm{cl}(P_2 \cup \{z\})$, so $x \in \mathrm{cl}_{M/z}(P_2)$. By 5.41.17, $y \in \mathrm{cl}(P_k \cup \{x\})$, so $x \in \mathrm{cl}(P_k \cup \{y, z\})$. Thus $(P_2 \cup \{x\}, P_3, \dots, P_{k-1}, P_k \cup \{y\})$ is a swirl-like flower in M/z and x is in the guts $P_2 \cup \{x\}$ and $P_k \cup \{y\}$. It now follows by Lemma 4.41 that M/z is k-coherent if and only if $M\backslash x/z$ is. As $M\backslash x/z$ is k-coherent, the sublemma follows. □

Relabel (x, y, z) by (r, s, t); relabel $(O_2, O_3, \dots, O_{k-1})$ by $(R_2, R_3, \dots, R_{k-1})$; relabel $(P_2, P_3, \dots, P_{k-1})$ by $(S_2, S_3, \dots, S_{k-1})$; relabel $(Q_2, Q_3, \dots, Q_{k-1})$ by $(T_2, T_3, \dots, T_{k-1})$; and let $Z = O_k \cap P_k \cap Q_k$. With this relabelling it is a matter of routine checking of the definition to confirm that $\{x, y, z\}$ is indeed a gang of three. □

At last we can perform the ritual incantation that completes the proof of Theorem 5.36.

Proof of Theorem 5.36. Assume that M/x is not a k-skeleton. Then there is an element $y \in E(M/x)$ such that either

(i) y is cofixed in M/x and $M/x, y$ is coherent, or
(ii) y is fixed in M/x and $M/x\backslash y$ is coherent.

By the dual of Lemma 5.40 (i) does not occur so that (ii) holds. But then, by Lemma 5.41 x is a member of a gang of three and the theorem follows. □

8. Removing a gang of three

Proving that k-skeletons in a class can be found by an inductive search is the topic of the next chapter. Elements in gangs of three are problematic in that we cannot remove them individually to keep the property of being a k-skeleton. In this section we show that we can remove the whole gang of three. Note that, while this is a 3-element move, viewed from a "bottom up" perspective, it is not particularly complicated. It amounts to extending a k-skeleton by adding two elements into the guts of petals of certain swirl-like flowers of order $k-1$ and performing a single coextension on the resulting matroid.

Theorem 5.42. *Let $\{r, s, t\}$ be a gang of three in the k-skeleton M. Then $M/r\backslash s, t$ is a k-skeleton.*

Given the gang of three $\{r, s, t\}$ there is an associated canonical partition of the ground set of M. In what follows we use the same labelling for this associated partition as the one given in the definition.

Lemma 5.43. *Let $\{r, s, t\}$ be a gang of three in the k-skeleton M.*

(i) *$(S_2 \cup \{t\}, S_3, \ldots, S_{k-1}, R \cup T \cup Z)$ is a swirl-like flower in M/r with t in the guts of the petals $S_2 \cup \{t\}$ and $R \cup T \cup Z$. Also $(T_2 \cup \{s\}, T_3, \ldots, T_{k-1}, R \cup S \cup Z)$ is a swirl-like flower in M/r with s in the guts of $T_2 \cup \{s\}$ and $R \cup S \cup Z$.*
(ii) *$M/r\backslash s, t$ is k-coherent.*
(iii) *If $\{r, p, q\}$ is another gang of three in M, then $\{r, p, q\} = \{r, s, t\}$.*

Proof. Part (i) is a routine consequence of the definition. Part (ii) follows from (i) and Lemma 5.32. Consider (iii). Say that $\{r, s', t'\}$ is a gang of three where $\{s', t'\} \neq \{s, t\}$. Then $\{s', t'\}$ is a 2-element petal in a tight swirl-like flower $(\{s', t'\}, Q_2, \ldots, Q_{k-1}, E(M) - (Q_2 \cup \cdots \cup Q_{k-1} \cup \{r, s', t'\}))$ of $M\backslash r$. As $\{s, t\}$ also has this property, we see that $\{s, t\} \cap \{s', t'\} = \emptyset$. As $\{r, s', t'\}$ is a gang of three, $\{s', t'\}$ is not a petal of a displayed swirl-like flower of M and it follows easily that $\{s', t'\} \subseteq Z$. It is also easily seen that $Q_2 \cup Q_3 \cup \cdots \cup Q_{k-1} \subseteq Z$. But $r \notin \mathrm{cl}(E(M) - (Q_2 \cup Q_3 \cup \cdots \cup Q_{k-1} \cup \{r, s', t'\}))$ and we obtain the contradiction that M is k-fractured. $\square$

We omit the easy proof of the next lemma.

Lemma 5.44. *Let a and b be elements of the 3-connected matroid M. Assume that a is in the guts of a pair of petals of a swirl-like flower of M if order at least 4.*

(i) *If b is fixed or cofixed in $M\backslash a$, then b is respectively fixed or cofixed in M.*
(ii) *If $M\backslash a, b$ is k-coherent, then $M\backslash b$ is k-coherent.*
(iii) *If $M\backslash a/b$ is k-coherent and M has no triangle containing a and b, then M/b is k-coherent.*

Proof of Theorem 5.42. By Lemma 5.43(ii) $M/r\backslash s, t$ is k-coherent. Assume that there is an element y that is fixed in $M/r\backslash s, t$ such that $M/r\backslash s, t, y$ is k-coherent. Then, by Lemma 5.44, y is fixed in M/r and $M/r\backslash y$ is k-coherent. Now by Lemma 5.41, there is a gang of three in M containing r and y, contradicting Lemma 5.43(iii). Thus, if y is fixed in $M/r\backslash s, t$, then $M/r\backslash s, t, y$ is not k-coherent.

Assume that y is cofixed in M/r and $M/r\backslash s, t/y$ is k-coherent. By Lemma 5.44(i), y is cofixed in M/r and hence cofixed in M. Assume that M/r has no triangle containing y and either s or t. By Lemma 5.44(iii), $M/r, y$ is k-coherent. All up, M/r and $M/r, y$ are k-coherent and y is cofixed in M/r, and we have contradicted the dual of Lemma 5.40. Otherwise, it is easily seen that $M/r, y$ is k-coherent up to a single parallel pair containing either s or t. Moreover, this parallel pair is coblocked by r. We omit the routine argument that proves that, in this case, M/y is k-coherent, contradicting the fact that y is cofixed in M. Thus, if y is cofixed in $M/r\backslash s, t$, then $M/r\backslash s, t/y$ is not k-coherent. Hence $M/r\backslash s, t$ is indeed a k-skeleton. $\square$

Chapter 6. A chain theorem for skeletons

The goal of this chapter is to prove that skeletons do not occur sporadically. In particular we prove the following theorem.

Theorem 6.1. *If M is a k-skeleton and $|E(M)| > 4$, then M has a k-skeleton minor M' with $|E(M) - E(M')| \leqslant 4$.*

In other words all k-skeletons in a minor-closed class can be found by an inductive search beginning with $U_{2,4}$ using 1-, 2-, 3-, or 4-element moves. It turns out that, in the case that 3- or 4-element moves are required, very specific structure arises, the full details of which are made explicit in Theorem 6.30.

We also prove in Corollary 6.12 that a swirl-like flower of order l in a k-skeleton has a rank-l free-swirl minor. This fact is used later in this paper. Apart from this, and from some of the easy lemmas, the results of this chapter are not used later in this paper. But they are of some independent interest; for example it is possible that they could be used to get explicit bounds for the number of inequivalent representations of 4-connected matroids over small prime fields such as $GF(7)$. Moreover, the material is so tightly interwoven with other material in this paper that it would be unwieldy to write it up separately. In any case the reader should feel perfectly relaxed about skipping this chapter.

We will say that a k-skeleton M is 1-*reduced* if $|E(M)| > 4$ and M has no element e such that either $M\backslash e$ or M/e is a k-skeleton. We know from Corollary 5.34 that, if p and q are comparable elements in a 1-reduced k-skeleton, then $\{p,q\}$ form a clonal class of size 2. In this case it often true that $M\backslash p/q$ is a k-skeleton, but unfortunately this is not always so. We say that a k-skeleton M is 2-*reduced* if it is 1-reduced and has no clonal pair $\{p,q\}$ such that $M\backslash p/q$ is a k-skeleton. We begin by getting a more explicit structural description of 1- and 2-reduced skeletons. To do this it will help to have a slightly revised version of 3-trees of matroids.

1. Augmented 3-trees

Let M be a 3-connected matroid and (D,C) be a 3-separation of M, where D is a quad. Then, for any partition (A,B) of D into 2-element subsets, the partition (A,B,C) is a tight swirl-like flower of M. Such a flower has order two as the 3-separation (D,C) displays all of the non-sequential 3-separations displayed by the flower. This is a somewhat degenerate situation, and in the definition of 3-trees given in Section 5, these flowers were specifically excluded from being displayed unless (A,B,C) refines to a larger swirl-like or spike-like flower $(A,B,P_1,P_2,\dots,P_l)$ in M, in which case, the refined flower certainly is displayed. An example of this situation arises when M is a swirl, in which case each consecutive pair of petals of the swirl is a quad of M.

The partition of D is somewhat less arbitrary if we are told that D partitions into 2-element clonal classes A and B. In this case it is easily seen that any 3-separation that crosses D crosses neither A nor B. Choosing to display the flower (A,B,C) now seems somewhat more natural. We next give a modified version of 3-trees in which such flowers are displayed. Recall appropriate definitions from Section 5. The following definition is sufficiently lengthy to challenge even a robust digestive system, but it is just a modification of the definition of 3-tree that guarantees that quads that partition into 2-element clonal classes get displayed by flower vertices. Let π be a partition of $E(M)$. We say that the π-labelled tree T is an *augmented* 3-*tree* for M if the following hold.

- (A1) For each edge e of T, the partition (X,Y) of E displayed by e is 3-separating, and, if e is incident with two bag vertices, then (X,Y) is a non-sequential 3-separation.
- (A2) Every non-bag vertex v is labelled either D or A; if v is labelled D, then there is a cyclic ordering on the edges incident with v.
- (A3) If a vertex v is labelled A, then either the partition of E displayed by v is a tight maximal anemone of order at least 3 or it has the form (A,B,C) where $A \cup B$ is a quad and A and B are 2-element clonal classes.

(A4) If a vertex v is labelled D, then either the partition of E displayed by v, with the cyclic order induced by the cyclic ordering on the edges incident with v, is a tight maximal daisy of order at least 3, or it has the form (A, B, C) where $A \cup B$ is a quad and A and B are 2-element clonal classes.
(A5) For every tight maximal flower of M of order three, there is an equivalent flower that is displayed by a vertex of T.
(A6) For every partition (A, B, C) of $E(M)$, where $A \cup B$ is a quad and A and B are 2-element clonal classes, there is a vertex of T that displays a, possibly trivial, refinement of (A, B, C).
(A7) If a vertex is incident with two edges e and f that display equivalent 3-separations, then the other ends of e and f are flower vertices, v has degree two, and v labels a nonempty bag.

Lemma 6.2. *Every* 3*-connected matroid has an augmented* 3*-tree.*

Proof. To facilitate the proof we say that a π-labelled tree is a *semi-augmented* 3-tree for M if it satisfies all properties of an augmented 3-tree except that $(A6)$ need not hold for all partitions (A, B, C) of M where $A \cup B$ is a quad and A and B are 2-element clonal classes. Note that 3-trees are semi-augmented 3-trees. Thus every 3-connected matroid has a semi-augmented 3-tree.

Let T' be a semi-augmented 3-tree for M. Let D be a quad of M consisting of 2-element clonal classes A and B and let $C = E(M) - D$. Assume that no refinement of (A, B, C) is displayed by T'. We claim that T' can be enhanced to a semi-augmented 3-tree that displays a refinement of (A, B, C).

Assume that the 3-separation (D, C) is sequential. Then C is sequential. In this case, T' consists of a single vertex and it is routinely seen that one obtains an augmented 3-tree for M by replacing T' with a tree T containing a single flower vertex displaying the flower (A, B, C). Thus we may assume that (D, C) is non-sequential.

Assume that M has a 3-separation (X, Y) crossing D that is not equivalent to (D, C). Then we may assume that $A \subseteq X$ and $B \subseteq Y$. In this case it is readily checked that $(X - A, A, B, Y - B)$ is a swirl-like, spike-like or Vámos-like flower of M of order 4. A flower equivalent to this is displayed by T', and it is readily checked that we may modify T' to obtain a semi-augmented 3-tree in which a refinement of $(X - A, A, B, Y - B)$ is displayed. Thus the claim holds in this case.

Assume that M has no 3-separation crossing D that is not equivalent to (D, C). Then a 3-separation equivalent to (D, C) is displayed by T and we see that D is contained in a bag associated with a leaf v of T. Add new vertices v', v_A and v_B to T' to obtain a tree T where v' is incident with v, v_A and v_B. In the new tree, v' is a flower vertex, the new bag at v is the bag in T' at v with D removed, and the bags at v_A and v_B are A and B respectively. Label this flower vertex A or D arbitrarily. The tree T is our required enhanced semi-augmented 3-tree and the claim holds in this case too.

The lemma now follows by induction. □

Note that an augmented 3-tree need not be a 3-tree, as it may violate property (ii) of 3-trees. Nonetheless we extend terminology for 3-trees to augmented 3-trees in an obvious way.

The new leaf bags that we create in an augmented tree are certainly petals of a spike-like or swirl-like flower. In fact this is true for any 2-element leaf bag.

Lemma 6.3. *Let T be an augmented* 3*-tree for a* 3*-connected matroid M and let P_1 be a* 2*-element leaf bag of M. Then M has a tight spike-like, swirl-like or Vámos-like flower $(P_1, P_2, \ldots, P_m)$ with at least three petals.*

Proof. Certainly P_1 is a petal of a tight flower $(P_1, P_2, \ldots, P_m)$ with at least three petals as this is true for any sequential 3-separating set that is a leaf bag of T. Say that $(P_1, P_2, \ldots, P_m)$ is not a swirl-like, spike-like or Vámos-like flower. Then, up to duality, we may assume that the flower is a paddle. But then $\sqcap(P_1, P_2) = 2$ so that $r(P_1 \cup P_2) = r(P_1) + r(P_2) - 2 = r(P_2)$ and it follows that $P_1 \subseteq \mathrm{cl}(P_2)$, contradicting the fact that the flower $(P_1, P_2, \ldots, P_m)$ is tight. □

2. 1-Reduced k-skeletons

In this section we develop some properties of k-skeletons where we have no single-element deletion or contraction available to maintain the property of being a k-skeleton. We first need to define another type of element.

Let M be a k-coherent matroid. Then the element e of M is *semi-feral* if either

(i) $M\backslash f$ is 3-connected and k-fractured, and M/f is not 3-connected, or
(ii) M/f is 3-connected and k-fractured, and $M\backslash f$ is not 3-connected.

Note that elements of k-wild triangles and triads are semi-feral. The goal of this section is to prove

Theorem 6.4. *Let M be a 1-reduced k-skeleton where $|E(M)| > 4$. Then the ground set of M consists of feral elements, semi-feral elements, members of gangs or cogangs of three, and 2-element clonal classes. Moreover, if T is an augmented 3-tree for M, then each leaf bag of T is a union of 2-element clonal classes of M.*

One part of Theorem 6.4 is clear.

Lemma 6.5. *Let M be a 1-reduced k-skeleton. Then the ground set of M consists of feral elements, semi-feral elements, members of gangs or cogangs of three and 2-element clonal classes.*

Proof. Say $e \in E(M)$. If M has an element f that is comparable with e, then, by Corollary 5.34, e is in a 2-element clonal class.

Assume that e is not comparable to any other element. If e is in a triangle or a triad T, then T is k-wild by Lemma 5.29 and therefore e is semi-feral. Assume that e is not in a triangle or a triad. Then either $M\backslash e$ or M/e is 3-connected. If neither $M\backslash e$ nor M/e is k-coherent, then, e is either feral or semi-feral. If M/e is k-coherent, then, by Theorem 5.36, e is a member of a gang of three and if $M\backslash e$ is k-coherent, then, by the dual of Theorem 5.36, e is a member of a cogang of three. □

Let x be an element of the 3-connected matroid M. Recall that x is *peripheral* if it belongs to a leaf bag of some 3-tree for M. We say that x is *strongly peripheral* if it belongs to a leaf bag of some augmented 3-tree for M.

It is shown in Lemmas 4.38 and 4.39 that feral elements and members of k-wild triangles are not peripheral. Note that if x is strongly peripheral, then x is peripheral. We omit the routine proof of the next lemma.

Lemma 6.6. *If M is a k-coherent matroid and x is a member of a gang or cogang of three in M, then x is not peripheral.*

On the other hand, it is possible for a semi-feral element of a k-coherent matroid to be peripheral. Fig. 6.1 illustrates an example. Indeed, for an arbitrary k-coherent matroid it is possible for a semi-feral element to be strongly peripheral. Our task is to show that this cannot happen for 1-reduced k-skeletons.

Lemma 6.7. *If f is a semi-feral element of the k-skeleton M then f is not comparable with any other element of M.*

Proof. Say f is semi-feral and is comparable with the element g of M. If f and g are not clones, then, by Corollary 5.34, either $M\backslash f$ or M/f is k-coherent, contradicting the definition of semi-feral element. Assume that f and g are clones. Then either $M\backslash f$ or M/f is k-coherent by Corollary 5.26, contradicting the definition of semi-feral elements. □

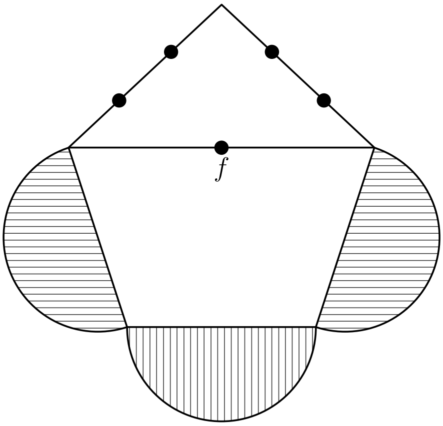

Fig. 6.1. A semi-feral element that is peripheral but not strongly peripheral.

Except in special circumstances, semi-feral elements belong to the guts of some vertical 3-separation.

Lemma 6.8. *If f is a semi-feral element of the k-skeleton M and $M\backslash f$ is 3-connected, then $\mathrm{si}(M/f)$ is not 3-connected unless f is in a costandard k-wild triangle.*

Proof. By the definition of semi-feral element M/f is not 3-connected. If $\mathrm{si}(M/f)$ is 3-connected, then f is in a triangle T that is k-wild by Lemma 5.29. If T is costandard, then the claim holds. Otherwise T is standard. In this case it follows from Lemma 4.22(iii) that f is in the guts of a vertical 3-separation so that $\mathrm{si}(M/f)$ is not 3-connected. □

Lemma 6.9. *If f is a semi-feral element of the 1-reduced k-skeleton M, then f is not strongly peripheral.*

Proof. Assume that f is semi-feral, where $M\backslash f$ is 3-connected. If f is in a k-wild triangle, then the lemma holds by Lemma 4.38. Thus we may assume that f is not in a k-wild triangle. By Lemma 6.8, f is in the guts of a vertical 3-separation $(X \cup \{f\}, Y)$ in M. Let $\mathbf{P} = (P_1, P_2, \dots, P_n)$ be a k-fracture of $M\backslash f$. By Lemma 6.7, Theorem 5.35 and the fact that f is not feral, and hence not in a bogan couple, we deduce that $\mathbf{P}$ is canonical. If either X or Y is contained in a petal of $\mathbf{P}$, then M is not k-coherent, so, up to labels, there is an $i \in \{2, 3, \dots, n-2\}$ such that $X = P_1 \cup P_2 \cup \dots \cup P_i$. If $i \notin \{2, n-2\}$, then it is clear that f is not peripheral. Therefore we may assume that $i \in \{2, n-2\}$ and, indeed, that $i = 2$.

The flower $(P_1 \cup P_2 \cup \{f\}, P_3, \dots, P_n)$ of M certainly needs to be displayed in T. Thus, if f is strongly peripheral, then it must be the case that $P_1 \cup P_2 \cup \{f\}$ is a leaf bag of T. Assume that P_1 is non-sequential. Then the swirl-like flower $(P_1, P_2, \{f\} \cup P_3 \cup \dots \cup P_n)$ is displayed by T, as P_1 is not equivalent to $P_1 \cup P_2$. Hence P_1 and P_2 are sequential. If $|P_1| > 2$, then P_1 contains either a triangle or triad. By Lemma 4.38 this triad or triangle is a clonal triple and we contradict the assumption that M is 1-reduced. Hence $|P_1| = |P_2| = 2$. Moreover, $(P_1, P_2, \{f\} \cup P_3 \cup \dots \cup P_n)$ is a swirl-like or spike-like flower, so that $P_1 \cup P_2$ is a quad of M.

Say $p \in P_1$. As $P_1 \cup P_2$ is a quad, $M\backslash p$ and M/p are k-coherent by Lemma 5.31. Neither of these matroids is a k-skeleton, and p is peripheral so is not in a gang or cogang of three. Therefore, by Theorem 5.36, p is comparable with some other element of M. It follows from Theorem 5.33 that p is in a clonal pair. Evidently this clonal pair must be P_1. Similarly P_2 is a clonal pair. Indeed P_1 and P_2 are 2-element clonal classes. By the definition of augmented 3-tree the flower $(P_1, P_2, \{f\} \cup P_3 \cup \dots \cup P_n)$ is displayed in M leading to the conclusion that f is indeed not strongly peripheral. □

We can now complete the proof of Theorem 6.4 which is nothing more than a summary of the facts that we have garnered so far.

Proof of Theorem 6.4. By Lemma 6.5 the ground set of a 1-reduced k-skeleton consists of feral elements, semi-feral elements, members of gangs and cogangs of three and 2-element clonal classes. By Lemmas 4.38, 4.39, 6.6 and 6.9, no feral element, semi-feral element, or member of a gang or cogang of three is strongly peripheral. Hence every leaf bag of an augmented 3-tree for M consists of 2-element clonal classes of M. $\square$

We also note another easy consequence of the results of this section.

Lemma 6.10. *If x is a strongly peripheral element of the k-skeleton M and x does not belong to a clonal pair, then either $M\backslash x$ or M/x is a k-skeleton.*

One consequence of Theorem 6.4 is that swirl-like flowers in k-skeletons have free-swirl minors. We turn attention to this now. We first prove a lemma that is a consequence of Tutte's Linking Theorem. In this chapter we only apply the swirl-like case. The spike-like case and the copaddle cases see applications in Chapter 9. Let $\mathbf{P} = (P_1, P_2, \dots, P_l)$ be a flower in the connected matroid M. A clonal pair $\{p_i, p_i'\}$ in P_i is $\mathbf{P}$-*strong* if $\kappa(\{p_i, p_i'\}, P_1 \cup \dots \cup P_{i-1} \cup P_{i+1} \cup \dots \cup P_l) = 2$.

Lemma 6.11. *Let M be a connected matroid with a flower $\mathbf{P} = (P_1, P_2, \dots, P_l)$ where $l \geqslant 3$ that is swirl-like (respectively spike-like, a paddle, or a copaddle). Assume that, for all $i \in \{1, 2, \dots, l\}$, the petal P_i contains a $\mathbf{P}$-strong clonal pair $\{p_i, q_i\}$. Then M contains a Δ_l-minor (respectively Λ_l-, $U_{2,2l}$- or $U_{2l-2,2l}$-minor). Moreover, in the spike-like or swirl-like case, the legs of the Δ_l- or Λ_l-minor are $\{p_1, q_1\}, \{p_2, q_2\}, \dots, \{p_l, q_l\}$.*

Proof. Let s be the first element of $\{1, 2, \dots, l\}$ such that $|P_s| > 2$. Let $Z = E(M) - P_s$. By Tutte's Linking Theorem, there is a minor M' on $Z \cup \{p_s, q_s\}$ such that $M'|Z = M|Z$ and such that $\lambda_{M'}(\{p_s, q_s\}) = 2$. One routinely checks that M' is 3-connected and that $\mathbf{P}' = (P_1, P_2, \dots, P_{s-1}, \{p_s, q_s\}, P_{s+1}, \dots, P_l)$ is a flower in M' having the same type that it does in M. It is easily checked that $\{p_i, q_i\}$ is a $\mathbf{P}'$-strong clonal pair for all $i \in \{1, 2, \dots, l\}$. The lemma now follows routinely. $\square$

Corollary 6.12. *Let M be a k-skeleton and let $l \geqslant 4$ be an integer. If M contains a swirl-like flower of order l, then M has a Δ_l-minor.*

Proof. Let $\mathbf{P} = (P_1, P_2, \dots, P_l)$ be a swirl-like flower of order l in M. Assume that M is not 1-reduced. Then there is an element x in a petal P_j of $\mathbf{P}$ such that, up to duality, $M\backslash x$ is a k-skeleton. By Lemma 5.32, $P_j - \{x\}$ is not a loose petal of the swirl-like quasi-flower $(P_1, \dots, P_{j-1}, P_j - \{x\}, P_{j+1}, \dots, P_l)$ of $M\backslash x$. In other words, the above partition is a tight swirl-like flower of order l in $M\backslash x$. It follows that we lose no generality in assuming that M is 1-reduced. In this case, by Theorem 6.4, each petal of $\mathbf{P}$ contains a clonal pair and it follows from Lemma 6.11 that M has a Δ_l-minor. $\square$

3. A miscellany

We have seen that 1-reduced k-skeletons are quite structured. The next task is to impose further structure on 2-reduced k-skeletons. Before doing that we develop some further terminology and prove a few lemmas that will be used later in this chapter. Let $\mathbf{P} = (P_1, P_2, \dots, P_m)$ be a flower in a matroid M and e be an element of the petal P_i of $\mathbf{P}$. Let N be a 3-connected matroid in $\{M\backslash e, M/e\}$. Then e *opens the flower* $\mathbf{P}$ *in* N or *opens the petal* P_i *of* $\mathbf{P}$ if, for some partition $(P_{i_1}, P_{i_2}, \dots, P_{i_t})$ of $P_i - \{e\}$, the partition $(P_1, P_2, \dots, P_{i-1}, P_{i_1}, \dots, P_{i_t}, P_{i+1}, \dots, P_m)$ is a flower in N whose order is greater than that of $\mathbf{P}$.

The first lemma is an easy consequence of Lemma 3.16.

Lemma 6.13. *Let $\mathbf{P} = (P_1, P_2, \dots, P_m)$ be a swirl-like flower of the k-coherent matroid M, where $m \geqslant 3$. Assume that $x \in P_1$ and that $M\backslash x$ is 3-connected and k-fractured. Let $\mathbf{Q} = (Q_1, Q_2, \dots, Q_n)$ be a k-fracture of $M\backslash x$. Then either*

(i) *the flower* $\mathbf{Q}$ *is obtained by opening the petal* P_i *of* $\mathbf{P}$*, or*
(ii) $P_2 \cup P_3 \cup \cdots \cup P_m \subseteq \hat{Q}_j$ *for some* $j \in \{1, 2, \ldots, n\}$.

Lemma 6.14. *Let* (X, Y) *be a* 3*-separation in a* k*-coherent matroid* M *where* $|X| \geqslant 4$*. Assume that* (X, Y) *does not refine to a swirl-like flower with at least three petals. If* $x \in X$ *and* $M\backslash x$ *is* 3*-connected and* k*-fractured by* $\mathbf{P}$*, then all but one petal of* $\mathbf{P}$ *is contained in* X*.*

Proof. Let $\mathbf{P} = (P_1, P_2, \ldots, P_n)$. We may assume that $\mathbf{P}$ is maximal. Consider the 3-separation $(X - \{x\}, Y)$ in $M\backslash x$. If $X - \{x\} \subseteq \hat{P}_i$ for some i, we obtain a contradiction to the fact that M is k-coherent. Otherwise, on the assumption that the lemma fails, we may assume, by possibly moving to a flower equivalent to $\mathbf{P}$, that there is an $i \in \{2, 3, \ldots, n-2\}$ such that $X - \{x\} = P_1 \cup P_2 \cup \cdots \cup P_i$. But then $(X, P_{i+1}, P_{i+2}, \ldots, P_n)$ is a flower in M that refines (X, Y). $\square$

Recall that a 3-connected matroid M is *uniquely* k*-fractured* if there is a flower $\mathbf{Q}$ such that, for every k-fracture $\mathbf{P}$ of M, we have $\mathbf{P} \preccurlyeq \mathbf{Q}$. In such a case $\mathbf{Q}$ may have order greater than k.

Lemma 6.15. *Let* M *be a matroid with a pair of elements* p *and* q *such that* $\mathbf{P} = (\{p, q\}, P_2, \ldots, P_m)$ *is a maximal tight swirl-like flower where* $m \geqslant 3$*. Say* $f \in P_i$ *for some* $i \in \{2, 3, \ldots, m\}$*, and* f *has the property that* $M\backslash f$ *is not* k*-coherent, but* $M\backslash p/q\backslash f$ *is* k*-coherent. Then the following hold.*

(i) $M\backslash f$ *is* 3*-connected and has a unique* k*-fracture* $\mathbf{Q}$.
(ii) $\mathbf{Q}$ *has order* k.
(iii) $\mathbf{Q}$ *is obtained by opening the petal* P_i *of* $\mathbf{P}$.

Proof. Assume that $M\backslash f$ is not 3-connected. Then, as $\{p, q\}$ is a clonal pair and $M\backslash p/q\backslash f$ is k-coherent and therefore 3-connected, we see that $\{p, q\}$ is a series pair of $M\backslash f$. Thus $\{p, q, f\}$ is a triad of M and hence f is a loose element of $\mathbf{P}$ in the coguts of $\{p, q\}$, which is, up to labels, contained in P_2. But then $(P_2 - \{f\}, P_3 \cup P_4 \cup \cdots \cup P_m)$ is a 2-separation of $M\backslash p/q\backslash f$, contradicting the fact that $M\backslash p/q\backslash f$ is 3-connected. Hence $M\backslash f$ is 3-connected.

Say $f \in P_i$. Observe that $(\{p, q\}, P_2, \ldots, P_{i-1}, P_i - \{f\}, P_{i+1}, \ldots, P_m)$ is a flower in $M\backslash f$ which refines to a maximal flower $\mathbf{Q} = (\{p, q\}, Q_2, \ldots, Q_s)$ in $M\backslash f$. Assume that $\mathbf{R} = (R_1, R_2, \ldots, R_t)$ is a maximal k-fracture of $M\backslash f$ that is not equivalent to $\mathbf{Q}$. By Lemma 3.16, up to equivalence and labels in $\mathbf{R}$, there is an $i \in \{2, 3, \ldots, s\}$ such that $Q_{i+1} \cup Q_{i+2} \cup \cdots \cup Q_s \cup \{p, q\} \cup Q_2 \cup \cdots \cup Q_{i-1} \subseteq R_1$. But, in this case $(R_1 - \{p, q\}, R_2, \ldots, R_t)$ is clearly a k-fracture of $M\backslash p/q\backslash f$. Hence the only possible k-fractures of $M\backslash f$ are flowers equivalent to $\mathbf{Q}$.

Pert (i) of the lemma follows immediately. If $\mathbf{Q}$ has order greater than k, then $(Q_2, Q_3, \ldots, Q_s)$ is a k-fracture of $M\backslash p/q\backslash f$, so (ii) holds. Certainly $\mathbf{Q}$ does not have the property given by Lemma 6.13(ii), so, by that lemma, $\mathbf{Q}$ is obtained by opening the petal P_i of $\mathbf{P}$. Hence part (iii) also holds. $\square$

While we may not be able to remove elements from a 1-reduced k-skeleton to keep a k-skeleton, we can always remove peripheral elements to keep k-coherence.

Lemma 6.16. *Let* M *be a* 1*-reduced* k*-skeleton and let* $\{p, q\}$ *be a clonal pair in a leaf bag* B *of an augmented* 3*-tree for* M*. Then* $M\backslash p$*,* M/p*, and* $M\backslash p/q$ *are* k*-coherent.*

Proof. By Theorem 6.4 B consists of 2-element clonal classes of M. By Corollary 5.26 we may assume, up to duality, that $M\backslash p$ is k-coherent. Assume that $M\backslash p/q$ is not k-coherent. As M is 1-reduced, there is an element f such that either

(a) f is fixed in $M\backslash p$ and the matroid $M\backslash p, f$ is k-coherent, or
(b) f is cofixed in $M\backslash p$ and the matroid $M\backslash p/f$ is k-coherent.

As q is not fixed in $M\backslash p$ and $M\backslash p/q$ is not k-coherent, we see that in either case $f \neq q$. By Theorem 6.4, f is not a clone of q. Now, if f is comparable with q, then by Corollary 5.34 M is not 1-reduced, so f is not comparable with q. In case (a), f is clearly fixed in M. In case (b) by Corollary 5.9, f is cofixed in M. By Theorem 6.4, B consists of 2-element clonal classes of M. Thus, in either case (a) or (b), we see that $f \notin B$.

Let $N = M\backslash p$ if case (a) holds and let $N = M/p$ if case (b) holds. Let $\mathbf{P} = (P_1, P_2, \ldots, P_n)$ be a k-fracture in N. Assume that $B \neq \{p, q\}$ so that $|B| \geqslant 4$. Assume that, for some $i \in \{2, 3, \ldots, n-2\}$, we have $B = P_1 \cup P_2 \cup \cdots \cup P_i$. Then $(P_1, P_2, \ldots, P_i, P_{i+1} \cup \cdots \cup P_n \cup \{f\})$ is a swirl-like flower in M, and this is easily seen to contradict the assumption that B is a peripheral bag of an augmented 3-tree for M. Thus we may assume that $B \subseteq P_1$. But now $P_1 - \{p\}$ contains a clonal pair and cannot be a set of loose elements of the flower $(P_1 - \{p\}, P_2, \ldots, P_n)$ of $N\backslash p$, contradicting the assumption that $N\backslash p$ is k-coherent.

Hence $B = \{p, q\}$. Now B is a 2-element bag of an augmented 3-tree for M. In this case, by Lemma 6.3, there exists a swirl-like, spike-like or Vámos-like flower $\mathbf{Q} = (\{p, q\}, Q_2, \ldots, Q_m)$ in M. For some $i \in \{2, 3, \ldots, m\}$, we have $f \in Q_i$. By Lemma 6.15, there is a partition $(Q_{i_1}, Q_{i_2}, \ldots, Q_{i_t})$ of Q_i such that

$$\big(\{p, q\}, Q_2, Q_3, \ldots, Q_{i-1}, Q_{i_1}, \ldots, Q_{i_t}, Q_{i+1}, \ldots, Q_m\big)$$

is a k-fracture of N. But $N\backslash p/q$ is k-coherent so

$$(Q_2, \ldots, Q_{i-1}, Q_{i_1}, \ldots, Q_{i_t}, Q_{i+1}, \ldots, Q_m)$$

is a swirl-like flower of order $k-1$ in $N\backslash p/q$. However $M\backslash p/q$ is not k-coherent, so by Lemma 4.43, $p \in \mathrm{cl}^{(*)}(\hat{Q})$ for some petal Q of the above flower. This is easily seen to contradict the maximality of $\mathbf{Q}$ in M.

Thus $N\backslash p/q$ is k-coherent and it is routinely verified that so too is N/q. $\square$

The last two lemmas of this section are perhaps oddly placed, but they are close to their first application.

Lemma 6.17. *Let M be a k-coherent matroid with an element z such that $M\backslash z$ is 3-connected with a k-fracture $(\{p_1, p_1'\}, P_2, \ldots, P_l)$. If $\{p_1, p_1'\}$ is fully closed in $M\backslash z$, then $M\backslash p_1$ is k-coherent.*

Proof. By Lemma 4.11, $M\backslash z, p_1$ is 3-connected. Therefore $M\backslash p_1$ is 3-connected.

Assume that $M\backslash z$ is not uniquely k-fractured. Let $(T_1, T_2, \ldots, T_m)$ be another k-fracture of $M\backslash z$. By Lemma 3.16, there is an $i \in \{1, 2, \ldots, l\}$ and a $j \in \{1, 2, \ldots, m\}$ such that $P_1 \cup P_2 \cup \cdots \cup P_{i-1} \cup P_{i+1} \cup \cdots \cup P_l \subseteq \hat{T}_j$, and $T_1 \cup T_2 \cup \cdots \cup T_{j-1} \cup T_{j+1} \cup \cdots \cup T_m \subseteq \hat{P}_i$. Evidently $i \neq 1$, so that $p_1 \in T_j$. Certainly $M\backslash z, p_1$ is 3-connected and $(T_1, T_2, \ldots, T_{j-1}, T_j - \{p_1\}, T_{j+1}, \ldots, T_m)$ is a k-fracture of $M\backslash z, p_1$. If this k-fracture induces a k-fracture $(T_1', T_2', \ldots, T_k')$ in $M\backslash p_1$, then, as $T_j - \{p_1\} \subseteq T_\mu'$, for some μ in $\{1, 2, \ldots, k\}$, we obtain the contradiction that $(T_1', T_2', \ldots, T_{\mu-1}', T_\mu' \cup \{p_1\}, T_{\mu+1}', \ldots, T_k')$ is a k-fracture of M.

From the above we deduce that any k-fracture of $M\backslash p_1$ is induced by the quasi-flower $(\{p_1'\}, P_2, \ldots, P_l)$ of $M\backslash z, p_1$ which we may assume is maximal. Thus, on the assumption that the lemma fails, there is a quasi-flower $\mathbf{O} = (O_1, O_2, \ldots, O_k)$, displayed by the above quasi-flower such that $(O_1, O_2, \ldots, O_{i-1}, O_i \cup \{z\}, O_{i+1}, \ldots, O_k)$ is a k-fracture of $M\backslash p_1$ for some $i \in \{1, 2, \ldots, k\}$. As $(\{p_1'\}, P_2, \ldots, P_l)$ is maximal, and $p_1' \in \mathrm{cl}^*_{M\backslash p_1, z}(P_2)$, we see that $P_2 \cup \{p_1'\} \subseteq \hat{O}_j$ for some $j \in \{1, 2, \ldots, k\}$, and also that $P_l \cup \{p_1'\} \subseteq \hat{O}_s$ for some $s \in \{1, 2, \ldots, k\}$. Note that $p_1 \in \mathrm{cl}(\{p_1'\} \cup P_2)$ and $p_1 \in \mathrm{cl}(\{p_1'\} \cup P_l)$. The only cases that are not almost immediately seen to lead to the contradiction that M is k-fractured are when, up to labels, we have, either (a) $j = s = i-1$ or (b) $j = i-1$ and $s = i+1$. By Lemma 3.31, case (b) leads to the contradiction that $(O_1, O_2, \ldots, O_{i-1}, O_i \cup \{z, p\},$

$O_{i+1}, \ldots, O_k)$ is a k-fracture of M. On the other hand, by considering the structure of swirl-like flowers, we may assume in case (a), that $(P_2 \cup \{p'_1\}) \cap \hat{O}_i = \emptyset$, so that, for some flower $(O'_1, O'_2, \ldots, O'_k)$, equivalent to $\mathbf{O}$, the partition $(O'_1, O'_2, \ldots, O'_{i-1} \cup \{p_1\}, O'_1 \cup \{z\}, O'_{i+1}, \ldots, O'_k)$ is a k-fracture of M. The lemma follows from this final contradiction. $\square$

We omit the proof of the next lemma which amounts to little more than observing the properties of a feral display.

Lemma 6.18. *Let f be a feral element of the k-coherent matroid M.*

(i) *If f blocks two petals of a k-fracture of $M\backslash f$, then there is a feral display for f in M^*.*
(ii) *If f coblocks two petals of a k-fracture of M/f, then there is a feral display for f in M.*

4. 2-Reduced skeletons

For 2-reduced skeletons we can strengthen the outcome of Theorem 6.4 somewhat. We say that a clonal pair $\{p, q\}$ of a 3-connected matroid M is *strongly peripheral* if it is contained in a leaf bag of an augmented 3-tree for M. Let $\{p, q\}$ be a strongly peripheral clonal pair of the 2-reduced k-skeleton M. For the remainder of this chapter, if $\{p, q\}$ is a clonal pair of M, then the matroid $M\backslash p/q$ will be denoted by M_{pq}. By Lemma 6.16, M_{pq} is k-coherent. As M_{pq} is not a k-skeleton, there is an element f such that either f is fixed in M_{pq} and $M_{pq}\backslash f$ is k-coherent, in which case we say that f is *pq-annoying for deletion*, or f is cofixed in M_{pq} and M_{pq}/f is k-coherent, in which case we say that f is *pq-annoying for contraction*. If either one of the cases holds we say that f is *pq-annoying*.

Lemma 6.19. *Let M be a 2-reduced k-skeleton and let $\{p, q\}$ be a strongly peripheral clonal pair of M.*

(i) *If f is pq-annoying for deletion, then f is fixed in M and $M\backslash f$ is 3-connected and k-fractured.*
(ii) *If f is pq-annoying for contraction, then f is cofixed in M and M/f is 3-connected and k-fractured.*

Proof. Assume that f is pq-annoying for deletion. Assume that $M\backslash f$ is not 3-connected. Then, as $M_{pq}\backslash f$ is 3-connected, $\{p, q, f\}$ is a triad of M and hence a clonal triple so that M is not 2-reduced. Hence $M\backslash f$ is 3-connected. Certainly f is not comparable with either p or q so, by Corollary 5.7, f is fixed in M. It now follows from the definition of k-skeleton that $M\backslash f$ is k-fractured. Thus (i) holds. Part (ii) is the dual of (i). $\square$

Lemma 6.20. *Let M be a 2-reduced k-skeleton and let B be a leaf bag of an augmented 3-tree for M. Then the following hold.*

(i) *B consists of a single clonal pair $\{p, q\}$.*
(ii) *M has a tight, maximal, swirl-like flower $\mathbf{P} = (B, P_2, \ldots, P_m)$ for some $m \geqslant 3$.*
(iii) *If f is pq-annoying for deletion, then f opens the flower $\mathbf{P}$ in $M\backslash f$.*

Proof. By Theorem 6.4, B consists of clonal classes of size 2. Let $\{p, q\}$ be a clonal pair contained in B. As M is 2-reduced, there is an element f of M_{pq} that is pq-annoying. We lose no generality in assuming that f is pq-annoying for deletion. By Lemma 6.19, $M\backslash f$ is 3-connected and k-fractured. Let $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_n)$ be a maximal k-fracture of $M\backslash f$.

Assume that $B \neq \{p, q\}$ so that $|B| \geqslant 4$. Assume that, for some $i \in \{2, 3, \ldots, n-2\}$ we have $B = Q_1 \cup Q_2 \cup \cdots \cup Q_i$. Then $(Q_1, Q_2, \ldots, Q_i, Q_{i+1} \cup \cdots \cup Q_n \cup \{f\})$ is a swirl-like flower in M, and this is easily seen to contradict the assumption that B is a peripheral bag of an augmented 3-tree for M. Thus we may assume that $B \subseteq Q_1$. But now $Q_1 - \{p, q\}$ contains a clonal pair of the 3-connected matroid $M_{pq}\backslash f$ and cannot be a set of loose elements of the flower $(Q_1 - \{p, q\}, Q_2, \ldots, Q_n)$ of $M_{pq}\backslash f$ contradicting the fact that $M_{pq}\backslash f$ is k-coherent. Therefore $B = \{p, q\}$ so that (i) holds.

By Lemma 6.3 M has a tight flower $\mathbf{P} = (\{p, q\}, P_2, \ldots, P_m)$ where $m \geqslant 3$. Say $f \in P_i$. By Lemma 6.15, any k-fracture of $M\backslash f$ is obtained by opening $\mathbf{P}$. This implies that $\mathbf{P}$ is swirl-like. (If $m = 3$, then $\mathbf{P}$ will be ambiguous.) Parts (ii) and (iii) of the lemma follow from these observations. $\square$

5. Removing a bogan couple

Our goal is to show that in a 2-reduced k-skeleton we can always find a 3- or 4-element move that preserves the property of being a k-skeleton. In this section we show that bogan couples lead to a good outcome in that, if a pq-annoying element belongs to a bogan couple, then we have a 4-element move.

Lemma 6.21. *Let $\{p, q\}$ be a strongly peripheral clonal pair of the 2-reduced k-skeleton M and let a be an element of M_{pq} that is pq-annoying. If a belongs to a bogan or cobogan couple $\{a, b\}$, then $M_{pq}\backslash a/b$ is a k-skeleton.*

Proof. Up to duality we may assume that $\{a, b\}$ is a bogan couple. Associated to the bogan couple $\{a, b\}$ is a partition $(R, S, T, \{a, b\})$ together with partitions $(R_1, R_2, \ldots, R_{k-2})$, $(S_1, S_2, \ldots, S_r)$ and $(T_1, T_2, \ldots, T_{k-2})$ of R, S and T respectively that form a bogan display for $\{a, b\}$. Note that we have labelled the display just as in the definition of bogan couple. The case where a is pq-annoying for deletion is not quite symmetric to the case where a is pq-annoying for contraction—or at least it requires argument to show that it is—but essentially the same proof works in either case. We prove the lemma in the case that a is pq-annoying for deletion. In this case $(R_1 \cup \{b\}, R_2, \ldots, R_{k-2}, S_1, \ldots, S_r, T)$ is a maximal swirl-like flower in $M\backslash a$ and b is in the coguts of the petals $R_1 \cup \{b\}$ and T. Note that, if $r > 1$, then the above partition with $\{p, q\}$ removed induces a k-fracture of $M_{pq}\backslash a$. Hence $r = 1$, so that $S = S_1$ and the flower is $\mathbf{F} = (R_1 \cup \{b\}, R_2, \ldots, R_{k-2}, S, T)$.

6.21.1. $\{p, q\} = R_i$ *for some* $i \in \{1, 2, \ldots, k-2\}$.

Subproof. As $\{p, q\}$ is a clonal pair, this set is contained in a petal of $\mathbf{F}$. Moreover this petal becomes a, possibly empty, set of loose elements in the induced flower in $M_{pq}\backslash a$. Given this, it is clear that $\{p, q\}$ is not contained in T. As $\{p, q\}$ is strongly peripheral and M is 2-reduced, by Lemma 6.20, there is a flower $\mathbf{O}$ displayed in an augmented 3-tree for M in which $\{p, q\}$ is a petal. Say $\{p, q\} \subseteq S$. The flower $\mathbf{O}$ conforms with the maximal flowers displayed by the bogan display for $\{a, b\}$ and it follows that $E(M) - S$ is contained in a petal of $\mathbf{O}$. But then $E(M) - S$ contains a strongly peripheral set other than $\{p, q\}$ and, again by Lemma 6.20, we see that $E(M) - S$ contains a clonal pair of M other than $\{p, q\}$ and is hence not a set of loose elements of a swirl-like flower in $M_{pq}\backslash a$. Thus $\{p, q\} \subseteq R_i$ for some $i \in \{1, 2, \ldots, k-2\}$. If R_i is not a petal of $\mathbf{F}$, then the previous argument applies with S replaced by R_i. Thus R_i is a petal of $\mathbf{O}$, that is $\{p, q\} = R_i$. $\square$

6.21.2. $M_{pq}\backslash a/b$ *is* 3*-connected.*

Subproof. Certainly $M_{pq}\backslash a$ is 3-connected. Let $(R'_1, R'_2, \ldots, R'_{k-3}) = (R_1, \ldots, R_{i-1}, R_{i+1}, \ldots, R_{k-2})$. Note that b is in the coguts of the petals R'_1 and T of the swirl-like flower $(R'_1 \cup \{b\}, R'_2, \ldots, R'_{k-3}, S, T)$ of $M_{pq}\backslash a/b$. Thus $M_{pq}\backslash a/b$ is 3-connected unless b is in a triangle $\{b, r, t\}$ where $r \in R'_1$ and $t \in T$. If $R_1 \neq \{p, q\}$, then $R'_1 = R_1$ and we deduce that b is in a triangle of M contradicting the fact that M/b is 3-connected. If $R_1 = \{p, q\}$, then $r \in \mathrm{cl}_M(\{p, q\})$ and $\{p, q, r\}$ is a triangle of M contradicting the fact that M is 2-reduced. $\square$

Assume that $M_{pq}\backslash a/b$ is not a k-skeleton. Then there is an element h such that either (i) h is fixed in $M_{pq}\backslash a/b$ and $M_{pq}\backslash a/b\backslash h$ is k-coherent, of (ii) h is cofixed in $M_{pq}\backslash a/b$ and $M_{pq}\backslash a/b, h$ is k-coherent. In what follows we assume that (i) holds noting that the argument in the case that (ii) holds is essentially identical.

6.21.3. *h is fixed in M.*

Subproof. Assume that h is not fixed in M. Let M' be a matroid obtained by independently cloning h by h'. For a set $Z \subseteq E(M)$, let Z' denote $Z \cup \{h'\}$ if $h \in Z$ and otherwise $Z' = Z$. It is easily checked that the bogan display for $\{a,b\}$ in M extends to a bogan display in M' where each member Z of the partitions of the display in M is replaced by Z'. But then one readily checks that $\{h,h'\}$ is independent in $M'\backslash p/q\backslash a/b$ so that h is not fixed in $M_{pq}\backslash a/b$. Thus h is indeed fixed in M. $\square$

We omit the routine verification of the next claim.

6.21.4. *$M\backslash h$ is 3-connected.*

As h is fixed in M and $M\backslash h$ is 3-connected, $M\backslash h$ is k-fractured. We will obtain a contradiction by showing that there is no sensible place for h. Assume first that $h \in R_j$ for some $j \in \{1,\dots,k-2\}$. Assume that a k-fracture for $M\backslash h$ is obtained by opening the petal R_j of the flower $(R_1, R_2, \dots, R_{k-2}, S \cup T \cup \{a,b\})$ of M. Say $(R_1,\dots,R_{j-1},R_{j1},\dots,R_{jl},R_{j+1},\dots,R_{k-2},S \cup T \cup \{a,b\})$ is such a k-fracture. In this case $(R_1,\dots,R_{j-1},R_{j1},\dots,R_{jl},R_{j+1},\dots,R_{k-2},S,T \cup \{b\})$ is a k-fracture of $M\backslash a$ whose order is greater than k and it follows that $M_{pq}\backslash a/b\backslash h$ is k-fractured. Otherwise $E(M) - R_j$ is contained in a petal of a k-fracture for $M\backslash h$ and again it follows that $M_{pq}\backslash a/b\backslash h$ is k-fractured.

Thus $h \notin R$ and, similarly, $h \notin T$. Hence $h \in S$. We next show that $|S| \geqslant 3$. Assume for a contradiction that $|S| = 2$, say $S = \{h,h'\}$. Then $(R_1 \cup \{b\}, R_2, \dots, R_{k-2}, \{h,h'\}, T)$ is a k-fracture of $M\backslash a$. One readily checks that $\{h,h'\}$ is fully closed in $M\backslash a$. Lemma 6.17 now applies and it follows from that lemma that $M\backslash h$ is k-coherent. Thus $|S| \geqslant 3$ as desired. As $M\backslash h$ is 3-connected and S is 3-separating in M we have $h \in \mathrm{cl}(S - \{h\})$.

If $E(M) - S$ is contained in a petal of a k-fracture for $M\backslash h$, then again it follows easily that $M_{pq}\backslash a/b\backslash h$ is k-fractured. Otherwise there is a partition (Z_1, Z_2) of $E(M) - S$ such that $\lambda_{M\backslash h}(Z_1) = \lambda_{M\backslash h}(Z_2) = 2$. As $h \in \mathrm{cl}(S - \{h\})$, neither Z_1 nor Z_2 is blocked by h so that $\lambda_M(Z_1) = \lambda_M(Z_2) = 2$, that is (S, Z_1, Z_2) is a flower in M. Such a partition contradicts the maximality of the flowers displayed in a bogan display. The lemma follows from this final contradiction. $\square$

6. Life in 2-reduced skeletons

In this section we obtain more information about the structure associated with a strongly peripheral clonal pair in a 2-reduced k-skeleton. Throughout this section $\{p,q\}$ will denote a peripheral clonal pair of the 2-reduced k-skeleton M. As M is 2-reduced, there is an element f of M that is pq-annoying. Up to duality we may assume that f is pq-annoying for deletion. By Lemma 6.20, there is a swirl-like flower $\mathbf{Q} = (Q_1, Q_2, \dots, Q_k)$ in $M\backslash f$, and an $s \in \{2,3,\dots,k-2\}$ such that

(i) $Q_t = \{p,q\}$ for some $t \in \{s+1, s+2, \dots, k\}$, and
(ii) $(Q_1 \cup Q_2 \cup \dots \cup Q_s \cup \{f\}, Q_{s+1}, \dots, Q_k)$ is a maximal swirl-like flower of M.

Let $Q_1' = Q_1 \cup Q_2 \cup \dots \cup Q_s \cup \{f\}$, and let $\mathbf{Q}' = (Q_1', Q_{s+1}, \dots, Q_k)$. As with M and $\{p,q\}$, the element f and the flowers $\mathbf{Q}$ and $\mathbf{Q}'$ will be fixed throughout this section.

We say that the element g is *f-bad for deletion* if f is fixed in $M_{pq}\backslash f$ and $M_{pq}\backslash f, g$ is k-coherent, and g is *f-bad for contraction* if f is cofixed in $M_{pq}\backslash f$ and $M_{pq}\backslash f/g$ is k-coherent. The element g is *f-bad* if it is f-bad for either deletion or contraction. The next lemma follows from the definition of k-skeleton.

Lemma 6.22. *If $M_{pq}\backslash f$ is not a k-skeleton, then $M_{pq}\backslash f$ has an element that is f-bad.*

Lemma 6.23. *If g is f-bad for deletion, then g is fixed in M, and if g is f-bad for contraction, then g is cofixed in M.*

Proof. Assume that g is f-bad for deletion. Then g is fixed in $M_{pq}\backslash f$ so that g is certainly fixed in M/p. But p is not comparable with g so g is fixed in M. Assume that g is f-bad for contraction. Then g is cofixed in $M_{pq}\backslash f$ so that g is cofixed in $M\backslash p, f$. If g is not cofixed in $M\backslash f$, then, by Corollary 5.7, $p \succcurlyeq g$ in $M\backslash f$ implying that $\{p, q, g\}$ is a triangle of M. Hence g is cofixed in $M\backslash f$. But as g is not comparable with any element of M, we see, by Corollary 5.9, that g is cofixed in M. $\square$

Assume that g is f-bad. If g is f-bad for deletion, then the symbol $*$ will denote deletion and if g is f-bad for contraction, then $*$ will denote contraction.

Lemma 6.24. *If g is f-bad, then $M * g$ and $M\backslash f * g$ are 3-connected.*

Proof. Assume that $M\backslash f * g$ is not 3-connected. Consider the 3-connected matroid $M\backslash f$. There is a 3-separation $(X \cup \{g\}, Y)$ of $M\backslash f$ for which g is either in the guts or the coguts according as to whether $*$ is contraction or deletion. For some $i \in \{1, 2, \dots, k\}$, we have $g \in Q_i$. We may assume that Q_i is fully closed. As $\mathbf{Q}$ is maximal, $(X \cup \{g\}, Y)$ conforms with $\mathbf{Q}$. Thus we may assume that $X \cup \{g\} \subseteq Q_i$. But then $(X, Y - \{p, q\})$ is clearly a 2-separation of $M_{pq}\backslash f * g$ contradicting the fact that this matroid is 3-connected. It follows from this contradiction that $M\backslash f * g$ is 3-connected.

Consider $M * g$. Assume that $M * g$ is not 3-connected. As $M\backslash f * g$ is 3-connected, $M * g$ has a single parallel pair containing f. Thus $*$ is contraction and M has a triangle T containing f and g. As g is cofixed in M, the triangle T is k-wild. Now $\mathrm{si}(M/g)$ is a 3-connected matroid, so, by Lemma 4.22, T is costandard. But it follows easily from the definition of k-wild display that, in this case, $M\backslash f/g$ has a swirl-like flower whose order is greater than k. But this implies that $M_{pq}\backslash f/g$ is not k-coherent. Therefore $M * g$ is also 3-connected. $\square$

The next lemma follows immediately from Lemmas 6.23 and 6.24.

Lemma 6.25. *If g is f-bad, then $M * g$ is 3-connected and k-fractured.*

Next we gain a little more information about the location of f-bad elements.

Lemma 6.26. *If g is f-bad, then $g \in Q_1 \cup Q_2 \cup \dots \cup Q_s$.*

Proof. Recall that $\mathbf{Q}'$ denotes the flower $(Q'_1, Q_{s+1}, \dots, Q_k)$ of M, where $Q'_1 = Q_1 \cup Q_2 \cup \dots \cup Q_s \cup \{f\}$. Assume that $g \in Q_j$, where $j \in \{s+1, s+2, \dots, k\}$. Assume that a k-fracture of $M * g$ is obtained by opening the petal Q_j of $\mathbf{Q}'$. Then the flower $(Q_1, Q_2, \dots, Q_{j-1}, Q_j - \{g\}, Q_{j+1}, \dots, Q_k)$ of $M\backslash f * g$ expands to a swirl-like flower whose order is strictly greater than k. It follows from this that $M_{pq}\backslash f * g$ is k-fractured. On the other hand, by Lemma 6.13, $E(M) - Q_j$ is contained in a petal P_1 of a k-fracture $(P_1, P_2, \dots, P_k)$ of $M * g$. In this case it is clear that $P_1 - \{p, q, f\}$ is not a set of loose elements of the swirl-like flower $(P_1 - \{p, q, f\}, P_2, \dots, P_k)$ of $M_{pq}\backslash f * g$ and again we see that this matroid is k-fractured. Thus $g \in Q'_1$, that is, $g \in Q_1 \cup Q_2 \cup \dots \cup Q_s$. $\square$

One possibility that leads to a good outcome is when f is semi-feral.

Lemma 6.27. *If f is semi-feral, then $M_{pq}\backslash f$ is a k-skeleton.*

Proof. Assume that f is semi-feral and assume that $M_{pq}\backslash f$ is not a k-skeleton. By Lemma 6.22, there is an element g that is f-bad.

We first prove that f is not in a costandard k-wild triangle. For ease of notation we relabel f to a and assume that a belongs to the costandard k-wild triangle $\{a, b, c\}$. With notation as in the definition of costandard k-wild triangle, we see that $M\backslash a$ has a unique k-fracture $(A_1, A_2, \dots, A_{k-2}, B \cup \{b\}, C \cup \{c\})$. As $\{a, b, c\}$ is costandard, the elements b and c are in the coguts of $B \cup \{b\}$ and $C \cup \{c\}$ respectively. As $\mathbf{Q}'$ and $(A_1, A_2, \dots, A_{k-2}, B \cup C \cup \{a, b, c\})$ are flowers in M

and both have petals that open to k-fractures in the uniquely k-fractured matroid $M\backslash a$, we deduce that $s = 2$, and that $(A_1, A_2, \dots, A_{k-2}, B \cup \{b\}, C \cup \{c\}) = (Q_3, Q_4, \dots, Q_k, Q_1, Q_2)$. Hence $\{p, q\} = A_i$ for some $i \in \{1, 2, \dots, k-2\}$. By Lemma 6.26, $g \in B \cup C \cup \{b, c\}$. Up to symmetry, the f-bad (that is a-bad) element g belongs to $B \cup \{b\}$. Say $g = b$. In this case M/g is not 3-connected, so, by Lemma 6.25, g must be f-bad for deletion. But g is in the coguts of B in $M\backslash a$, so $M\backslash a, g$ is not 3-connected, contradicting Lemma 6.24. Thus $g \neq b$. Recall that there is a partition $(B_1, B_2, \dots, B_k)$ of B such that $(B_1, B_2, \dots, B_k, A \cup C \cup \{a, b, c\})$ is a swirl-like flower of M. We have $g \in B_i$ for some $i \in \{1, 2, \dots, k-2\}$. By Lemma 6.13, a k-fracture for $M * g$ is either obtained by opening the petal B_i of the flower $(B_1, B_2, \dots, B_k, A \cup C \cup \{a, b, c\})$ of M, or has the property that $E(M) - B_i$ is contained in a petal of this k-fracture. In either case it is easily deduced that $M_{pq}\backslash a * f = M\backslash g\backslash p/q\backslash a$ is either not 3-connected or is k-fractured. It follows that f is not in a costandard k-wild triangle.

As f is not in a costandard k-wild triangle, by Lemma 6.8, f is in the guts of a vertical 3-separation (X, Y) of M, where $f \in X$. If (X, Y) crosses $(Q_1', Q_{s+1} \cup Q_{s+2} \cup \dots \cup Q_k)$, then it is easily seen that we either contradict the maximality of the flower $\mathbf{Q}'$ in M, or we find that $\mathbf{Q}'$ is not canonical, contradicting Lemma 5.32. If X properly contains Q_1', then, again as $\mathbf{Q}'$ is maximal, we deduce that $Y = Q_i \cup Q_{i+1} \cup \dots \cup Q_j$, where, up to labels, $s < i \leqslant j \leqslant k$. But then, as $x \in \mathrm{cl}(Y)$, we see again that f is loose in $\mathbf{Q}'$, contradicting Lemma 5.32. Thus $X \subseteq Q_1'$. Now $\lambda_{M\backslash f}(X - \{f\}) = 2$. If $X - \{f\} \subseteq Q_i$ for some $i \in \{1, 2, \dots, s\}$, then we obtain the contradiction that M is k-fractured. Otherwise, as $\mathbf{Q}'$ is maximal, we see that, up to labels, $X - \{f\} = Q_i \cup Q_{i+1} \cup \dots \cup Q_j$, where $1 \leqslant i \leqslant j < s$, so that $(Q_1 \cup Q_2 \cup \dots \cup Q_{s-1}, Q_s, Q_{s+1}, \dots, Q_k)$ is a flower in M, contradicting the maximality of $\mathbf{Q}'$. Therefore $X = Q_1'$. It follows that $f \in \mathrm{cl}(Q_{s+1} \cup Q_{s+2} \cup \dots \cup Q_k)$ and hence $(Q_1, Q_2, \dots, Q_s, Q_{s+1} \cup \dots \cup Q_k \cup \{f\})$ is a swirl-like flower in M.

Consider the f-bad element g. By Lemma 6.26 $g \in Q_i$ for some $i \in \{1, 2, \dots, s\}$. By Lemma 6.13, a k-fracture for $M * g$ either opens the petal Q_i of $(Q_1, Q_2, \dots, Q_s, Q_{s+1} \cup \dots \cup Q_k \cup \{f\})$ or has the property that $E(M) - Q_i$ is contained in a petal of a k-fracture of $M * g$. Routine arguments show that both cases lead to the contradiction that $M_{pq}\backslash f * g$ is not k-coherent. □

The next lemma follows from Lemma 6.27 and Theorem 6.4.

Lemma 6.28. *If $M_{pq}\backslash f$ is not a k-skeleton, then f is either a feral element or is in a gang of three.*

It is good news if M has a gang of three since, by Theorem 5.42, we can remove the whole gang of three and keep the property of being a k-skeleton. We also know that if f belongs to a bogan couple we can obtain a 4-element reduction. The assumption in the next lemma that M has no gangs or cogangs of three and that f does not belong to a bogan couple is probably not necessary, but it does simplify the argument a little.

Lemma 6.29. *Assume that M has no gangs of cogangs of three and that f is not in a bogan couple. If g is f-bad, then any k-fracture of $M * g$ is obtained by opening the petal Q_1' of $\mathbf{Q}'$. Moreover $M * g$ is uniquely k-fractured.*

Proof. By Lemma 6.25, $M * g$ is 3-connected and k-fractured. By Lemma 6.26, $g \in Q_1'$. Let $\mathbf{R} = (R_1, R_2, \dots, R_u)$ be a k-fracture of $M * g$. Assume that $\mathbf{R}$ is not obtained by opening the petal Q_1' of $\mathbf{Q}$. By Lemma 6.13, we may assume, up to labels, that R_1 is fully closed and that $R_1 \supseteq Q_{s+1} \cup Q_{s+2} \cup \dots \cup Q_k$. Moreover it is routinely seen that $u = k$.

For some $i \in \{1, 2, \dots, s\}$, we have $g \in Q_i$.

6.29.1. $|Q_i| > 2$.

Subproof. Assume that $|Q_i| = 2$. As f is not in a bogan couple, the flower $\mathbf{Q}$ is canonical, so that Q_i is fully closed. Now, by Lemma 6.17, $M\backslash g$ is k-coherent. The claim follows from this contradiction. □

Assume that $f \in R_j$.

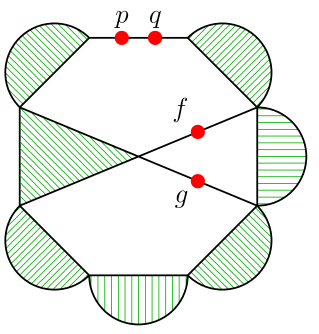

Fig. 6.2. Illustration of a 4-element move.

6.29.2. *The operation $*$ is contraction and $|R_j| = 2$.*

Subproof. By Lemma 6.24, $M\backslash f * g$ is 3-connected and by 6.29.1, $|Q_i| > 2$. Hence $(Q_1, Q_2, \dots, Q_{i-1}, Q_i - \{g\}, Q_{i+1}, \dots, Q_k)$ is a flower in $M\backslash f * g$. It follows easily that, if $*$ is deletion, then $g \in \mathrm{cl}_{M\backslash f}(Q_i)$ and, if $*$ is contraction, then $g \in \mathrm{cl}^*_{M\backslash f}(Q_i)$.

Note that $(R_1, R_2, \dots, R_{j-1}, R_j - \{f\}, R_{j+1}, \dots, R_k)$ is a quasi-flower in $M\backslash f * g$. As $R_1 \supseteq Q_{s+1} \cup Q_{s+2} \cup \dots \cup Q_k$, and $M_{pq}\backslash f * g$ is k-coherent, we see that $R_j - \{f\}$ is a set of loose elements of $(R_1, R_2, \dots, R_{j-1}, R_j - \{f\}, R_{j+1}, \dots, R_k)$. Assume that $Q_i - \{g\}$ is not contained in R_1. Then $(R_2 \cup R_3 \cup \dots \cup R_k) - \{f\} \subseteq Q_i$. But then one readily concludes that $f \in \mathrm{cl}_M(Q_i)$ giving the contradiction that M is k-fractured. Therefore $Q_i - \{g\} \subseteq R_1$.

By the observation that $g \in \mathrm{cl}_{M\backslash f}(Q_i)$, or $g \in \mathrm{cl}^*_{M\backslash f}(Q_i)$, according as to whether $*$ is deletion or contraction, we deduce that $(R_1 \cup \{g\}, R_2, \dots, R_{j-1}, R_j - \{f\}, R_{j+1}, \dots, R_k)$ is a flower in $M\backslash f$.

Now $f \in \mathrm{cl}_{M*g}(R_j - \{f\})$. If $f \in \mathrm{cl}_M(R_j - \{f\})$, then $(R_1 \cup \{g\}, R_2, \dots, R_k)$ is a k-fracture of M. Hence $*$ is contraction.

It is also the case that $R_j - \{f\}$ is a set of loose elements in the flower $(R_1 \cup \{g\}, R_2, \dots, R_{j-1}, R_j - \{f\}, R_{j+1}, \dots, R_k)$ of $M\backslash f$. Let l be an initial element of $R_j - \{f\}$ in this flower. For some h, we have $l \in Q_h$. Assume that l is a guts element. Then, by Lemma 4.13, l does not expose any 3-separations in $M\backslash f, l$. This means that $(Q_1, Q_2, \dots, Q_{h-1}, Q_h - \{l\}, Q_{h+1}, \dots, Q_k)$ is a maximal flower in $M\backslash f, l$. Moreover, either $M\backslash f, l$ is k-coherent or this is a unique k-fracture of $M\backslash f, l$. It follows from either Lemma 4.43 or Corollary 4.46 that $M\backslash l$ is k-coherent. But it is readily seen that l is fixed in M contradicting the definition of a k-skeleton. Thus l is a coguts element.

Assume that $|R_j| > 2$. Note that $f \in \mathrm{cl}_{M/g}(R_j - \{f\})$. Again, by Lemma 4.13, l does not expose any 3-separations in $M\backslash f/l$, so that $(Q_1, Q_2, \dots, Q_{h-1}, Q_h - \{l\}, Q_{h+1}, \dots, Q_k)$ is a maximal flower in $M\backslash f/l$. But l is cofixed in $M\backslash f$ and f is not comparable with any element of M, so l is cofixed in M. Therefore M/l is k-fractured. Again, by Lemma 4.43 or Corollary 4.46, we see that f is in the closure in M/l of a petal of the above flower. Recall that $g \in Q_i$ and that $f \in \mathrm{cl}_{M/g}(R_j - \{f\})$. From these facts it follows that we must have $f \in \mathrm{cl}_{M/l}(Q_i)$, so that $l \in \mathrm{cl}_M(Q_i \cup \{f\})$. Let l' be the other end of $R_j - \{f\}$, then it is also the case that $l' \in \mathrm{cl}_M(Q_i \cup \{f\})$ and, indeed, that $l' \in \mathrm{cl}_M(Q_i \cup \{l\})$. But $Q_i - \{g\} \subseteq R_1$, and $(R_1, R_2, \dots, R_{j-1}, R_j - \{f\}, R_{j+1}, \dots, R_k)$ is a swirl-like flower. By Lemma 3.9, $l' \notin \mathrm{cl}_{M/g}(R_1 \cup \{l\})$, so $l' \notin \mathrm{cl}_M(Q_1 \cup \{l\})$. From this contradiction we can finally deduce that $|R_j| = 2$. $\square$

By 6.29.2, f is in a fully closed 2-element petal of a k-fracture of M/g. But f is feral and we have a contradiction to Lemma 6.17. $\square$

7. The chain theorem

At last we are in a position to prove the more detailed version of Theorem 6.1. Fig. 6.2 illustrates one situation where a 4-element move is needed. This is the type of situation that the proof of Theorem 6.30 converges to.

Theorem 6.30. *Let M be a nonempty k-skeleton. Then at least one of the following holds.*

(i) *There is an element e such that either $M\backslash e$ or M/e is a k-skeleton.*
(ii) *There is a clonal pair $\{p,q\}$ such that $M\backslash p/q$ is a k-skeleton.*
(iii) *There is a gang of three $\{r,s,t\}$ such that $M/r\backslash s,t$ is a k-skeleton.*
(iv) *There is a cogang of three $\{r,s,t\}$ such that $M\backslash r/s,t$ is a k-skeleton.*
(v) *The ground set of M consists of clonal classes of size two, feral elements and semi-feral elements.*

Moreover, in the case that (v) *holds, but* (i)–(iv) *do not, then every leaf bag of an augmented* 3*-tree for M contains exactly one clonal pair, and for any such clonal pair $\{p,q\}$ at least one of the following holds.*

(vi) *There is a feral or semi-feral element f such that either $M\backslash p/q\backslash f$ or $M\backslash p/q/f$ is a k-skeleton.*
(vii) *There is a pair f, g of feral elements of M such that $M\backslash p/q\backslash f/g$ is a k-skeleton.*

For the remainder of this section we assume that we are under the hypotheses of Theorem 6.30. We also assume that none of (i), (ii), (iii) or (iv) holds. By Theorems 6.4 and 5.42, (v) holds. Moreover M is 2-reduced, so by Lemma 6.20, every leaf bag of an augmented 3-tree for M consists of a single clonal pair. Let $\{p,q\}$ be a peripheral clonal pair of M. It remains to prove that either (vi) or (vii) holds. As (ii) does not hold there is an element that is pq-annoying. By Lemmas 6.27 and 6.21 we may also assume the following.

(a) Any element that is pq-annoying is feral.
(b) No element that is pq-annoying belongs to a bogan couple.

Let f be an element of M that is pq-annoying. Up to duality we may assume that f is pq-annoying for deletion. We are now effectively under the hypotheses of Section 6 and in what follows we use the notational conventions of that section. Thus there is a swirl-like flower $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_k)$ in $M\backslash f$, and an $s \in \{2,3,\ldots,k-2\}$ such that

(i) $Q_t = \{p,q\}$ for some $t \in \{s+1,\ldots,k\}$, and
(ii) $(Q_1 \cup Q_2 \cup \cdots \cup Q_s \cup \{f\}, Q_{s+1}, \ldots, Q_k)$ is a maximal swirl-like flower of M.

We also let $Q_1' = Q_1 \cup Q_2 \cup \cdots \cup Q_s \cup \{f\}$ and $\mathbf{Q}' = (Q_1', Q_{s+1}, \ldots, Q_k)$.

Assume that (vi) does not hold. Then there is an element g that is f-bad. By, Lemma 6.26, $g \in Q_i$ for some $i \in \{1,2,\ldots,s\}$. Let $\mathbf{Q}'' = (Q_1, Q_2, \ldots, Q_{i-1}, Q_i - \{g\}, Q_{i+1}, \ldots, Q_k)$.

Lemma 6.31. $\mathbf{Q}''$ *uniquely k-fractures $M\backslash f * g$.*

Proof. Consider the quasi-flower $\mathbf{Q}''$ of $M\backslash f * g$. If this refines to a flower of order greater than k, then it is clear that $M_{pq}\backslash f * g$ is k-fractured. Hence either $\mathbf{Q}''$ is a maximal flower of order k in $M\backslash f * g$, or $Q_i - \{g\}$ is a set of loose elements of this flower. Consider the former case. Then, either the lemma holds, of there is another k-fracture of $M\backslash f * g$. But then $M_{pq}\backslash f * g$ is certainly not k-coherent. Thus, if the lemma fails, the latter case holds. Assume that we are in this case.

The flower $\mathbf{Q}'' = (Q_1, Q_2, \ldots, Q_{i-1}, Q_i - \{g\}, Q_{i+1}, \ldots, Q_k)$ has order $k-1$. By Lemma 6.29, a k-fracture of $M * g$ is obtained by opening the petal Q_1' of $\mathbf{Q}'$. Thus $\mathbf{Q}''$ induces a k-fracture in $M * g$. By Lemma 4.44 there is a set Q' of loose elements of $\mathbf{Q}''$ that has the property that $Q' \cup \{f\}$ is a tight petal of the induced k-fracture in $M * g$. As f is not in a bogan couple, the flower $\mathbf{Q}$ is canonical, so the only loose elements of $\mathbf{Q}''$ are in $Q_i - \{g\}$. Therefore $Q' = Q_i - \{g\}$, so that $(Q_1, Q_2, \ldots, Q_{i-1}, (Q_i - \{g\}) \cup \{f\}, Q_{i+1}, \ldots, Q_k)$ is a k-fracture of $M * g$, giving the contradiction that $(Q_1, Q_2, \ldots, Q_{i-1}, Q_i \cup \{f\}, Q_{i+1}, \ldots, Q_k)$ is a k-fracture of M. The lemma follows from this contradiction. $\square$

As $\mathbf{Q}''$ uniquely fractures $M\backslash f * g$, and $M * g$ is k-fractured, there is a petal Q of a flower equivalent to $\mathbf{Q}'$ such that $f \in \mathrm{cl}_{M*g}(Q)$. Certainly $Q \neq \{p,q\}$ and it follows that $M_{pq} * g$ is k-coherent. This establishes the next lemma.

Lemma 6.32. *The element g is pq-annoying and hence g is feral and not in a bogan couple. Consequently a k-fracture of $M * g$ is canonical.*

Lemma 6.33. *The element g is f-bad for contraction.*

Proof. Assume that g is f-bad for deletion. By Lemma 6.31, $\mathbf{Q}''$ uniquely k-fractures $M\backslash f, g$. But $M\backslash g$ is k-fractured, so f is in the closure of a petal of some flower equivalent to $\mathbf{Q}''$. One consequence of this is that $s = 2$ (otherwise either $f \in \mathrm{cl}(Q_1 \cup Q_2 \cup \cdots \cup Q_{s-1})$ or $f \in \mathrm{cl}(Q_2 \cup Q_3 \cup \cdots \cup Q_s)$ so that $\mathbf{Q}' = (Q_1 \cup Q_2 \cup \cdots \cup Q_s \cup \{f\}, Q_{s+1}, \ldots, Q_k)$ is not a maximal flower in M). We may now also assume that $i = 2$, so that $\mathbf{Q}'' = (Q_1, Q_2 - \{g\}, Q_3, \ldots, Q_k)$ and $Q_1' = Q_1 \cup Q_2$. Another consequence is that for some maximal fan H between Q_1 and $Q_2 - \{g\}$, we have $f \in \mathrm{cl}(Q_1 \cup H)$.

We now consider possibilities for the fan H. Let $Q_1'' = Q_1 - H$, $Q_2'' = Q_2 - (H \cup \{g\})$, and assume that $H = (h_1, h_2, \ldots, h_t)$ is ordered from Q_1 to $Q_2 - \{g\}$ in $\mathbf{Q}''$. Thus the quasi-flower $(Q_1'', H, Q_2'', Q_3, \ldots, Q_k)$ in $M\backslash f, g$ is equivalent to $\mathbf{Q}''$. Moreover $f \in \mathrm{cl}(Q_1'' \cup H)$ and $g \in \mathrm{cl}(Q_2'' \cup H)$. Note also that we have symmetry between f and g.

6.33.1. *The elements h_1 and h_t are coguts elements of H.*

Subproof. Assume that h_1 is a guts element. Then $h_1 \in \mathrm{cl}(Q_1'')$ so that h_1 is a loose element of the k-fracture $(Q_1'', H \cup Q_2'' \cup \{g\}, Q_3, \ldots, Q_k)$ of $M\backslash f$. This contradicts the fact that f is not in a bogan or cobogan couple. Thus h_1 is a coguts element of H, and so too is h_t. □

6.33.2. $f \notin \mathrm{cl}(Q_1'' \cup \{h_1, h_2, \ldots, h_{t-1}\})$ *and* $g \notin \mathrm{cl}(Q_2'' \cup \{h_2, h_3, \ldots, h_t\})$.

Subproof. If $f \in \mathrm{cl}(Q_1'' \cup \{h_1, h_2, \ldots, h_{t-1}\})$, then h_t is a loose element of a k-fracture of $M\backslash g$ contradicting the fact that g is not in a bogan or cobogan couple. The claim follows from this observation and the symmetry between f and g. □

6.33.3. $f \in \mathrm{cl}(Q_1'' \cup \{h_t\})$ *and* $g \in \mathrm{cl}(Q_2'' \cup \{h_1\})$.

Subproof. Assume that $f \notin \mathrm{cl}(Q_1'' \cup \{h_t\})$. Note that this means that $t > 1$. Consider $M\backslash f, g/h_t$. It follows from Lemma 4.13 and the fact that $M\backslash f, g$ is uniquely k-fractured by $\mathbf{Q}''$, that $M\backslash f, g/h_t$ is uniquely k-fractured by $(Q_1'', \{h_1, h_2, \ldots, h_{t-1}\}, Q_2'', Q_3, \ldots, Q_k)$. Now $g \in \mathrm{cl}_{M/h_t}(Q_2'' \cup \{h_1, h_2, \ldots, h_{t-1}\})$, but $g \notin \mathrm{cl}_{M/h_t}(Q_2'' \cup \{h_2, \ldots, h_{t-1}\})$. Therefore there is a circuit C in M/h_t such that $h_1 \in C$ and $C \subseteq \{h_1, h_2, \ldots, h_{t-1}\} \cup Q_2''$. It follows that the flower $(Q_1'', \{g, h_1, h_2, \ldots, h_{t-1}\} \cup Q_2'', Q_3, \ldots, Q_k)$ of $M\backslash f/h_t$ is canonical. Evidently it is also the unique k-fracture of $M\backslash f/h_t$. Now $f \notin \mathrm{cl}_M(Q_1'' \cup \{h_t\})$, so $f \notin \mathrm{cl}_{M/h_t}(Q_1'')$, and f is certainly not in the closure of any other petal of the flower. We conclude, by Corollary 4.46, that M/h_t is k-coherent. But M is a 2-reduced k-skeleton with no gangs or cogangs of three. So this means that h_t is in a clonal pair. But h_t evidently has no clone and the claim follows from this contradiction. □

6.33.4. $|H| \in \{1, 3\}$.

Subproof. Otherwise F contains a triangle T in $M\backslash f, g$ and hence in M. By Lemma 5.39 T cannot be k-wild. But T certainly cannot be a clonal triple of M and the claim follows. □

6.33.5. *The elements of H are feral.*

Subproof. It is easily seen that the elements of H are either feral or semi-feral. We omit the routine verification that they are not semi-feral. □

We first consider that case that $|H| = 3$. Let $\mathbf{Q}''' = (Q_1'' \cup \{f, h_3\}, Q_2'' \cup \{g, h_1\}, Q_3, \ldots, Q_k)$.

6.33.6. *$\mathbf{Q}'''$ is a k-fracture of $M\backslash h_2$.*

Proof. Consider $M\backslash f, g, h_2/h_3$. Observe that $(Q_1'', Q_2'' \cup \{h_3\}, Q_3, \dots, Q_k)$ is a swirl-like flower in this matroid and that h_3 is in the coclosure of Q_1''. But $\{h_1, h_3\}$ is a series pair in $M\backslash f, g, h_2$ so that $(Q_1'', Q_2'' \cup \{h_1, h_3\}, Q_3, \dots, Q_k)$ is a swirl-like flower in $M\backslash f, g, h_2$. Moreover, $h_3 \in \mathrm{cl}^*_{M\backslash f,g,h_2}(Q_1'')$, so that the partition $(Q_1'' \cup \{h_3\}, Q_2'' \cup \{h_1\}, Q_3, \dots, Q_k)$ has the property that the union of every consecutive pair of sets in the cyclic order is exactly 3-separating. (This partition is not technically a flower according to our definition as the series pair is split between two petals.) By 6.33.3, $f \in \mathrm{cl}(Q_1'' \cup \{h_3\})$ and $g \in \mathrm{cl}(Q_1'' \cup \{h_1\})$. By 6.33.5, $M\backslash h_2$ is 3-connected. Therefore $\mathbf{Q}'''$ is a flower in $M\backslash h_2$. It now follows easily that it is swirl-like and tight, establishing the claim. □

Note that h_2 blocks both $Q_1'' \cup \{f, h_3\}$ and $Q_2'' \cup \{g, h_1\}$. Thus, by Lemma 6.18, there is a k-fracture $\mathbf{P} = (P_1, P_2, \dots, P_m)$ of M/h_2 such that $\mathbf{Q}'''$ and $\mathbf{P}$ form a feral display for f_2 in M^*. Mimicking the notation in the definition of feral display as closely as possible, we may assume, for some $i \in \{3, 4, \dots, m-1\}$, that $P_2 \cup P_3 \cup \dots \cup P_i \subseteq Q_2'' \cup \{g, h_1\}$, and $P_{i+1} \cup P_{i+2} \cup \dots \cup P_m \subseteq Q_1'' \cup \{f, h_3\}$. Let $Z_1 = P_1 \cap (Q_1'' \cup \{f, h_3\})$ and $Z_2 = P_1 \cap (Q_2'' \cup \{g, h_1\})$.

6.33.7. *$\{f, h_3\} \subseteq Z_1$ and $\{g, h_1\} \subseteq Z_2$.*

Proof. Say $f \in P_j$ for some $j \in \{i+1, i+2, \dots, m\}$. As f is feral it follows from Lemma 6.17 that $|P_j| \geqslant 3$. Using this fact and the fact that f is feral, we deduce that f is not on the guts of P_j. It now follows from Lemma 6.13, or Lemma 6.14, that any k-fracture of $M\backslash f$ is either obtained by opening the petal P_i in the flower $(P_1 \cup P_2 \cup \dots \cup P_i \cup \{h_2\}, P_{i+1}, \dots, P_m)$ or has the property that $E(M) - P_j$ is contained in a petal. Such a k-fracture is certainly not $\mathbf{Q}$, the flower that we know uniquely k-fractures $M\backslash f$. Hence $f \in Z_1$ and, symmetrically, $g \in Z_2$.

Assume that $h_3 \in P_j$. Arguing as before we have $|P_j| > 2$. Also h_3 is not in the coguts of P_j as otherwise it is not feral. Thus $h_3 \notin \mathrm{cl}^*(E(M) - P_j)$. But then, as $f \notin P_j$ we see that $h_3 \notin \mathrm{cl}^*_M(\{h_1, h_2, f, g\})$ so that $h_3 \notin \mathrm{cl}^*_{M\backslash f,g}(\{h_1, h_2\})$, contradicting the fact that $\{h_1, h_2, h_3\}$ is a triad of $M\backslash f, g$. Therefore $h_3 \in Z_1$ and symmetrically $h_1 \in Z_2$ as required. □

By 6.33.7, $\{h_1, h_3, f, g\} \subseteq P_1$. But, by the properties of the feral display in M^*, the element h_2 coblocks P_1 in M, so that $h_2 \notin \mathrm{cl}^*(P_1)$. Therefore $h_2 \notin \mathrm{cl}^*_M(\{h_1, h_3, f, g\})$ and again we contradict the fact that $\{h_1, h_2, h_3\}$ is a triad in M.

Assume that $|H| = 1$ so that $H = \{h_1\}$. Clearly $(Q_1'' \cup \{f\}, Q_2'' \cup \{g\}, Q_3, \dots, Q_k)$ is a k-fracture of M/h_1. Moreover, one readily checks that it is unique. It follows that M_{pq}/h_1 is k-coherent and hence h_1 is pq-annoying. Therefore h_1 is not in a bogan or cobogan couple. By 6.33.5, h_1 is feral. Moreover h_1 coblocks both $Q_1'' \cup \{f\}$ and $Q_2'' \cup \{g\}$. Thus, by Lemma 6.18, there is a feral display for h_1 in M. By examining the definition of feral display we observe that there are petals P_i and P_{i+1} of a k-fracture of $M\backslash h_1$ such that $P_i \subseteq Q_1'' \cup \{f\}$, $P_{i+1} \subseteq Q_2'' \cup \{g\}$, $\lambda_M(P_i) = \lambda_M(P_{i+1}) = 2$, and $\sqcap_M(P_i, P_{i+1}) = 1$. But $\sqcap_M(Q_1'', Q_2'') = 0$, so we may assume without loss of generality that $f \in P_i$. But $f \notin \mathrm{cl}(Q_1'')$, so that f is a coloop of $M|P_i$ and hence $\lambda_{M\backslash f}(P_i - \{f\}) = 1$. As $M\backslash f$ is 3-connected, it must be the case that $|P_i| = 2$. As h_1 is not in a bogan or cobogan couple, the flower $\mathbf{P}$ is canonical. Therefore P_i is fully closed in $M\backslash h_1$. But now Lemma 6.17 implies the contradiction that h_1 is not feral. All cases lead to a contradiction and the lemma follows. □

From now on we assume that g is f-bad for contraction. Recall that $Q_1' = Q_1 \cup Q_2 \cup \dots \cup Q_s$ and, for some $i \in \{1, 2, \dots, s\}$, we have $g \in Q_i$. We may assume, up to labels, that $g \notin Q_1$.

Lemma 6.34. *$i = s = 2$.*

Proof. Assume that $s \neq 2$. By Lemma 6.31, $\mathbf{Q}'' = (Q_1, Q_2, \dots, Q_{i-1}, Q_i - \{g\}, Q_{i+1}, \dots, Q_k)$ uniquely k-fractures $M\backslash f/g$. But M/g is k-fractured. Thus, by Lemma 4.45, f is in the span of a petal of a flower equivalent to $\mathbf{Q}''$. Certainly the petal is not equivalent to $Q_i - \{g\}$ as otherwise M is k-fractured.

An elementary argument, based on the fact that $\mathbf{Q}'$ is a maximal flower in M shows that we may now assume that $i = s$ and that f is in the span of a petal equivalent to Q_1. As $\mathbf{Q}$ is a canonical flower in $M\backslash f$, and $s \neq 2$, the petal Q_1 is fully closed in $\mathbf{Q}''$. Hence $(Q_1 \cup \{f\}, Q_2, \dots, Q_{s-1}, Q_s - \{g\}, Q_{s+1}, \dots, Q_k)$ is a k-fracture of M/g.

Consider the k-fracture $\mathbf{Q} = (Q_1, Q_2, \dots, Q_k)$ of $M\backslash f$. Certainly f blocks Q_s and, as $\mathbf{Q}' = (Q_1 \cup Q_2 \cup \dots \cup Q_s \cup \{f\}, Q_{s+1}, \dots, Q_k)$ is a maximal swirl-like flower in M, it also must be the case that f blocks $Q_2 \cup Q_3 \cup \dots \cup Q_s$. Thus f is not 2-spanned by $\mathbf{Q}$. Now, by the definition of a feral display, one sees that f has a feral display in M rather than M^*. As f blocks Q_s, we see that, with the labelling given in the definition of feral display, we have $(P_1, P_2, \dots, P_i, P_{i+1}, \dots, P_k) = (Q_s, Q_{s+1}, \dots, Q_k, Q_1, \dots, Q_k)$, where $P_i = Q_k$. It now follows from the properties of a feral display that $f \in \mathrm{cl}(Q_s \cup Q_{s+1} \cup \dots \cup Q_k)$. Therefore $f \in \mathrm{cl}_{M/g}((Q_s - \{g\}) \cup Q_{s+1} \cup \dots \cup Q_k)$. But this implies that f is a loose element of the k-fracture $(Q_1 \cup \{f\}, Q_2, \dots, Q_{s-1}, Q_s - \{g\}, Q_{s+1}, \dots, Q_k)$, of M/g contradicting the fact that g is not in a bogan couple. The lemma follows from this contradiction. $\square$

We now know that $\mathbf{Q}'' = (Q_1, Q_2 - \{g\}, Q_3, \dots, Q_k)$ is a k-fracture in $M\backslash f/g$. As in the proof of the previous lemma we have a, possibly empty, fan H of loose elements between Q_1 and $Q_2 - \{g\}$ such that $f \in \mathrm{cl}_{M/g}(Q_1 \cup H)$. Let $Q_1'' = Q_1 - H$ and $Q_2'' = Q_2 - (H \cup \{g\})$. Assume that $H = (h_1, h_2, \dots, h_t)$ is ordered from Q_1 to $Q_2 - \{g\}$ in $\mathbf{Q}''$. In other words, we have a quasi-flower $(Q_1'', H, Q_2'', Q_3, \dots, Q_k)$ in $M\backslash f/g$ is equivalent to $\mathbf{Q}''$. Moreover $(Q_1'' \cup H \cup \{f\}, Q_2'', Q_3, \dots, Q_k)$ and $(Q_1'', Q_2'' \cup H \cup \{g\}, Q_3, \dots, Q_k)$ are k-fractures of M/g and $M\backslash f$ respectively. As f and g are both pq-annoying, these k-fractures are canonical.

Our next goal is to get rid of the irritating fan H.

Lemma 6.35. *The flower $\mathbf{Q}''$ is canonical. In particular, the fan H is empty.*

Proof. It is easily seen that the only possible loose elements of $\mathbf{Q}''$ are in H. Assume that $H \neq \emptyset$.

6.35.1. *The element h_1 is a guts element of H and h_t is a coguts element. Consequently $|H| \geqslant 2$.*

Subproof. Consider h_t, the last element of H. If h_t is a guts element of H, then $h_t \in \mathrm{cl}_{M\backslash f/g}(Q_2'')$ so that $h_t \in \mathrm{cl}_{M/g}(Q_2'')$, contradicting the fact that the k-fracture $(Q_1'' \cup H \cup \{f\}, Q_2'', Q_3, \dots, Q_k)$ of M/g is canonical. Hence h_t is a coguts element of H and, dually, h_1 is a guts element of H. $\square$

We omit the easy proof of the next claim.

6.35.2. *The elements of H are feral.*

Consider possible k-fractures of $M\backslash h_1$.

6.35.3. *There is an $i \in \{1, 2, \dots, t\}$ such that*

$$\left(Q_1'' \cup \{f, h_2, h_3, \dots, h_i\}, Q_2'' \cup \{g, h_{i+1}, h_{i+2}, \dots, h_t\}, Q_3, \dots, Q_k\right)$$

is a k-fracture of $M\backslash h_1$.

Subproof. Consider the matroid $M\backslash h_1/g$. It is easily seen that $M\backslash h_1/g$ is 3-connected. As $h_1 \in \mathrm{cl}_{M/g}(Q_1'')$, and $f \in \mathrm{cl}_{M/g}(Q_1'' \cup H)$, we have $f \in \mathrm{cl}_{M/g}(Q_1'' \cup \{h_2, h_3, \dots, h_t\})$. Thus $(Q_1'' \cup \{f, h_2, h_3, \dots, h_t\}, Q_2'', Q_3, \dots, Q_k)$ is a k-fracture of $M\backslash h_1/g$. Let i be the least integer such that $(Q_1'' \cup \{f, h_2, h_3, \dots, h_i\}, Q_2'' \cup \{h_{i+1}, h_{i+2}, \dots, h_t\}, Q_3, \dots, Q_k)$ is also a k-fracture of $M\backslash h_1/g$.

As h_1 is feral, $M\backslash h_1$ is k-fractured. Let $(O_1 \cup \{g\}, O_2, \dots, O_k)$ be a k-fracture of $M\backslash h_1$. By the structure of swirl-like flowers either $g \in \mathrm{cl}(O_k \cup O_1)$ or $g \in \mathrm{cl}(O_1 \cup O_2)$. Up to labels we may assume that $g \in \mathrm{cl}(O_1 \cup O_2)$. The partition $(O_1 \cup O_2, O_3, \dots, O_k)$ is a swirl-like flower in the 3-connected matroid $M\backslash h_1/g$. Assume that this flower is not comparable with $(Q_1'' \cup \{f, h_2, h_3, \dots, h_i\}, Q_2'' \cup \{h_{i+1}, h_{i+2}, \dots, h_t\}, Q_3, \dots, Q_k)$. By Lemma 3.16, either $O_1 \cup O_2$ is contained in the full closure of a petal of $(Q_1'' \cup \{f, h_2, h_3, \dots, h_i\}, Q_2'' \cup \{h_{i+1}, h_{i+2}, \dots, h_t\}, Q_3, \dots, Q_k)$ or $O_3 \cup O_4 \cup \dots \cup O_k$ is contained in the full closure of a petal of this flower. The latter case routinely leads to a contradiction of the fact that M_{pq}/g is k-coherent. The only non-contradictory possibility in the former case is if $O_1 \subseteq Q_2'' \cup \{h_{i+1}, h_{i+2}, \dots, h_t\}$. But, in this case, the claim holds. $\square$

Consider the k-fracture $(Q_1'' \cup \{f, h_2, h_3, \dots, h_i\}, Q_2'' \cup \{g, h_{i+1}, h_{i+2}, \dots, h_t\}, Q_3, \dots, Q_k)$ of $M\backslash h_1$ given by 6.35.3. As h_1 is a feral element of M, there is a k-fracture $\mathbf{P} = (P_1, P_2, \dots, P_m)$ of M/h_1 such that $\mathbf{P}$ and $(Q_1'' \cup \{f, h_2, h_3, \dots, h_i\}, Q_2'' \cup \{g, h_{i+1}, h_{i+2}, \dots, h_t\}, Q_3, \dots, Q_k)$ form a feral display in M or M^*. Observe that h_1 is spanned in M by $(Q_1'' \cup \{f, h_2, h_3, \dots, h_i\}) \cup (Q_2'' \cup \{g, h_{i+1}, h_{i+2}, \dots, h_t\})$ but not by either $Q_1'' \cup \{f, h_2, h_3, \dots, h_i\}$ or $Q_2'' \cup \{g, h_{i+1}, h_{i+2}, \dots, h_t\}$, as otherwise M is k-fractured. This shows that we have a feral display in M. By the definition of feral display, we may assume that there is an $l \in \{2, 3, \dots, m-1\}$ such that $P_2 \cup P_3 \cup \dots \cup P_l \subseteq Q_2'' \cup \{g, h_{i+1}, h_{i+2}, \dots, h_t\}$, $P_{l+1} \cup P_{l+2} \cup \dots \cup P_m \subseteq Q_1'' \cup \{f, h_1, h_2, \dots, h_i\}$, and $Q_3 \cup Q_4 \cup \dots \cup Q_k \subseteq P_1$. As usual we let $Z_1 = (Q_1'' \cup \{f, h_1, h_2, \dots, h_i\}) \cap P_1$, and $Z_2 = (Q_2'' \cup \{g, h_{i+1}, h_{i+2}, \dots, h_t\}) \cap P_1$.

6.35.4. *$Q_1'' \cup \{f, h_1, h_2, \dots, h_i\} \subseteq Z_1$ and $Q_2'' \cup \{g, h_{i+1}, h_{i+2}, \dots, h_t\} \subseteq Z_2$.*

Subproof. Say $f \notin Z_1$. Then $f \in P_u$ for some $u \in \{l+1, l+2, \dots, m\}$. But then, by Lemma 6.13 or Lemma 6.14, $(Q_1, Q_2, \dots, Q_k)$ is not a k-fracture of $M\backslash f$. Thus $f \in Z_1$ and similarly $g \in Z_2$.

If $\{h_2, h_3, \dots, h_t\} \subseteq Z_1 \cup Z_2$, then the claim holds. Assume otherwise. Let ω be the first element of $\{1, 2, \dots, t\}$ such that $h_\omega \notin Z_1 \cup Z_2$. Say $\omega \leqslant i$. Then $Q_1'' \cup \{f, g, h_1, h_2, \dots, h_{\omega-1}\} \subseteq E(M) - (P_{l+1} \cup P_{l+2} \cup \dots \cup P_m)$, and $h_\omega \in \mathrm{cl}^{(*)}_{M\backslash f/g}(Q_1'' \cup \{f, g, h_1, h_2, \dots, h_{\omega-1}\})$, so that $h_\omega \in \mathrm{cl}^{(*)}_M(E(M) - (P_{l+1} \cup P_{l+2} \cup \dots \cup P_m))$. It follows from this that, if $m \neq l+1$, then h_ω is a loose element of a flower of M of order at least three, contradicting Lemma 5.32. Say $l+1 = m$. By Lemma 6.17, $|P_m| > 2$. Thus h_w is in the guts or coguts of a 3-separation of M, contradicting the fact that h_w is feral. The same argument works in the case that $\omega > i$ and the sublemma follows. $\square$

By the properties of a feral display, $\sqcap_M(P_l, P_{l+1}) = 1$. But, by 6.35.4, $P_l \subseteq Q_1''$ and $P_{l+1} \subseteq Q_2''$. Hence $\sqcap_M(Q_1'', Q_2'') \geqslant 1$, that is, $\sqcap_{M\backslash f}(Q_1'', Q_2'') \geqslant 1$. But $g \notin \mathrm{cl}(Q_1'') \cap \mathrm{cl}(Q_2'')$, so $\sqcap_{M\backslash f/g}(Q_1'', Q_2'') \geqslant 1$. However the existence of the coguts element h_t of the fan H in $M\backslash f/g$ ensures that $\sqcap_{M\backslash f/g}(Q_1'', Q_2'') = 0$.

It follows from the above contradiction that a feral display does not exist for h_1 so that h_1 is not feral, and we can at last conclude that $H = \emptyset$. $\square$

The next lemma shows, thankfully, that we do not have to dig any deeper.

Lemma 6.36. *There is no element h such that h is fixed in $M_{pg}\backslash f/g$ and $M_{pq}\backslash f/g\backslash h$ is k-coherent.*

Proof. Assume that the lemma fails so that we have an element h that is fixed in $M_{pg}\backslash f/g$ and $M_{pq}\backslash f/g\backslash h$ is k-coherent. It is clear that h is either feral or semi-feral, and, in the semi-feral case, $M\backslash f$ is 3-connected. In other words, $M\backslash h$ is 3-connected and k-fractured. By Lemma 6.35, $\mathbf{Q}'' = (Q_1, Q_2 - \{g\}, Q_3, \dots, Q_k)$ canonically k-fractures $M\backslash f/g$, and this k-fracture is unique. Moreover $(Q_1 \cup \{f\}, Q_2 - \{g\}, Q_3, \dots, Q_k)$ and $(Q_1, Q_2, \dots, Q_k)$ are unique canonical k-fractures of M/g and $M\backslash g$ respectively. Recall also that $Q_t = \{p, q\}$ for some $t \in \{3, 4, \dots, k\}$.

We omit the argument that shows that either $h \in Q_1$ or $h \in Q_2 - \{g\}$ which is similar to, but easier than, the arguments below.

Consider the case that $h \in Q_2 - \{g\}$. We know that $(Q_1 \cup \{f\}, Q_2 - \{g\}, Q_3, \dots, Q_k)$ is a canonical unique k-fracture of M/g and $h \in Q_2 - \{g\}$. Consider $M/g\backslash h$. This matroid is clearly 3-connected. If, for some partition (Q_2', Q_2'') of $Q_2 - \{g\}$, the partition $(Q_1 \cup \{f\}, Q_2', Q_2'', Q_3, \dots, Q_k)$ is a tight swirl-like flower of $M/g\backslash h$, then it is clear that $M_{pq}\backslash f/g\backslash h$ is not k-coherent. Hence $(Q_1 \cup \{f\}, Q_2 - \{g, h\}, Q_3, \dots, Q_k)$ is a maximal swirl-like quasi-flower in $M/g\backslash h$. This flower is either a k-fracture, or else $Q_2 - \{g, h\}$ is a set of loose elements of the flower. In the latter case we have $M/g\backslash h$ is k-coherent, as any other fracture would again contradict the fact that $M_{pq}\backslash f/g\backslash h$ is k-coherent. As $M\backslash h$ is k-fractured, it follows that $(Q_1 \cup \{f\}, Q_2 - \{h\}, Q_3, \dots, Q_k)$ is a maximal k-fracture of $M\backslash h$. Note that $g \in \mathrm{cl}(Q_1 \cup \{f\})$, as otherwise $(Q_1 \cup \{f\}, Q_2, \dots, Q_k)$ is a k-fracture of M. But $g \in Q_2 - \{h\}$, so that g is loose in $(Q_1 \cup \{f\}, Q_2 - \{h\}, Q_3, \dots, Q_k)$. This gives the contradiction that $\{g, h\}$ is a bogan couple.

Consider the case that $h \in Q_1$. Assume that $M\backslash f, h$ is not 3-connected. Then, as $M\backslash f, h/g$ is 3-connected, there is a series pair $\{g, g'\}$ in $M\backslash f, h$ so that $\{g, g', h\}$ is a triad of $M\backslash f$. But $\mathbf{Q}$ is a canonical flower of $M\backslash f$ and it follows that any triad must be contained in a single petal of $\mathbf{Q}$, contradicting the fact that $h \in Q_1$ and $g \in Q_2$. Thus $M\backslash f, h$ is 3-connected. Arguing as in the previous case we see that $(Q_1 - \{h\}, Q_2, \dots, Q_k)$ either uniquely k-fractures $M\backslash f, h$ or $Q_1 - \{h\}$ is a set of loose elements of this flower. But $M\backslash h$ is k-fractured. In the case that $M\backslash f, h$ is k-fractured, it must be the case that this k-fracture is $(Q_1 - \{h\}, Q_2, Q_3, \dots, Q_k)$, as otherwise it is easily checked that $M_{pq}\backslash f/g\backslash h$ is k-fractured. In either case we deduce that $M_{pq}\backslash f, h$ is k-coherent and we have contradicted Lemma 6.33. $\square$

Assume that $M_{pq}\backslash p/q$ is not a k-skeleton. By Lemma 6.36, there is no element h such that either h is fixed in $M_{pq}\backslash f/g$ and $M_{pq}\backslash f/g\backslash h$ is k-coherent. By the symmetry between f and g under duality, it also follows from Lemma 6.36, that there is no element h such that h is cofixed in $M_{pq}\backslash f/g$ and $M_{pq}\backslash f/g, h$ is k-coherent. Thus $M_{pq}\backslash f/g$ is indeed a k-skeleton. This completes the proof of Theorem 6.30.

Chapter 7. Paths of 3-separations

1. Introduction

Recall that if X and Y are disjoint sets of elements of the matroid M, then $\kappa(X, Y)$ denotes the minimum of $\lambda(X', Y')$ over all partitions (X', Y') of $E(M)$ with $X \subseteq X'$ and $Y \subseteq Y'$. A *path* of 3-separations in a matroid M is a partition $\mathbf{P} = (P_0, P_1, \dots, P_l)$ of $E(M)$ into subsets such that $\kappa(P_0, P_l) = 2$ and $\lambda(P_0 \cup \dots \cup P_i) = 2$, for all $i \in \{0, 1, \dots, l-1\}$. If $i \in \{0, 1, \dots, l\}$, then P_i is a *step* of the path; P_0 and P_l are *end steps* and otherwise P_i is an *internal step*.

We allow the possibility that internal steps can be empty. A *solid* path has no empty steps. If $\mathbf{P}$ has $l+1$ nonempty steps, then the *length* of $\mathbf{P}$ is l.

Recall that Λ_m denotes the rank-m free spike and recall that $\mathcal{E}(q)$ denotes the class of matroids that has no $U_{2,q+2}$-, $U_{q,q+2}$- or Λ_q-minor. In this chapter we begin the task of controlling the structure of a k-skeleton in $\mathcal{E}(q)$. Our primary goal is to prove that k-skeletons in $\mathcal{E}(q)$ cannot have arbitrarily long paths of 3-separations. In other words, we prove

Theorem 7.1. *Let $k \geqslant 5$ and $q \geqslant 2$ be integers. Then there is a function $f_{7.1}(k, q)$ such that, if M is a k-skeleton with a path of 3-separations of length $f_{7.1}(k, q)$, then $M \notin \mathcal{E}(q)$.*

We also prove a number of other lemmas that will be of use later in the paper. Before ploughing on into the technicalities of the proof we note a corollary of Theorem 7.1.

Corollary 7.2. *Let $k \geqslant 5$ and $q \geqslant 2$ be integers. Let M be a k-coherent matroid with a path of 3-separations $\mathbf{P}$ of length $f_{7.1}(k, q) + 2$ such that each step of $\mathbf{P}$ contains an element that is neither fixed nor cofixed. Then $M \notin \mathcal{E}(q)$.*

Proof. Assume that M satisfies the hypotheses of the corollary. Let $n = f_{7.1}(k, q) + 2$. Then M has a path $\mathbf{P} = (P_0, P_1, \dots, P_n)$ such that each step of $\mathbf{P}$ contains an element that is neither fixed nor cofixed and both P_0 and P_n contain at least two such elements. If M is a k-skeleton, the result follows immediately, so assume that M is not a k-skeleton. Then, up to duality, we may assume that there is a fixed element z such that $M\backslash z$ is k-coherent. For some $i \in \{0, 1, \dots, l\}$, we have $z \in P_i$. If x

is not fixed in M, then x is not fixed in $M\backslash z$. Assume that x is not cofixed in M. Then, as z is fixed in M, it is not the case that z is freer than x in M. Thus, by Corollary 5.7, x is not cofixed in $M\backslash x$. It is now clear that $(P_0, \ldots, P_{i-1}, P_i - \{z\}, P_{i+1}, \ldots, P_n)$ is a path of 3-separations in $M\backslash z$ each step of which contains an element that is not fixed or cofixed. The result now follows from an obvious induction. $\square$

2. Some preliminaries

We begin by developing some more terminology for paths of 3-separations. A nonempty internal step P_i of $\mathbf{P} = (P_0, P_1, \ldots, P_l)$ is *prime* if there is no partition (P_{i1}, P_{i2}) of P_i into nonempty subsets such that $(P_0, P_1, \ldots, P_{i-1}, P_{i1}, P_{i2}, P_{i+1}, \ldots, P_l)$ is a path of 3-separations of M. The path $\mathbf{P}$ is *maximal* if all of its nonempty internal steps are prime. It may be that an internal step P_i contains a single element p_i, in which case it is a *singleton* step. A singleton step $P_i = \{p_i\}$ is a *guts* step or a *coguts* step if it is respectively in the guts or coguts of the 3-separation $(P_0 \cup P_1 \cup P_1 \cup \cdots \cup P_i, P_{i+1} \cup P_{i+2} \cup P_{i+2} \cup \cdots \cup P_l)$ of M. To simplify notation we often denote the singleton step $\{p_i\}$ by p_i. We also use P_i^- to denote the set $P_0 \cup P_1 \cup \cdots \cup P_{i-1}$ and P_i^+ to denote the set $P_{i+1} \cup P_{i+2} \cup \cdots \cup P_l$.

The next lemma follows from the fact that the connectivity function of a matroid is monotone under minors.

Lemma 7.3. *Let $\mathbf{P} = (P_0, P_1, \ldots, P_l)$ be a path of 3-separations in the matroid M and let M' be a minor of M on the set E'. If $\kappa_{M'}(P_0 \cap E', P_l \cap E') = 2$, then $\mathbf{P}' = (P_0 \cap E', P_1 \cap E', \ldots, P_l \cap E')$ is a path of 3-separations in M'.*

There are no surprises in the next elementary lemma. It essentially says that if we keep the same local connectivity, then we keep the same guts.

Lemma 7.4. *Let Y and Z be disjoint sets of a matroid M and let y be an element of Y. Assume that $Z' \subseteq Z$, and $\sqcap(Y, Z') = \sqcap(Y, Z)$. Then $y \in \mathrm{cl}(Z')$ if and only if $y \in \mathrm{cl}(Z)$.*

Proof. One direction is clear. For the other direction, say that $y \in \mathrm{cl}(Z)$. Assume, for a contradiction, that $y \notin \mathrm{cl}(Z')$. We consider two cases. For the first assume that $y \notin \mathrm{cl}(Y - \{y\})$. Then $\sqcap(Z \cup \{y\}, Y - \{y\}) = \sqcap(Z, Y) - 1$, and $\sqcap(Z' \cup \{y\}, Y - \{y\}) = (r(Z') + 1) + (r(Y) - 1) - r(Z' \cup Y) = \sqcap(Z', Y)$. This contradicts the fact that $\sqcap(Z' \cup \{y\}, Y - \{y\}) \leqslant \sqcap(Z \cup \{y\}, Y - \{y\})$.

For the other case, assume that $y \in \mathrm{cl}(Y - \{y\})$. Then $\sqcap(Z \cup \{y\}, Y - \{y\}) = \sqcap(Z, Y)$, and $\sqcap(Z' \cup \{y\}, Y - \{y\}) = (r(Z') + 1) + r(Y) - r(Z' \cup Y) = \sqcap(Z', Y) + 1$. Again we contradict the fact that $\sqcap(Z' \cup \{y\}, Y - \{y\}) \leqslant \sqcap(Z \cup \{y\}, Y - \{y\})$. $\square$

We will also need the following slight strengthening of Tutte's Linking Theorem.

Corollary 7.5. *Let (A, Z, B) be a partition of the ground set of a matroid M where $\kappa(A, B) = k$. Then there is a partition $\{I, J\}$ of Z such that $\sqcap(A, I) = \sqcap(B, I) = 0$, and $\lambda_{M/I\backslash J}(A, B) = k$.*

Proof. By Tutte's Linking Theorem there is a partition (I, J) of $E(M) - (A \cup B)$ such that $\lambda_{M/I\backslash J}(A, B) = \kappa_M(A, B)$. Assume that amongst all partitions with this property we have chosen I to have least cardinality. Let $N = M\backslash J$. Then, for all $z \in I$ there is a k-separation (A', B') of N such that $A \subseteq A'$, $B \subseteq B'$ and z is in the coguts of (A', B'). A routine uncrossing argument shows that there is an ordering $(z_1, z_2, \ldots, z_l)$ of I, with the property that $(A \cup \{z_1, \ldots, z_i\}, \{z_{i+1}, \ldots, z_l\} \cup B)$ is a k-separation of N with z_i in the coguts for all $i \in \{1, 2, \ldots, l\}$. But then $r(A \cup I) = r(A) + |I|$ and $r(B \cup I) = r(B) + |I|$ giving the corollary. $\square$

Let $\mathcal{U}(q)$ denote the class of matroids with no $U_{2,q+2}$-minor and $\mathcal{U}^*(q)$ denote the class of matroids with no $U_{q,q+2}$-minor. We use the following result of Kung [16].

Lemma 7.6. *If M is a simple rank-r matroid in $\mathcal{U}(q)$, then M has at most $(q^r - 1)/(q - 1)$ elements.*

An easy corollary of Lemma 7.6 is

Corollary 7.7. *If M is a simple matroid in $\mathcal{U}(q)$, and $A \subseteq E(M)$ has $\lambda(A) = r$, then there are at most $(q^r - 1)/(q - 1)$ elements in $\mathrm{cl}(A) - A$.*

3. Strands and paths of clonal pairs

In this section we develop some straightforward properties of paths of 3-separations. In doing this we typically do not need strong connectivity assumptions about the underlying matroid. Recall that in Chapter 3 flowers were defined for connected, but not necessarily 3-connected matroids.

Strands Let (A, B) be a partition of the ground set of the matroid M. A B-*strand* is a minimal subset X of B such that $\sqcap(A, X) = 1$. Say $a \in A$. Note that, if X is a B-strand, and $a \in \mathrm{cl}(X)$, then $X \cup \{a\}$ is a circuit. Note also, that if Y is a subset of B with $\sqcap(Y, A) = 1$ and $a \in \mathrm{cl}(Y)$, then there is a strand $X \subseteq Y$ such that $a \in \mathrm{cl}(X)$.

Lemma 7.8. *Let (A, x, B) be a path of 3-separations in the matroid M. If there exists a B-strand X such that $x \in \mathrm{cl}(X)$, then x is fixed in M.*

Proof. Assume that X is a B-strand with $x \in \mathrm{cl}(X)$. Then $x \in \mathrm{cl}(B)$. If $x \notin \mathrm{cl}(A)$, then $(A, B \cup \{x\})$ is a 2-separation in M contradicting the definition of path of 3-separations. Therefore $x \in \mathrm{cl}(A)$.

Let M' be a matroid obtained by cloning x by x'. Then $x' \in \mathrm{cl}_{M'}(A)$, so $\sqcap(A \cup \{x, x'\}, X) = \sqcap(A \cup \{x\}, X) = 1$. But $\{x, x'\} \subseteq \mathrm{cl}_{M'}(X)$, so that x and x' are parallel in M'. Hence x is fixed in M. $\square$

Let (A, x, B) be a path of 3-separations of M, where x is a guts singleton. Then x is *fixed from the right* if there is a B-strand X such that $x \in \mathrm{cl}(X)$ and x is *fixed from the left* if there is an A-strand X such that $x \in \mathrm{cl}(X)$.

The next lemma follows from Theorem 6.2 and Lemma 6.3 of [8]. See also [2].

Lemma 7.9. *Let (A, B) be an exact 3-separation of the matroid M. Then there exists an extension M' of M by the element x' such that $x' \in \mathrm{cl}_{M'}(A) \cap \mathrm{cl}_{M'}(B)$ and x' is not fixed in M'.*

Lemma 7.10. *Let (A, x, B) be a path of 3-separation, where x is a guts singleton. Then x is fixed in M if and only if x is fixed from the left or fixed from the right.*

Proof. If x is fixed from the left or right, then x is fixed in M by Lemma 7.8. Consider the converse. Assume that x is fixed in M. By Lemma 7.9 it is possible to extend M by an element x' to obtain a matroid M' with a path of 3-separations (A, x, x', B) such that x' is a guts singleton and x' is not fixed in M'. As x is fixed in M', there is a circuit C of M containing x such that $x' \notin \mathrm{cl}_{M'}(C)$. Consider M'/x'. As $x' \notin \mathrm{cl}_{M'}(C)$, we see that C is a circuit of M'/x'. Moreover, x is in the guts of the 2-separation $(A \cup \{x\}, B)$ of M'/x'. But now (A, B) is a separation of $M'/x', x$ and $C - \{x\}$ is a circuit of this matroid. Thus $C - \{x\} \subseteq A$ or $C - \{x\} \subseteq B$. Assume the latter holds. As $x \in \mathrm{cl}(C - \{x\})$, we see that $\sqcap(A \cup \{x\}, C - \{x\}) \geqslant 1$. But $x' \notin \mathrm{cl}(C - \{x\})$, so, by Lemma 7.4, $\sqcap(A \cup \{x\}, C - \{x\}) < 2$. Hence $C - \{x\}$ is a B-strand, and x is fixed from the right. $\square$

Lemma 7.11. *Let (A, x, B) be a path of 3-separations of the matroid M, where x is a guts singleton. Assume that B has a partition $\{B_1, B_2, \ldots, B_l\}$ such that*

(i) *$(A \cup \{x\}, B_1, B_2, \ldots, B_l)$ is a swirl-like or spike-like flower with at least three petals, and*
(ii) *B_1 is a clonal pair.*

Then x is fixed from the right in (A, x, B) if and only if either $x \in \mathrm{cl}(B_1)$ or $x \in \mathrm{cl}(B_l)$.

Proof. Say $x \in \mathrm{cl}(B_1)$ or $x \in \mathrm{cl}(B_l)$. Then, as $\sqcap(B_1, A) = \sqcap(B_l, A) = 1$, we see that x is fixed from the right.

Assume that x is fixed from the right. Then there is a B-strand X whose closure contains x so that $X \cup \{x\}$ is a circuit. For the first case, assume that $X \cap B_1 \neq \emptyset$. Then, as $X \cup \{x\}$ is a circuit, and B_1 is a clonal pair, we see that $B_1 \subseteq \mathrm{cl}(X)$. Now

$$1 = \sqcap(B_1, A) \leqslant \sqcap\big(\mathrm{cl}(X), A\big) = 1,$$

so that $\sqcap(B_1, A) = \sqcap(\mathrm{cl}(X), A)$. By Lemma 7.4, $x \in \mathrm{cl}(B_1)$, as required. For the other case assume that $X \cap B_1 = \emptyset$. Then

$$1 = \sqcap(B_2 \cup B_3 \cup \cdots \cup B_l, A) = \sqcap(X, A) = \sqcap(B_l, A) = 1,$$

and again by Lemma 7.4, it follows that $x \in \mathrm{cl}(B_l)$. □

Lemma 7.12. *Let (A, x, B) be a path of 3-separations of the matroid M, where x is a guts singleton. Assume that x is not fixed from the right. Say $I \subseteq B$, and $\sqcap(A, I) = 0$. Then, in M/I, the element x is not parallel to any element of A.*

Proof. Assume that $y \in B$ and $\{x, y\}$ is a parallel pair in M/I. Then there is a circuit C of M such that $\{x, y\} \subseteq C \subseteq \{x, y\} \cup I$. Now $1 \leqslant \sqcap(C - \{x\}, A) \leqslant \sqcap(I, A) + 1 = 1$, so that $\sqcap(C - \{x\}, A) = 1$. Thus $C - \{x\}$ is a B-strand and $x \in \mathrm{cl}(C - \{x\})$. This contradicts the fact that x is not fixed from the right. □

Lemma 7.13. *Let $\mathbf{P}$ be a path of 3-separations with a set $\{x_1, x_2, \ldots, x_l\}$ of singleton guts elements, none of which are fixed from the right. Then M has a $U_{2,l}$-minor on the set $\{x_1, x_2, \ldots, x_l\}$.*

Proof. By an appropriate concatenation we may assume that

$$\mathbf{P} = (P_0, x_1, P_1, x_2, P_2, \ldots, P_{l-1}, x_l, P_l).$$

Note that $\mathbf{P}$ need not be a solid path in that it may have some empty steps. By Corollary 7.5, there is a partition $\{I, J\}$ of P_{l-1} such that $\sqcap(P_0 \cup \{x_1\} \cup P_1 \cup \cdots \cup P_{l-2} \cup \{x_{l-1}\}, I) = \sqcap(I, \{x_l\} \cup P_l) = 0$, and $\lambda_{M/I\backslash J}(P_0 \cup \{x_1\} \cup \cup \cdots \cup P_{l-2} \cup \{x_{l-1}\}, \{x_l\} \cup P_l) = 2$. Evidently $(P_0, x_1, P_1, \ldots, P_{l-2}, x_{l-1}, x_l, P_l)$ is a path of 3-separations in $M/I\backslash J$. By Lemma 7.12, x_{l-1} is not parallel to x_l. Moreover, each element of $\{x_1, x_2, \ldots, x_l\}$ is a guts singletons in this path and is not fixed from the right. Repeating this process we obtain a minor M' with a path of 3-separations $P' = (P_0, x_1, x_2, \ldots, x_l, P_l)$, where the members of $\{x_1, x_2, \ldots, x_l\}$ are guts singletons and x_i is not parallel to x_j for all distinct $i, j \in \{1, 2, \ldots, l\}$. Evidently $M'|\{x_1, x_2, \ldots, x_l\} \cong U_{2,l}$. □

Displayed flowers Let $\mathbf{P} = (P_0, P_1, \ldots, P_l)$ be a path of 3-separations of the matroid M and let $\mathbf{Q}$ be a flower of M. Then $\mathbf{Q}$ is *displayed* by $\mathbf{P}$ if each petal of $\mathbf{Q}$ is a union of steps of $\mathbf{P}$. The flowers displayed by $\mathbf{P}$ are partially ordered by refinement. The flower $\mathbf{Q}$ is a *maximal* displayed flower of $\mathbf{P}$ if it is maximal in this partial order. Note that a maximal displayed flower need not be a maximal flower of M and flowers that are incomparable in this partial order may be comparable in the usual partial order of flowers.

Lemma 7.14. *Let $\mathbf{P} = (P_0, P_1, \ldots, P_l)$ be a path of 3-separations of the connected matroid M with the property that $\lambda(P_i) \geqslant 2$ for all $i \in \{0, 1, \ldots, l\}$. Let $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_r)$ be a maximal displayed flower of $\mathbf{P}$. Then there exist i, j such that $0 \leqslant i \leqslant j < l$, and $\{Q_1, Q_2, \ldots, Q_r\} = \{P_i^-, P_i, P_{i+1}, \ldots, P_j, P_j^+\}$.*

Proof. Certainly $\mathbf{Q}$ has petals Q_α and Q_β containing P_0 and P_l respectively. We first show that any other petal of $\mathbf{Q}$ contains exactly one step of $\mathbf{P}$. Assume otherwise. Say that Q is a petal of $\mathbf{Q}$ other than Q_α or Q_β that contains more than one step of $\mathbf{P}$. Assume that $P_i, P_j \subseteq Q$, where $i < j$. Then $(P_i^- \cup P_i, P_i^+)$ is not displayed by $\mathbf{Q}$ and a routine uncrossing argument shows that there is a flower displayed by $\mathbf{P}$ that refines $\mathbf{Q}$ and displays this 3-separation, contradicting the fact that $\mathbf{Q}$ is a maximal displayed flower of $\mathbf{P}$.

Consider Q_α, the petal containing P_0. Let i be the greatest integer such that $P_i \subseteq Q_\alpha$. We now show that $Q_\alpha = P_0 \cup P_1 \cup \cdots \cup P_i$. Assume that this is not the case. Then there is a j with $0 < j < i$ such that $P_j \not\subseteq Q_\alpha$. In this case an uncrossing argument using the 3-separation $(P_j^- \cup P_j, P_j^+)$ again shows that $\mathbf{Q}$ is not a maximal displayed flower of $\mathbf{P}$. Hence Q_α and Q_β respectively contain an initial and a terminal sequence of steps of $\mathbf{P}$ and the lemma follows. $\square$

A straightforward uncrossing argument also proves

Lemma 7.15. *Let $\mathbf{P} = (P_0, P_1, \ldots, P_l)$ be a path of 3-separations of the connected matroid M with the property that $\lambda(P_i) \geqslant 2$ for all $i \in \{0, 1, \ldots, l\}$. If P is a step of $\mathbf{P}$, then P is a petal of at most one maximal displayed flower.*

Special paths of 3-separations We say that a flower in a matroid is a *spiral* if it is either swirl-like or spike-like. A path of 3-separations $(P_0, P_1, \ldots, P_l)$ is *special* if

(i) (P_i^-, P_i, P_i^+) is a spiral for all $i \in \{1, 2, \ldots, l-1\}$, and
(ii) $(P_0, P_1, \ldots, P_l)$ displays no 4-petal flowers.

We begin by characterising special paths of length 4. We first note an elementary operation that can be performed on special paths of 3-separations.

Lemma 7.16. *Let (A, B, C, D) be a special path of 3-separations. Then so too is (B, A, C, D).*

Proof. As $(A, B, C \cup D)$ is a spiral, $\lambda(B) = 2$, so that (B, A, C, D) is a path of 3-separations. The flowers displayed by (B, A, C, D) are the same as the flowers displayed by (A, B, C, D), so that the path displays no 4-petal flowers. To show that the path (B, A, C, D) is special, we need to show that (i) $(B \cup A, C, D)$ and (ii) $(B, A, C \cup D)$ are spirals. As (A, B, C, D) is special $(A \cup B, C, D)$ is a spiral, so that (i) holds. We have already observed that $(A, B, C \cup D)$ is a spiral and this spiral is equivalent or $(B, A, C \cup D)$ so that (ii) holds. $\square$

We say that the special path (B, A, C, D) is obtained from (A, B, C, D) by *switching*. Evidently (D, C, B, A) is also a special path of 3-separations and we say that it is obtained from (A, B, C, D) by *reversal*. The special path (A, B, C, D) has

Type I: if $\sqcap(A, C) = \sqcap(B, C) = 1$ and $\sqcap(B, D) = 0$,
Type II: if $\sqcap(B, C) = 1$ and $\sqcap(A, C) = \sqcap(B, D) = 0$, and
Type III: if $\sqcap(A, C) = \sqcap(B, C) = \sqcap(B, D) = 0$.

It is easily seen that no sequence of switches and reversals can convert a special path of one type into another type.

Lemma 7.17. *If (A, B, C, D) is a special path of 3-separations, then some sequence of switches and reversals converts (A, B, C, D) into a path of Type I, II, or III.*

Proof. As $(A, B, C \cup D)$ is a spiral in M, we have $\sqcap(A, B) = 1$ and similarly $\sqcap(C, D) = 1$. Moreover $\sqcap(A, C) \leqslant \sqcap(A, C \cup D) = 1$. Thus $\sqcap(A, C)$, $\sqcap(A, D)$, $\sqcap(B, C)$ and $\sqcap(C, D)$ are all at most 1.

Say that $\sqcap(A, C) = \sqcap(B, C) = 1$. Assume that $\sqcap(B, D) = 1$. Then

$$\lambda(A \cap C) = r(A \cup C) + r(B \cup D) - r(M)$$

$$\begin{aligned}&= \big(r(A)+r(C)-1\big)+\big(r(B)+r(D)-1\big)-r(M)\\&= r(A\cup B)+r(C\cup D)-r(M)\\&= 2.\end{aligned}$$

It follows from this that (A, B, C, D) is a flower and it is easily checked that this flower is spike-like, contradicting the definition of special path. Thus $\sqcap(B, D) = 0$, and (A, B, C, D) has Type I.

Assume that (A, B, C, D) cannot be converted into a path of Type I or Type III by a sequence of switchings and reversals. Then we may assume, up to switchings and reversals, that $\sqcap(A, C) = 0$ and $\sqcap(B, C) = 1$. If $\sqcap(B, D) = 1$, then (A, B, C, D) converts into a path of Type I. Assume that $\sqcap(B, D) = 0$. If $\sqcap(A, D) = 1$, then it is easily checked that (A, B, C, D) is a swirl-like flower. Hence $\sqcap(A, D) = 0$ and (A, B, C, D) has Type II. □

We omit the straightforward rank calculation that proves the next lemma.

Lemma 7.18. *If (A, B, C, D) is a special path of 3-separations of Type I, II or III in M, then (B, A, D, C) is a special path of 3-separations of Type I, II or III respectively in M^*.*

Paths of clonal pairs Let $\{p, q\}$ be a clonal pair in a matroid M. Then $\{p, q\}$ is *M-strong* if $\lambda_M(\{p, q\}) = 2$. Let $\mathbf{P} = (P_0, P_1, \ldots, P_l)$ be a path of 3-separations of the connected matroid M. Then $\mathbf{P}$ is a *path of clonal pairs* if

(i) $\mathbf{P}$ is maximal;
(ii) P_i is an M-strong clonal pair for all $i \in \{1, 2, \ldots, l-1\}$;
(iii) (P_i^-, P_i, P_i^+) is a flower for all $i \in \{1, 2, \ldots, l-1\}$.

Note that (iii) follows from (ii) except for the fact that there is nothing in the definition of a path of 3-separations to prevent the possibility of M having a 2-separating set that crosses $P_0 \cup P_l$. Such a 2-separating set causes no difficulties structurally, but, given our definition of flower it does prevent $\mathbf{P}$ from having displayed flowers. This triviality having been dealt with we move on.

Let $\mathbf{P} = (P_0, P_1, \ldots, P_l)$ be a path of clonal pairs. Say $i \in \{1, 2, \ldots, l-1\}$ and $P_i = \{p_i, q_i\}$. If the flower (P_i^-, P_i, P_i^+) is a paddle or copaddle, then $(P_0, P_1, \ldots, P_{i-1}, p_i, q_i, P_{i+1}, \ldots, P_l)$ is a path of 3-separations contradicting the maximality of $\mathbf{P}$. Thus (P_i^-, P_i, P_i^+) is a spiral. The next corollary follows from this observation and Lemmas 7.17 and 7.18.

Lemma 7.19. *Let $\mathbf{P} = (P_0, P_1, \ldots, P_l)$ be a path of clonal pairs. Then every flower displayed by $\mathbf{P}$ is a spiral. Moreover, there are integers $0 \leqslant i_1 \leqslant \cdots \leqslant i_m < l$ such that $(Q_1, Q_2, \ldots, Q_r)$ is a maximal displayed flower of $\mathbf{P}$ if and only if*

$$\{Q_1, Q_2, \ldots, Q_r\} = \big\{P^-_{i_j+1}, P_{i_j+1}, \ldots, P_{i_{j+1}}, P^+_{i_{j+1}}\big\}$$

for some $j \in \{1, 2, \ldots, m\}$.

Via Lemma 7.19, the maximal displayed flowers of a path of clonal pairs can be canonically associated with a partition of $(P_0, P_1, \ldots, P_{l-1})$ into consecutive sets of steps. We call this partition the *flower partition* of $\mathbf{P}$ in M. We omit the routine proof of the next lemma.

Lemma 7.20. *Let $\mathbf{P} = (P_0, P_1, \ldots, P_l)$ be a path of clonal pairs of the matroid M. If $t \in \{1, 2, \ldots, l-1\}$ and $P_t = \{p_t, q_t\}$ is a petal of a displayed flower of $\mathbf{P}$ with at least four petals, then the following hold.*

(i) *$(P_0, P_1, \ldots, P_{t-1}, P_{t+1}, \ldots, P_l)$ is a path of clonal pairs in $M\backslash p_t/q_t$.*

(ii) *If $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_s)$ is the flower partition of* $\mathbf{P}$ *in* M*, where* $P_t \in \mathcal{P}_j$*, then* $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_{j-1}, \mathcal{P}_j - \{P_t\}, \mathcal{P}_{j+1}, \dots, \mathcal{P}_l)$ *is the flower partition of the path of clonal pairs* $(P_0, P_1, \dots, P_{t-1}, P_{t+1}, \dots, P_l)$ *in* $M\backslash p_t/q_t$.

The next corollary follows from Lemma 7.20 and an elementary calculation.

Corollary 7.21. *Let* $s \geqslant 1$ *and* $k \geqslant 5$ *be integers and let* $\mathbf{P} = (P_0, P_1, \dots, P_l)$ *be a path of clonal pairs of the connected matroid* M*. Assume that* $\mathbf{P}$ *does not display any spiral with* k *petals. If* $l > (k-3)(s-1)$*, then* M *has a minor with a special path of clonal pairs of length* s*.*

Special paths of clonal pairs At the risk of overkill we point out that a *special path of clonal pairs* is a special path of 3-separations that is also a path of clonal pairs. We omit the easy proof of the next lemma.

Lemma 7.22. *Let* $\mathbf{P} = (P_0, P_1, \dots, P_l)$ *be a special path of clonal pairs of the connected matroid* M*. If* $P_i = \{p_i, q_i\}$ *is an internal step of* $\mathbf{P}$*, then* $(P_0, P_1, \dots, P_{i-1}, \{q_i\} \cup P_i^+)$ *and* $(P_i^- \cup \{q_i\}, P_{i+1}, \dots, P_l)$ *are both special paths of clonal pairs in both* $M\backslash p_i$ *and* M/p_i*.*

Lemma 7.23. *Let* $(P_0, \{p_1, q_1\}, \{p_2, q_2\}, P_3)$ *be a special path of clonal pairs in the connected matroid* M*. Then for some* $N \in \{M, M^*\}$*, either*

(i) q_1 *is not fixed from the right in the path* $(P_0, q_1, \{p_2, q_2\}, P_3)$ *of* N/p_1*, or*
(ii) q_2 *is not fixed from the left in the path* $(P_0, \{p_1, q_1\}, q_2, P_3)$ *of* N/p_2*.*

Proof. Consider the path $(P_0, \{p_1, q_1\}, \{p_2, q_2\}, P_3)$. By Lemma 7.17 a sequence of switches and reversals converts it into a path or Type I, II or III. If the lemma holds for a given path, then it certainly holds for the reversal of that path. It is now easily seen that we lose no generality in assuming that there is a path (A, B, C, D) of Type I, II or III such that $\{A, B\} = \{P_0, \{p_1, q_1\}\}$ and $\{C, D\} = \{\{p_2, q_2\}, P_3\}$. By Lemma 7.18, we may further assume that $D = \{p_2, q_2\}$. Then $\sqcap_M(P_0, \{p_2, q_2\}) = \sqcap_M(\{p_1, q_1\}, \{p_2, q_2\}) = 0$. Hence $q_2 \notin \mathrm{cl}_{M/p_2}(P_0)$ and $q_2 \notin \mathrm{cl}_{M/p_2}(\{p_1, q_1\})$. Now, by Lemma 7.11, q_2 is not fixed from the left in $(P_0, \{p_1, q_1\}, q_2, P_3)$. $\square$

Lemma 7.24. *Let* s *be a positive integer and let* $\mathbf{P} = (P_0, P_1, \dots, P_l)$ *be a special path of clonal pairs of the connected matroid* M*. If* $l > 8s$*, then* M *has a* $U_{2,s}$*- or a* $U_{s-2,s}$*-minor.*

Proof. Assume that $l > 8s$. By Corollary 7.21, we may assume, up to reversal and taking duals, that $\mathbf{P}$ has a subsequence $(P_{i_1} = \{p_{i_1}, q_{i_1}\}, P_{i_2} = \{p_{i_2}, q_{i_2}\}, \dots, P_{i_s} = \{p_{i_s}, q_{i_s}\})$ of internal steps with the following property: for $j \in \{1, 2, \dots, s\}$, the element q_{i_j} is not fixed from the right in the path $(P_0, P_1, \dots, P_{i_j-1}, q_{i_j}, P_{i_j+1}, \dots, P_l)$ of 3-separations of M/p_{i_j}. By Lemma 7.22, $(P_0 \cup P_1 \cup \dots \cup P_{i_j-1} \cup \{q_{i_j}\}, P_{i_j+1}, \dots, P_l)$ is a special path of clonal pairs in this matroid. Define $\mathbf{Q} = (Q_0, Q_1, \dots, Q_l)$ as follows. If $r \in \{i_1, i_2, \dots, i_s\}$, then $Q_r = \{q_r\}$ and otherwise $Q_r = P_r$. This is clearly a path of 3-separations in $M/p_{i_1}, p_{i_2}, \dots, p_{i_s}$ and the elements $\{q_{i_1}, q_{i_2}, \dots, q_{i_s}\}$ are guts singletons that are not fixed from the right. By Lemma 7.13, M has a $U_{2,s}$-minor. $\square$

Corollary 7.25. *There is a function* $f_{7.25}(k, q)$ *such that, if* $n \geqslant f_{7.25}(k, q)$ *and* $\mathbf{P} = (P_0, P_1, \dots, P_n)$ *is a path of clonal pairs in a connected matroid* M *that does not display any swirl-like flower of order* k*, then* $M \notin \mathcal{E}(q)$*.*

Proof. Let $s = \max\{k, q\}$ and let $f_{7.25}(k, q) = ((s+2) - 3)8s$. Certainly $\mathbf{P}$ does not display any swirl-like flower with $s+2$ petals. If $\mathbf{P}$ displays a spike-like flower with $s+2$ petals, then all but at most two of the petals of this flower are clonal pairs so that M has a Λ_s- and hence a Λ_q-minor. Therefore we may assume that $\mathbf{P}$ does not display any spiral with $s+2$ petals.

By Corollary 7.21, M has a minor with a special path of clonal pairs of length $8s+1$. By Lemma 7.24, M has a $U_{2,q+2}$- or a $U_{q,q+2}$-minor and we conclude that $M \notin \mathcal{E}(q)$. $\square$

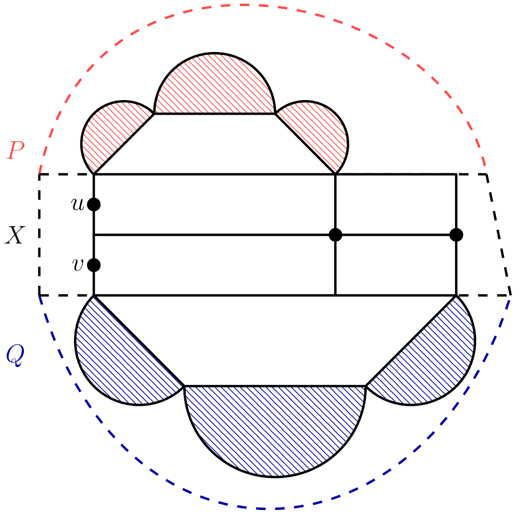

Fig. 7.1. A feral pack in the path (P, X, Q).

4. Feral elements in paths

To prove Theorem 7.1 we would like to reduce a long path of 3-separations in a k-skeleton to one in which the internal steps are either singletons or clonal pairs. But there are obstacles to doing this caused by the presence of feral elements.

Let X be a prime step in a path of 3-separations in the k-coherent matroid M. Then X is a *feral pack* if no element of X is in a triangle or triad, X contains a single clonal pair $\{u, v\}$, and $X - \{u, v\}$ consists of feral elements of M. Fig. 7.1 illustrates a feral pack. Note that $X - \{u, v\}$ is a bogan couple. We know of no examples of feral packs where $X - \{u, v\}$ is neither a bogan nor a cobogan couple.

Recall that if $Z \subseteq E(M)$, then $\mathrm{coh}(Z)$ denotes the set $Z - \mathrm{fcl}(E(M) - Z)$.

Lemma 7.26. *Let Z be a 3-separating set of the k-coherent matroid M such that no member of Z is in a triangle or a triad and $|Z| \geqslant 4$. Then the following hold.*

(i) *There is an element $z \in Z$ such that either $M\backslash z$ or M/z is k-coherent.*
(ii) *If Z contains a clonal pair $\{u, v\}$, then there is an element $z \in Z - \{u, v\}$ such that either $M\backslash z$ or M/z is k-coherent.*

Proof. As no member of Z is in a triangle or a triad, Z is non-sequential. Thus a 3-separation equivalent to Z is displayed in a 3-tree for M. It follows that Z contains a peripheral set P. Assume that P is non-sequential, then by Lemma 4.40(ii), if $z \in \mathrm{coh}(P)$, either $M\backslash z$ or M/z is k-coherent. As $|\mathrm{coh}(P)| \geqslant 4$ there is an element $z \in \mathrm{coh}(P) - \{u, v\}$ and the lemma holds. Assume that P is sequential. Then, as no element of Z is in a triangle or a triad, $|P| = 2$. By Corollary 4.10, if $p \in P$, then either $M\backslash p$ or M/p is k-coherent and the lemma holds unless $P = \{u, v\}$. If $P = \{u, v\}$, then it is clear that Z contains another peripheral set P' and the lemma holds by considering this peripheral set. $\square$

It follows immediately from Lemma 7.26 that if X is a feral pack of the k-coherent matroid M and $Z \subseteq X$ is 3-separating, then $|Z| \leqslant 2$. Because of this the next lemma applies to feral elements of feral packs.

Lemma 7.27. *(See Fig. 7.2.) Let M be a k-coherent matroid and let (P, X, Q) be a path of 3-separations in M where X is a prime step. Assume that no element of X is in a triangle or a triad, and that there is no 3-separating set $Z \subseteq X$ with $|Z| \geqslant 3$. Let f be a feral element of X. Let $(P_1, P_2, \ldots, P_m)$ and $(Q_1, Q_2, \ldots, Q_n)$ be*

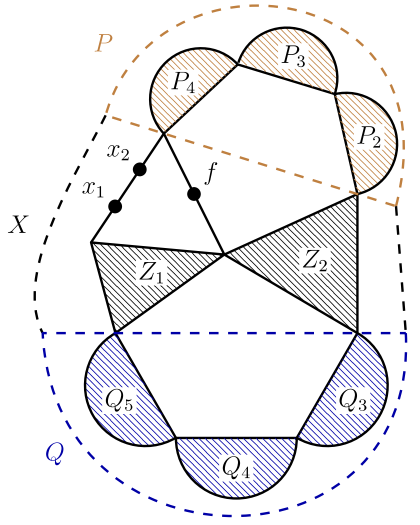

Fig. 7.2. Illustration for Lemma 7.27.

maximal k-fractures of $M\backslash f$ and M/f respectively. Then there is a partition $\{\{x_1, x_2\}, Z_1, Z_2\}$ of $X - \{f\}$ such that, up to labels, duality and equivalence of flowers, the following hold.

(i) *$m = k$, $P_1 = Q \cup Z_1 \cup Z_2$, $P_2 \cup P_3 \cup \cdots \cup P_{k-1} = P$, $P_k = \{x_1, x_2\}$, and $(Q \cup X, P_2, P_3, \ldots, P_{k-1})$ is a swirl-like flower of order $k - 1$ in M.*
(ii) *$n = k$, $Q_3 \cup Q_4 \cup \cdots \cup Q_k = Q$, $Q_1 = Z_1 \cup \{x_1, x_2\}$, $Q_2 = Z_2 \cup P$, and $(P \cup X, Q_3, Q_4, \ldots, Q_k)$ is a swirl-like flower of order $k - 1$ in M.*
(iii) *$M\backslash x_1$ is k-coherent.*

Proof. Up to duality we may assume that $(P_1, P_2, \ldots, P_m)$ and $(Q_1, Q_2, \ldots, Q_n)$ form a feral display for f. It follows that $n = k$ and that $(Q_1 \cup Q_2 \cup \{f\}, Q_3, \ldots, Q_k)$ is a maximal swirl-like flower of M. Now consider the way that the 3-separations $(P, X \cup Q)$ and $(P \cup X, Q)$ interact with this flower. If neither P nor Q is contained in a petal of the flower, then one deduces that X contains a 3-separating set equivalent to the non-sequential 3-separating set $Q_1 \cup Q_2 \cup \{f\}$. Thus, up to switching the labels of P and Q, we may assume that P is contained in a petal Q'. Assume that $Q' \neq Q_1 \cup Q_2 \cup \{f\}$. Then we deduce that $Q_1 \cup Q_2 \cup \{f\} \subseteq X$, contradicting the assumption that X contains no 3-separating set Z with $|Z| \geqslant 3$. Hence $P \subseteq Q_1 \cup Q_2 \cup \{f\}$. If $P \cup X \nsubseteq Q_1 \cup Q_2 \cup \{f\}$, we contradict the assumption that X is prime. Thus $P \cup X \subseteq Q_1 \cup Q_2 \cup \{f\}$. Now $(Q_1, Q_2, \ldots, Q_k)$ is a maximal flower in M/f, and $((P \cup X) - \{f\}, Q)$ is a 3-separation. We know that $(P \cup X) - \{f\} \subseteq Q_1 \cup Q_2$. If $(P \cup X) - \{f\}$ is contained in either $\mathrm{fcl}(Q_1)$ or $\mathrm{fcl}(Q_2)$, we obtain the contradiction that M is k-fractured. Therefore $(P \cup X) - \{f\}$ is equivalent to $Q_1 \cup Q_2$, and we may assume that $(P \cup X) - \{f\} = Q_1 \cup Q_2$.

Consider the flower $(P_1, P_2, \ldots, P_n)$ of $M\backslash f$. As we have a feral display for f, we have $P_1 \supseteq Q_3 \cup Q_4 \cup \cdots \cup Q_k$ and, for some $i \geqslant 3$, the partition $(P_2, P_3, \ldots, P_i, P_{i+1} \cup \cdots \cup P_n \cup \{f\})$ is a swirl-like flower of order i in M. Arguing as before we deduce that $P_2 \cup P_3 \cup \cdots \cup P_i = P$. Now $P_{i+1} \cup P_{i+2} \cup \cdots \cup P_n \subseteq X$ and $\lambda_M(P_{i+1} \cup P_{i+2} \cup \cdots \cup P_n) = 2$. By the assumption that X contains no 3-separating set Z with $|Z| \geqslant 3$, we must have $n = k$, $i = k - 1$, and $|P_n| = 2$. Let $P_k = P_n = \{x_1, x_2\}$. From the properties of a feral display we see that there is a partition (Z_1, Z_2) of $X - \{x_1, x_2\}$ such that $Q_1 = Z_1 \cup \{x_1, x_2\}$ and $Q_2 = Z_2 \cup P$. This shows that parts (i) and (ii) of the lemma hold.

Consider part (iii). The partition $(P_1, P_2, \ldots, P_k)$ is a k-fracture of $M\backslash f$. Moreover, $\{x_1, x_2\}$ is a fully-closed petal of this k-fracture. It now follows from Lemma 6.17 that $M\backslash x_1$ is k-coherent. $\square$

The next lemma is elementary.

Lemma 7.28. *Let S be a sequential 3-separating set in a 3-connected matroid M where $|S|, |E(M) - S| \geqslant 3$ and let (U, V) be a non-sequential 3-separation of M. Then either $S \subseteq \mathrm{fcl}(U)$ or $S \subseteq \mathrm{fcl}(V)$.*

Proof. As S is sequential it contains a triangle or triad T such that $S \subseteq \mathrm{fcl}(T)$. If $|U \cap T| \geqslant 2$, then $S \subseteq \mathrm{fcl}(U \cap T)$, so that $S \subseteq \mathrm{fcl}(U)$. The lemma follows from this observation. □

The next lemma shows that feral packs are essentially the only obstruction to obtaining a satisfactory simplification of a path of 3-separations.

Lemma 7.29. *Let M be a k-coherent matroid and let (P, X, Q) be a path of 3-separations where X is a prime step and $|P|, |Q| \geqslant 3$. Let $X' = (\mathrm{fcl}(P \cup X) \cap Q) \cup (\mathrm{fcl}(Q \cup X) \cap P)$. Assume that there is no fixed element in X' such that $M\backslash x$ is k-coherent and no cofixed element in X' such that M/x is k-coherent.*

(i) *If $|X| \geqslant 2$, then there exists $x \in X$ such that either $M\backslash x$ or M/x is k-coherent.*
(ii) *If $|X| \geqslant 3$, X contains a clonal pair $\{u, v\}$, and X is not a feral pack, then there exists $x \in X - \{u, v\}$ such that either $M\backslash x$ or M/x is k-coherent.*

Proof. If we are in case (i) of the lemma, let $Y = X$ and if we are in case (ii), let $Y = X - \{u, v\}$. Assume that there is no element $y \in Y$ such that either $M\backslash y$ or M/y is k-coherent. Note that, as X is prime, both P and Q are fully closed.

7.29.1. *No element of Y is in both a triangle and a triad.*

Subproof. If $y \in Y$ is in both a triangle and a triad, then y is in a fan F with at least four elements. As $|X| > 1$ and X is prime, it is easily seen that M is not a wheel or a whirl. Hence F has ends. If e is an end of F, then either $M\backslash e$ or M/e is k-coherent by Corollary 4.6 and e is respectively fixed or cofixed. Evidently either $e \in X'$ or $e \in X$. The former case contradicts our assumption about the elements of X'. Consider the latter case. The element e is either fixed or cofixed, so $e \notin \{u, v\}$. Thus $e \in Y$, contradicting our assumption about the elements of Y. □

7.29.2. *No element of X belongs to a triangle or triad.*

Subproof. If the sublemma fails, then we may assume that there is a triangle $T = \{a, b, c\}$ that meets X.

Assume that T is not k-wild. Note that the elements of T are in $X \cup X'$. Say $a \in X'$. We may assume that $a \in P$. Then $a \in \mathrm{cl}(P - \{a\})$, but $T \nsubseteq \mathrm{cl}(P - \{a\})$, so by Lemma 5.16, a is fixed in M. By 7.29.1, and the fact that T is not k-wild, we see that there is an element $z \in T$ such that $M\backslash z$ is k-coherent. The only way that this does not contradict assumptions about the members of $X' \cup Y$ is to have $z \in \{u, v\}$. But in this case $a \notin \{u, v\}$. By Corollary 5.25, $M\backslash a$ is k-coherent again contradicting our assumptions about elements of $Y \cup X'$.

Assume that T is k-wild. By 7.29.1, T is either a standard or costandard k-wild triangle. Let $(A_1, A_2, \dots, A_{k-2}, B_1, B_2, \dots, B_{k-2}, C_1, C_2, \dots, C_{k-2})$ be a k-wild display for T, where $A = A_1 \cup A_2 \cup \dots \cup A_{k-2}$, $B = B_1 \cup B_2 \cup \dots \cup B_{k-2}$ and $C = C_1 \cup C_2 \cup \dots \cup C_{k-2}$. Let (U, V) be a non-sequential 3-separation of M. If neither U nor V is contained in any of $\mathrm{fcl}(A)$, $\mathrm{fcl}(B)$ or $\mathrm{fcl}(C)$, then one deduces by uncrossing that one of the flowers of M displayed by the k-wild display is not maximal. This contradicts the fact that M is k-coherent. This means that for any non-sequential 3-separation (U, V), up to labels, U is contained in either $\mathrm{fcl}(A)$, $\mathrm{fcl}(B)$ or $\mathrm{fcl}(C)$. If $|P \cup Q| = 4$, the sublemma is clear. Thus we may assume that $|P| > 2$. Note that $T \nsubseteq \mathrm{fcl}(P)$. If P is non-sequential, then we see that we can assume that $P \subseteq \mathrm{fcl}(A)$ by the above observation. If P is sequential, the same conclusion follows from Lemma 7.28. If $|Q| > 2$, then we may assume that $Q \subseteq \mathrm{fcl}(B)$ so that X contains a set equivalent to C, and we see that X contains at least three peripheral sets and it is clear that Y contains a peripheral set $Z \subseteq C$

whose full closure avoids $\{u, v\}$. If $p \in \mathrm{fcl}(Z) \cap P$, then p is a loose element of the swirl-like flower $(A_1, A_2, \ldots, A_{k-2}, B \cup C \cup T)$ of M. But then, by Lemma 4.13, up to duality, X' has a fixed element whose deletion preserves k-coherence. Hence $\mathrm{fcl}(Z)$ avoids P and similarly avoids Q. Thus $\mathrm{fcl}(Z) \subseteq Y$. But now, by Corollary 4.37, Y contains an element that can be either deleted or contracted to preserve k-coherence. We omit the easy analysis for the case that $|Q| = 2$. $\square$

7.29.3. *There is no* 3*-separating set* $Z \subseteq X$ *with* $|Z| \geqslant 3$.

Subproof. Assume that Z is such a set. If Z is sequential then it contains a triangle or triad contradicting 7.29.2. Thus Z is non-sequential and $|Z| \geqslant 4$. In this case it follows from Lemma 7.26 Z that contains an element $z \notin \{u, v\}$ such that $M\backslash z$ or M/z is k-coherent. This contradiction establishes the sublemma. $\square$

7.29.4. *For all* $y \in X$*, both* $M\backslash y$ *and* M/y *are* 3*-connected.*

Subproof. If the lemma fails, then, as no element of X is in a triangle or triad, we may assume up to duality that, for some $y \in X$, the element y is in the guts of a vertical 3-separation $(Y_1 \cup \{y\}, Y_2)$. If $Y_1 \subseteq P$, then $y \in \mathrm{cl}(P)$, contradicting the fact that P is fully closed. Hence neither Y_1 nor Y_2 is contained in either P or Q. If $P \subseteq Y_i$ and $Q \subseteq Y_j$ for some permutation (i, j) of $(1, 2)$, then $(P, Y_i - P, \{y\}, Y_j - Q, Q)$ is a path of 3-separations contradicting the fact that P is prime. Thus (Y_1, Y_2) crosses at least one of $(P, X \cup Q)$ and $(P \cup X, Q)$.

Up to symmetry, we may now assume that (Y_1, Y_2) crosses $(P, X \cup Q)$ and further that, if (Y_1, Y_2) crosses $(P \cup X, Q)$, then all vertical 3-separations of M with y in the guts cross both $(P, X \cup Q)$ and $(P \cup X, Q)$. Without loss of generality, $|P \cap Y_1| \geqslant 2$. So, by uncrossing, we see that $\lambda((X \cup Q) \cap Y_2) \leqslant 2$ and $\lambda((X \cup Q) \cap (Y_2 \cup \{y\})) \leqslant 2$. If $r((X \cup Q) \cap Y_2) \geqslant 3$, then $(P \cup Y_1 \cup \{y\}, (X \cup Q) \cap Y_2)$ is a vertical 3-separation with y in the guts that does not cross $(P, X \cup Q)$. Moreover this 3-separation crosses $(P \cup X, Q)$ if and only if (Y_1, Y_2) does and we have contradicted the assumption about the choice of (Y_1, Y_2). Thus $r((X \cup Q) \cap Y_2) \leqslant 2$. If $r((X \cup Q) \cap Y_2) = 2$, then $y \in \mathrm{cl}((X \cup Q) \cap Y_2)$ and y is in a triangle. Hence $|(X \cup Q) \cap Y_2| = 1$, so that $|P \cap Y_2| > 1$. We may now repeat the argument to conclude that $|Y_1 \cap (X \cup Q)| = 1$, so that $|X \cup Q| = 3$ giving numerous contradictions, one of which is to the fact that y is not in a triangle or triad. $\square$

Assume that $Y = X$. Then each element of X is feral. But this contradicts Lemma 7.26. This contradiction shows that part (i) holds. Assume that $Y = X - \{u, v\}$. Then each element of $X - \{u, v\}$ is a feral element and X is a feral pack. This contradiction shows that (ii) holds. $\square$

The next lemma shows that feral packs cannot hunt in teams in a k-skeleton. In fact they cannot hunt in teams in arbitrary paths of 3-separations, but it is easier to prove for k-skeletons.

Lemma 7.30. *Let* M *be a* k*-skeleton with a maximal path* $\mathbf{P} = (P_0, P_1, \ldots, P_l)$ *of* 3*-separations. Assume that* P_i *is a feral pack for some* $i \in \{1, 2, \ldots, l-2\}$*. Then* $\lambda(P_{i+1}) = 2$ *and* $(P_{i+1}^-, P_{i+1}, P_{i+1}^+)$ *is a spiral. In particular,* P_{i+1} *is not a feral pack.*

Proof. By Lemma 7.27, M has a maximal swirl-like flower $\mathbf{R} = (R_1, R_2, \ldots, R_{k-1})$, where $R_1 = P_0 \cup P_1 \cup \cdots \cup P_i$. By Lemma 5.32, swirl-like flowers of order $(k-1)$ in k-skeletons are canonical. Consider the 3-separation $(P_{i+1}^- \cup P_{i+1}, P_{i+1}^+)$ of M. If P_{i+1}^+ is contained in a petal of $\mathbf{R}$, then we contradict the fact that $\mathbf{P}$ is maximal. It follows that a 3-separation equivalent to this is displayed by $\mathbf{R}$, and, as $\mathbf{R}$ is canonical, this 3-separation must be precisely $(P_{i+1}^- \cup P_{i+1}, P_{i+1}^+)$. Thus $P_{i+1}^- \cup P_{i+1}$ is a union of consecutive petals of $\mathbf{R}$ and it follows that $(P_{i+1}^-, P_{i+1}, P_{i+1}^+)$ is a concatenation of $\mathbf{R}$ so that $(P_{i+1}^-, P_{i+1}, P_{i+1}^+)$ is a spiral and $\lambda(P_{i+1}) = 2$. It follows routinely that P_i is not a feral pack. $\square$

5. Proof of the main theorem

We are almost in a position to achieve the primary goal of this chapter and prove Theorem 7.1. We need three more lemmas.

Lemma 7.31. *Let M be a k-skeleton and (P_0, p_1, P_2, P_3) be a path of 3-separations in M where p_1 is a fixed guts singleton and P_2 is prime. Then $(P_0 \cup \{p_1\}, P_2, P_3)$ is a tight spiral in M.*

Proof. As M is a skeleton, no element of M is in both a triangle and a triad. We first prove that $M\backslash p_1$ is 3-connected. Assume that $r(P_0) = 2$. Then $P_0 \cup \{p_1\}$ is a triangle. As p_1 is fixed, this triangle is k-wild by Lemma 5.29. But p_1 is not in a 4-element fan, so this triangle is either standard or costandard in which case $M\backslash p_1$ is 3-connected.

Assume that $r(P_0) > 2$. Then $\mathrm{co}(M\backslash p_1)$ is 3-connected by Bixby's Lemma. Assume that p_1 is in a triad T. As M/p_1 is not 3-connected, T is not k-wild and therefore T is a clonal triple by Lemma 5.29, contradicting the fact that p_1 is fixed in M.

Thus $M\backslash p_1$ is 3-connected and, as p_1 is fixed in M, we see that $M\backslash p_1$ is k-fractured. Let $\mathbf{R} = (R_0, R_1, \dots, R_m)$ be a maximal k-fracture of $M\backslash p_1$. It is easily seen that, for some $j \in \{0, 1, \dots, m-2\}$, we have, up to labels in $\mathbf{R}$, that $P_0 = R_0 \cup \dots \cup R_j$ and that $(P_0 \cup \{p_1\}, R_{j+1}, \dots, R_m)$ is a maximal swirl-like flower of M. By Lemma 5.32, this flower has no loose elements so that $P_0 \cup \{p_1\}$ is fully closed. This means that P_2 is not a singleton and it follows routinely that $(P_0 \cup \{p_1\}, P_2, P_3)$ is a tight spiral in M. □

Recall that $\mathcal{U}_q$ and $\mathcal{U}_q^*$ denote the classes of matroids with no $U_{2,q+2}$- and $U_{q,q+2}$-minor respectively.

Lemma 7.32. *Let M be a k-coherent matroid in $\mathcal{U}(q) \cap \mathcal{U}^*(q)$ with a path $\mathbf{P} = (P_0, P_1, \dots, P_l)$ of 3-separations. Then there is a function $f_{7.32}(m, q)$ such that, if at least $f_{7.32}(m, q)$ internal steps of $\mathbf{P}$ contain clonal pairs, then M contains a k-coherent minor with a path of clonal pairs of length m.*

Proof. Let $m \geqslant 2$ be an integer and let $f_{7.32}(m, q) = 2(q + 2)(m + 1)$. Assume that $n \geqslant f_{7.32}(m, q)$ and that, for this value of m, the matroid M is a minor-minimal counterexample to the lemma. Given the hypotheses of the lemma, we have a distinguished set S of n clonal pairs such that each step of the path contains at most one clonal pair in S. By taking an appropriate concatenation we may assume that P_0 and P_l each contain a clonal pair in S. By Lemma 7.29 and the minimality assumption, we see that each internal step of $\mathbf{P}$ is either a singleton, a clonal pair or a feral pack. A clonal pair $P_i = \{p_i, q_i\}$ may be non-prime so that $(P_0, P_1, \dots, P_{i-1}, p_i, q_i, P_{i+1}, \dots, P_l)$ is also a path of 3-separations, or else it may be prime. For the moment we do not refine the path by splitting non-prime clonal pairs.

If M has a fixed element x such that $M\backslash x$ is k-coherent, then it is clear that the induced path in $M\backslash x$ has the properties of $\mathbf{P}$ described above. By this observation, its dual and the minimality assumption we deduce that M is a k-skeleton. It now follows by Lemma 7.30, that if $i \in \{2, 3, \dots, l-2\}$ and P_i is a feral pack, then both P_{i-1} and P_{i+1} are prime clonal pairs. Thus at most $n/2$ of the steps that contain clonal pairs are feral packs. We now sacrifice the clonal pairs in feral packs. Let S' denote the set of clonal pairs in S that are not in feral packs. Then S' contains at least $n/2$ clonal pairs. Let M' be a minor-minimal k-coherent minor of M whose ground set contains S' and let $\mathbf{Q} = (Q_0, Q_1, \dots, Q_{l'})$ be the path of 3-separations induced by $\mathbf{P}$ in M'. Note that $\mathbf{Q}$ is well defined since both P_0 and P_l contain clonal pairs in S'. By Lemma 7.29, the internal steps of $\mathbf{Q}$ are either clonal pairs or singletons. It is also clear that M' is a k-skeleton and that if $Q_i = \{q_i\}$ is a singleton internal step, then neither $M'\backslash q_i$ nor M'/q_i is k-coherent.

7.32.1. *If $2 \leqslant i \leqslant l' - 2$ and $Q_i = \{q_i\}$ is a guts singleton, then q_i is not fixed in M'.*

Subproof. Assume that q_i is a fixed guts singleton. By Lemma 7.31 $(Q_i^+ \cup \{q_i\}, Q_{i+1}, Q_{i+1}^+)$ is a tight spiral flower in M, so that Q_{i+1} is a prime clonal pair. Evidently $q_i \notin \mathrm{cl}(Q_{i+1})$ and $q_i \notin \mathrm{cl}(Q_{i+1}^+)$. Thus, by Lemma 7.11, q_i is not fixed from the right in M'. A symmetric argument shows that q_i is not fixed from the left in M'. By Lemma 7.10, q_i is not fixed in M. □

Refine $\mathbf{Q}$ to a maximal path by splitting the non-prime clonal pairs in internal steps to obtain a path $\mathbf{Q}' = (Q'_0, Q'_1, \ldots, Q'_l)$. By 7.32.1 and its dual, all the singletons of this path are either unfixed guts singletons or uncofixed coguts singletons. If there are at least $2(q+2)$ such singletons, then, by Lemma 7.13, M has either a $U_{2,q+2}$- or a $U_{q,q+2}$-minor contradicting the fact that $M \in \mathcal{U}(q) \cap \mathcal{U}^*(q)$. Thus there are at most $2(q+2)-1$ such singletons, so that $\mathbf{Q}'$ has a section of at least $f_{7.32}(q,m)/2(q+2) = m+1$ consecutive prime clonal pairs. We conclude that M has a k-coherent minor with a path of clonal pairs of length m contradicting the assumption that M was a counterexample. □

Finally, a lemma about the way the gangs of three can appear in paths. The lemma follows easily from the structure of the 3-separations of M around a gang of three and we omit the proof.

Lemma 7.33. *Let (P, X, Q) be a path of 3-separations in the k-coherent matroid M. If $\{x, y, z\}$ is a gang of three in M and $\{x, y, z\} \cap X \neq \emptyset$, then $\{x, y, z\} \subseteq X$ and $|X| \geqslant 4$.*

We are at last in a position to prove Theorem 7.1 which we restate here for convenience.

Theorem 7.34. *Let $k \geqslant 5$ and $q \geqslant 2$ be integers. Then there is a function $f_{7.34}(k,q)$ such that, if M is a k-skeleton with a path of 3-separations of length $f_{7.34}(k,q)$, then $M \notin \mathcal{E}(q)$.*

Proof. Let $f_{7.34}(k,q) = 2f_{7.32}(f_{7.25}(k,q),q) + 2(q+2) + 1$. Assume that M is a minor-minimal counterexample to the theorem in that M is a k-skeleton in $\mathcal{E}(q)$ with a path of 3-separations of length $n = f_{7.34}(k,q)$, but no proper k-skeleton minor of M has such a path. Let $\mathbf{P} = (P_0, P_1, \ldots, P_n)$ be a path of 3-separations of length n in M. We may assume that each internal step of $\mathbf{P}$ is prime.

7.34.1. *Each internal step of* $\mathbf{P}$ *is either a singleton or contains a clonal pair.*

Subproof. Assume that P_i is an internal step that has at least two elements but no clonal pair. As M is a k-skeleton, M has no fixed element x such that $M\backslash x$ is k-coherent and dually M has no cofixed element such that M/x is k-coherent. Thus, by Lemma 7.29(i), there is an element $x \in P_i$ such that either $M\backslash x$ or M/x is k-coherent. If either $M\backslash x$ or M/x is a k-skeleton, then $(P_0, P_1, \ldots, P_{i-1}, P_i - \{x\}, P_{i+1}, \ldots, P_n)$ is a path of 3-separations in a proper k-skeleton minor of M, contradicting the choice of M. So neither $M\backslash x$ nor M/x is a k-skeleton. If x is comparable with another element, then, by Theorem 5.33, either $M\backslash x$ or M/x is a k-skeleton. Thus x is not comparable with any other element of M. In this case, by Theorem 5.36, it must be the case that, up to duality, x is in a gang of three. Say $\{x, y, z\}$ is such a gang of three. By Lemma 7.33, $\{x, y, z\} \subseteq P_i$ and $|P_i| \geqslant 4$. But now, by Theorem 5.42, $M/x\backslash y, z$ is a k-skeleton, and $(P_0, P_1, \ldots, P_{i-1}, P_i - \{x, y, z\}, P_{i+1}, \ldots, P_n)$ is a path of 3-separations of length n in a proper k-skeleton minor of M, again contradicting the choice of M. □

If there are at least $q+2$ unfixed guts singletons or $q+2$ uncofixed coguts singletons in $\mathbf{P}$, then, by Lemma 7.13, M has a $U_{2,q+2}$- or a $U_{q,q+2}$-minor. Thus there are at least $n-2(q+2)$ steps that are either fixed guts singletons, cofixed coguts singletons or contain a clonal pair. By Lemma 7.31 and its dual, if $2 < i < n-1$, and $P_i = \{p_i\}$ is a fixed guts singleton or cofixed coguts singleton, then neither P_{i-1} nor P_{i+1} are singletons and therefore both of these sets contain clonal pairs. Hence there are at least $(n-2(q+2)-1)/2$ internal steps that contain clonal pairs, that is, at least $f_{7.32}(f_{7.25}(k,q),q)$ internal steps contain clonal pairs. By Lemma 7.32, M has a k-coherent minor with a path of clonal pairs of length $f_{7.25}(k,q)$. But now, by Corollary 7.25, M is not in $\mathcal{U}(q) \cap \mathcal{U}^*(q)$, contradicting the assumption that M was a counterexample to the theorem. □

6. A last lemma

We conclude this chapter with a lemma on paths of 3-separations that focusses on displayed swirl-like flowers without making assumptions about the k-coherence of the matroid. We will use the following straightforward result.

Lemma 7.35. *Let $(P_1, P_2, \dots, P_l)$ be a maximal swirl-like flower of order $l \geqslant 5$ in the 3-connected matroid M. Assume that P_1 is fully closed. Then the following hold.*

(i) *M has a 3-connected minor on $P_2 \cup P_3 \cup \dots \cup P_l$.*
(ii) *If M' is a 3-connected minor of M on $P_2 \cup P_3 \cup \dots \cup P_l$, and (X, Y) is a 3-separation of M', then either $(X \cup P_1, Y)$ or $(X, Y \cup P_1)$ is a 3-separation of M.*

Proof. Part (i) follows from Theorem 4.36 and an easy induction. Consider (ii). It is easily seen that $(P_2, P_3, \dots, P_l)$ is a maximal flower in M'. Say (X, Y) is a 3-separation of M'. If (X, Y) is displayed by $(P_2, P_3, \dots, P_l)$, then (ii) follows easily. Otherwise we may assume that $X \subseteq \mathrm{fcl}_{M'}(P_i)$ for some $i \in \{2, 3, \dots, l\}$. But then $(X, Y \cup P_1)$ is clearly a 3-separation of M. $\square$

Let $\mathbf{P} = (P_0, P_1, \dots, P_n)$ be a path of 3-separations in the connected matroid M and let $\{p_i, q_i\}$ be a clonal pair contained in P_i for some $i \in \{0, 1, \dots, n\}$. Then $\{p_i, q_i\}$ is $\mathbf{P}$-*strong* if $\kappa(\{p_i, q_i\}, P_0 \cup P_1 \cup \dots \cup P_{i-1} \cup P_{i+1} \cup \dots \cup P_n) = 2$. In other words, $\{p_i, q_i\}$ is $\mathbf{P}$-strong if there is no 2-separating set A with the property that $\{p_i, q_i\} \subseteq A \subseteq P_i$.

Lemma 7.36. *Let $\mathbf{P} = (P_0, P_1, \dots, P_n)$ be a path of 3-separations in the connected matroid M each step of which contains a $\mathbf{P}$-strong clonal pair and let k be an integer. Then there is a function $f_{7.36}(k, q)$ such that, if $\mathbf{P}$ displays no swirl-like flower of order k, and $n \geqslant f_{7.36}(k, q)$, then $M \notin \mathcal{E}(q)$.*

Proof. Assume that $k \geqslant 5$. Let $f_{7.36}(k, q) = f_{7.1}(k, q) + 2$. We claim that the lemma holds with this definition of $f_{7.36}(k, q)$.

Assume not. Let M be a counterexample to the lemma, so that M contains a path $\mathbf{P} = (P_0, P_1, \dots, P_l)$ of 3-separations satisfying the hypotheses of the lemma where $l \geqslant f_{7.36}(k, q)$, and $M \in \mathcal{E}(q)$.

Assume that M is chosen to have a ground set of minimum cardinality. It follows routinely from the definitions of path of 3-separations and $\mathbf{P}$-strong clonal pair that, under this assumption, M is 3-connected. We now show that M is k-coherent. Assume not. Then there is a swirl-like flower $\mathbf{Q} = (Q_1, Q_2, \dots, Q_t)$ of order $t \geqslant k$ in M. A subflower of $\mathbf{Q}$, displayed by $\mathbf{P}$ has order at most $k - 1$. It follows from this fact and Lemma 7.14, that, up to flowers equivalent to $\mathbf{Q}$, there is a step P_i of $\mathbf{P}$ such that $Q_j \cup Q_{j+1} \subseteq P_i$ for some $j \in \{1, 2, \dots, t\}$. Let $\{p, p'\}$ be a $\mathbf{P}$-strong clonal pair in P_i. Then we may assume that Q_j avoids $\{p, p'\}$. It is also easily seen that we may assume that P_i is fully closed. By Lemma 7.35(i), we may remove elements from P_i to obtain a 3-connected minor M' of M. But it follows from Lemma 7.35(ii) that $(P_1, P_2, \dots, P_{i-1}, P_i - Q_j, P_{i+1}, \dots, P_l)$ is a path of 3-separations in M' satisfying the hypotheses of the lemma and we have contradicted the minimality of the choice of $|E(M)|$.

Thus M is k-coherent. This contradicts Corollary 7.2. In the case that $k \leqslant 5$, the lemma holds by letting $f_{7.36}(k, q) = f_{7.1}(5, q) + 2$. $\square$

Finally we note a result that is a more-or-less immediate corollary of Lemma 7.36 and Lemma 6.11.

Corollary 7.37. *Let $\mathbf{P}$ be a path of 3-separations of length n in the connected matroid M in $\mathcal{E}(q)$, each step of which contains a $\mathbf{P}$-strong clonal pair. Then there is a function $f_{7.37}(m, q)$ such that, if $n \geqslant f_{7.37}(m, q)$, then M has a Δ_m-minor, each leg of which is equal to $\{p_i, q_i\}$ for some i.*

Chapter 8. Taming a skeleton

1. Introduction

In the previous chapter we proved that a k-skeleton in $\mathcal{E}(q)$ cannot have an arbitrarily long path of 3-separations. In this chapter we continue the process of controlling structure in skeletons. The goal is to prove the following theorem.

Theorem 8.1. *There is a function $f_{8.1}(m,k,q)$ such that, if M is a k-skeleton in $\mathcal{E}(q)$ with at least $f_{8.1}(m,k,q)$ elements, then M has a 4-connected minor whose ground set contains a set of m pairwise-disjoint clonal pairs.*

With this theorem in hand we can forget about k-skeletons and focus on 4-connected matroids with many clonal pairs. We begin by learning more about 3-trees associated with k-skeletons in $\mathcal{E}(q)$. We already know that they cannot contains arbitrarily long induced paths. If v is a flower vertex of such a tree, then, as M is a k-skeleton the associated flower cannot have high order as it induces a path of 3-separations in M. But so far, there is nothing to control the degree of a bag vertex.

2. Potatoes

In this section we prove the following theorem.

Theorem 8.2. *Let M be a k-skeleton in $\mathcal{E}(q)$, and let T be a 3-tree for M. Then there is a function $f_{8.2}(m,k,q)$ such that, if T has at least $f_{8.2}(m,k,q)$ vertices, then M has a 4-connected minor with a set of m pairwise-disjoint clonal pairs.*

As noted above we need to control the degree of bag vertices in a 3-tree for a k-skeleton. Such a vertex v typically exemplifies a more highly connected part of the matroid, attached to which are the 3-separating sets displayed by v. This motivates the definition of "potato" that we now give.

Let M be a 3-connected matroid and $n \geqslant 0$ be an integer. A *potato* of M is a partition $\{P_1, P_2, \dots, P_n\}$ of a subset of $E(M)$ such that the following hold.

(i) $\lambda(P_i) = 2$, $|P_i| \geqslant 3$, and $|E(M) - P_i| \geqslant 5$, for all $i \in \{1,2,\dots,n\}$.
(ii) If (X,Y) is a 3-separation of M, then, for some $i \in \{1,2,\dots,n\}$, either X or Y is contained in P_i.
(iii) $\lambda(P_i \cup P_j) \geqslant 3$ for all distinct i and j in $\{1,2,\dots,n\}$.

The set $E(M) - (P_1 \cup P_2 \cup \dots \cup P_n)$ is the *core* of the potato and each P_i is an *eye* of the potato. A potato may have an empty core. If M is 4-connected, then the empty set of eyes defines a potato. If $n \geqslant 4$, then condition (iii) is redundant, but if $n \in \{2,3\}$, then (iii) says that a potato cannot be a 3-petal flower. Note that the eyes of a potato are fully-closed sets.

Readers familiar with matroid tangles (see for example [9]) will recognise that, except for trivially small matroids, a tangle τ of order 4 can be associated with a potato $\mathbf{P}$ of the 3-connected matroid M. If $\lambda(X,Y) \leqslant 2$, then $(X,Y) \in \tau$ if either $|X| \leqslant 2$, or X is contained in an eye of $\mathbf{P}$. Of course, not all tangles of order 4 can be derived from a potato in this way.

Our main interest is in potatoes in k-coherent matroids and k-skeletons. If $\mathbf{P}$ is a potato of the k-coherent matroid M, then one annoying possibility is that one of its eyes is a k-wild triangle or triad. Let $\{a,b,c\}$ be a k-wild triangle of M. Assume that $\{a,b,c\}$ is an eye of $\mathbf{P}$. Then $\{a,b,c\}$ is certainly not in a 4-element fan, so that it is either standard or costandard, in which case there is a partition $(A,B,C,\{a,b,c\})$ of $E(M)$ such that A, B, and C are each non-trivial 3-separating sets. If $\{a,b,c\}$ is standard, then $(A \cup \{a\}, B \cup C \cup \{b,c\})$ is a 3-separation that crosses $\{a,b,c\}$, so it must be the case that $\{a,b,c\}$ is costandard. In this case there must be eyes that contain A, B and C, so the potato must be $\{A,B,C,\{a,b,c\}\}$. Using the above notation we have proved the following lemma.

Lemma 8.3. *If $\{a,b,c\}$ is a k-wild triangle of the k-coherent matroid M with associated partition $(A,B,C,\{a,b,c\})$, and $\{a,b,c\}$ is an eye of the potato $\mathbf{P}$, then $\{a,b,c\}$ is costandard and $\mathbf{P} = \{A,B,C,\{a,b,c\}\}$. In particular $\mathbf{P}$ has four eyes, an empty core, and at most one eye of $\mathbf{P}$ is a k-wild triangle or triad.*

Such a result does not hold for a costandard k-wild triangle in an arbitrary k-coherent matroid the converse of Lemma 8.3 need not hold as, for example, there may be a single element in $\mathrm{cl}(A) \cap B$. But this result does hold for k-skeletons. We omit the easy proof.

Lemma 8.4. *If $\{a, b, c\}$ is a costandard k-wild triangle of the k-skeleton M with associated partition $(A, B, C, \{a, b, c\})$, then $\{A, B, C, \{a, b, c\}\}$ is a potato in M.*

The next lemma helps us to find potatoes in k-skeletons.

Lemma 8.5. *Let M be a k-skeleton and let A_1 and A_2 be 3-separating sets each having at least three elements. If $\lambda(A_1 \cup A_2) > 2$ and $A_1 \cap A_2 \neq \emptyset$, then either A_1 or A_2 is a k-wild triangle or triad.*

Proof. Assume that $A_1 \cap A_2 \neq \emptyset$. As $\lambda(A_1) = \lambda(A_2) = 2$ and $\lambda(A_1 \cup A_2) > 2$, we have $\lambda(A_1 \cap A_2) < 2$. Thus $|A_1 \cap A_2| = 1$. Say $A_1 \cap A_2 = \{a\}$. As A_1 and $E(M) - A_2$ are 3-separating and the union of these sets avoids at least two elements of M, we have $\lambda(A_1 \cap (E(M) - A_2)) \leqslant 2$, that is, $\lambda(A_1 - \{a\}) = 2$ and similarly $\lambda(A_2 - \{a\}) = 2$. Now $a \in \mathrm{cl}^{(*)}(A_1 - \{a\})$ and, up to duality, we may assume that $a \in \mathrm{cl}(A_1 - \{a\})$. Evidently $a \notin \mathrm{cl}^*(A_2 - \{a\})$, so that we also have $a \in \mathrm{cl}(A_2 - \{a\})$.

We now show that a is fixed in M. Say that $\sqcap(A_1, A_2) = 2$. Then

$$\lambda(A_1 \cup A_2) = \lambda(A_1) + \lambda(A_2) - \sqcap(A_1, A_2) - \sqcap^*(A_1, A_2) \leqslant 2.$$

Thus $\sqcap(A_1, A_2) = 1$. But $\sqcap(A_1 - \{a\}, A_2 - \{a\}) = \sqcap(A_1, A_2)$, and it follows by Lemma 5.11 that a is fixed in M.

If either A_1 or A_2 is a triangle, then, by Lemma 5.29, that triangle is k-wild and the lemma holds. Therefore we may assume that neither A_1 nor A_2 is a triangle and indeed that $r(A_1), r(A_2) \geqslant 3$. Thus $\mathrm{si}(M/a)$ is not 3-connected. Now by Bixby's Lemma and the fact that M has no 4-element fans, the matroid $M\backslash a$ is 3-connected and therefore k-fractured; that is, a is semi-feral. Let $\mathbf{P} = (P_1, P_2, \ldots, P_m)$ be a maximal fracture of $M\backslash a$. As a is semi-feral, a is in the guts of a 3-separation. If this 3-separation is not well displayed by $\mathbf{P}$, then we obtain the contradiction that M is not k-coherent. It follows that there exists $j \in \{2, 3, \ldots, m-1\}$ such that a is in the guts of $(P_1 \cup P_2 \cup \cdots \cup P_j, P_{j+1} \cup P_{j+2} \cup \cdots \cup P_m)$. As M is a k-skeleton and a is fixed in M, it follows from Theorem 5.35 that $\mathbf{P}$ is canonical. Note that, if $A_1 - \{a\}$ is contained in a petal of $\mathbf{P}$, then $a \notin \mathrm{cl}(A_1 - \{a\})$. Thus $A_1 - \{a\}$ and similarly $E(M\backslash a) - A_1$ is not contained in a petal of $\mathbf{P}$. Thus, by Lemma 3.32, $A_1 - \{a\}$ is equal to a union of petals of $\mathbf{P}$. It is easily seen that this can only hold if, up to labels, $A_1 - \{a\} = P_1 \cup P_2 \cup \cdots \cup P_j$. Similarly we must have $A_2 - \{a\} = P_{j+1} \cup P_{j+2} \cup \cdots \cup P_m$. But then we obtain the contradiction that $\sqcap(A_1, A_2) = 2$. □

The next two lemmas show that we may shrink the eyes of a potato $\mathbf{P}$ to find a 4-connected minor of a k-skeleton with a clonal pair for each eye of $\mathbf{P}$.

Lemma 8.6. *Let $\mathbf{P} = \{P_1, P_2, \ldots, P_n\}$ be a potato of the k-skeleton M. Then there is a k-skeleton minor M' of M with a potato $\mathbf{P}' = \{P'_1, P'_2, \ldots, P'_n\}$ such that the following hold.*

(i) *The core of $\mathbf{P}'$ is equal to the core of $\mathbf{P}$.*
(ii) *$P'_i \subseteq P_i$ for $i \in \{1, 2, \ldots, n\}$.*
(iii) *At most one eye of $\mathbf{P}'$ does not contain a clonal pair.*

Proof. Let P be an eye of $\mathbf{P}$. Assume that $|P| \geqslant 4$. If P is sequential with at least four elements, then P contains a triangle or triad that is certainly not k-wild and again P contains a clonal pair. Assume that P is non-sequential. Then P contains a strongly peripheral set Z. If Z contains no clonal pairs, then, by Lemma 6.10, there is an element $z \in Z$ such that $M\backslash z$ or M/z is a k-skeleton. Up to duality we may assume that $M\backslash z$ is a k-skeleton. Assume that $P = P_1$. We now show that $\{P_1 - \{z\}, P_2, \ldots, P_n\}$ is potato of $M\backslash z$.

Condition (iii) of a potato clearly holds. Consider condition (i). If this fails, then $|E(M\backslash z) - P_i| < 5$ for some $i > 1$, that is $|E(M) - P_i| = 5$. But, in this case, $n = 2$ and the core of $\mathbf{P}$ has at most one element, meaning that $\lambda_M(P_1 \cup P_2) < 3$ contradicting the definition of a potato.

For the more substantial case consider property (ii) of a potato. Note that, as $M\backslash z$ is 3-connected, $z \in \mathrm{cl}(P_1 - \{z\})$. Let (X, Y) be a 3-separation of $M\backslash z$. If X contains $P_1 - \{z\}$, then $(X \cup \{z\}, Y)$ is a 3-separation of M and, by the definition of potato, $Y \subseteq P_i$ for some $i \in \{1, 2, \dots, n\}$, so that (ii) holds in this case. Thus we may assume that (X, Y) crosses $(P_1 - \{z\}, E(M) - P_1)$. We now consider subcases of this case. Assume that $\lambda_{M\backslash z}(X - P_1) = \lambda_{M\backslash z}(Y - P_1) = 2$. These 3-separating sets are not blocked by z and it follows that $(X - P_1, Y - P_1, P_1)$ is a partition of $E(M)$ into 3-separating sets. From the definition of potato, $|E(M) - P_1| \geqslant 5$, so we may assume that $|X - P_1| \geqslant 3$. Consider the 3-separation $(X - P_1, Y \cup P_1)$. By property (ii) of a potato, we may assume that $X - P_1$ is contained in P_2. But then we have the contradiction that $Y - P_1$ is contained in both P_3 and P_4. Therefore it is not the case that $\lambda_{M\backslash z}(X - P_1) = \lambda_{M\backslash z}(Y - P_1) = 2$.

If $\lambda_{M\backslash z}(X - P_1) < 2$, then $|X - P_1| = 1$ so that $P_1 - \{z\}$ is not fully closed in $M\backslash z$ contradicting the fact that P_1 is fully closed in M.

For the last, somewhat irritating, subcase we may assume that $\lambda_{M\backslash z}(X - P_1) > 2$. This means that $|Y \cap (P_1 - \{z\})| = 1$. But now, $P_1 - \{z\}$ and Y are 3-separating sets in the k-skeleton $M\backslash z$ that meet, and $\lambda((P_1 - \{z\}) \cup Y) > 3$. By Lemma 8.5 we deduce that Y is a k-wild triangle or triad of $M\backslash z$. Say $Y = \{a, b, c\}$, where $\{a\} = Y \cap (P_1 - \{z\})$. Note that, as $\mathbf{P}$ is a potato in M, the set $\{a, b, c\}$ is blocked in M, so $\{a, b, c\}$ is a triad, and $a \in \mathrm{cl}^*(P_1 - \{z\})$. Let $(A_1, A_2, \dots, A_{k-2}, B_1, B_2, \dots, B_{k-2}, C_1, C_2, \dots, C_{k-2})$ be a k-wild display for $\{a, b, c\}$ and let $A = A_1 \cup A_2 \cup \dots \cup A_{k-2}$, $B = B_1 \cup B_2 \cup \dots \cup B_{k-2}$ and $C = C_1 \cup C_2 \cup \dots \cup C_{k-2}$. We have $a \in \mathrm{cl}^*_{M\backslash z}(P_1 - \{z, a\})$, but $P_1 - \{z\}$ is fully closed in $M\backslash z$. It is readily checked that, since M is a k-skeleton, this implies that $P_1 - \{z\} = A \cup \{a\}$. In this case $\{a, b, c\}$ is a standard k-wild triad. Therefore $B \cup \{b\}$ and $C \cup \{c\}$ are 3-separating. Certainly they are not triangles or triads. Thus we may assume that for some distinct i and j, we have $B \cup \{b\} \subseteq P_i$ and $C \cup \{c\} \subseteq P_j$. Hence $\mathbf{P} = (P_1, P_i, P_j) = (A \cup \{a, z\}, B \cup \{b\}, C \cup \{c\})$. But now $\lambda_M(P_1 \cup P_i) = 2$, contradicting the definition of a potato. Therefore $\{P_1 - \{z\}, P_2, \dots, P_n\}$ is indeed a potato in $M\backslash z$.

The process may be repeated to obtain a k-skeleton minor M' of M with a potato $\mathbf{P}' = \{P'_1, P'_2, \dots, P'_n\}$ such that each eye either contains a clonal pair of M' or is a triangle or triad. If one of these is k-wild then the lemma holds by Lemma 8.3. Otherwise each triangle or triad is a clonal triple and the lemma holds in this case too. □

Lemma 8.7. *Let $\mathbf{P} = \{P_1, P_2, \dots, P_n\}$ be a potato of the 3-connected matroid M with the property that at most one eye of $\mathbf{P}$ does not contain a clonal pair. Then M contains a 4-connected minor containing the core of $\mathbf{P}$ and a set of $n - 1$ pairwise-disjoint clonal pairs.*

Proof. Assume that the potato $\mathbf{P} = \{P_1, P_2, \dots, P_n\}$ has core C. Let $\{p, q\}$ be a clonal pair in P_1. By Tutte's Linking Theorem there is a 3-connected minor N of M on $\{p, q\} \cup P_2 \cup \dots \cup P_n \cup C$ such that $N|(P_2 \cup P_3 \cup \dots \cup P_n \cup C) = M|(P_2 \cup P_3 \cup \dots \cup P_n \cup C)$. We show that $\{P_2, P_3, \dots, P_n\}$ is a potato in N.

Properties (i) and (iii) of a potato are clear. Consider (ii). Let (X, Y) be a 3-separation in N. Assume that $\{p, q\} \subseteq X$. Then $(X \cup P_1, Y)$ is a 3-separation in M. But $X \neq \{p, q\}$, so $X \cup P_1$ is not contained in an eye of $\mathbf{P}$. Therefore $Y \subseteq P_i$ for some $i \in \{2, 3, \dots, n\}$ as required. Assume that $p \in X$ and $q \in Y$. As p and q are clones, we may assume, up to duality, that both p and q are in the guts of (X, Y). Thus $\{p, q\} \subseteq \mathrm{cl}(Y - \{q\})$. If $|Y - \{q\}| = 2$, then $N|(Y \cup \{p\}) \cong U_{2,4}$ and, if $y \in Y - \{q\}$, then $(P_1 \cup \{y\}, E(M) - (P_1 \cup \{y\}))$ is a 3-separation in M that violates condition (ii). Therefore $(X \cup \{q\}, Y - \{q\})$ is a 3-separation of N. Arguing as above we have $Y \subseteq P_i$ for some $i \in \{2, 3, \dots, n\}$, so that $\{p, q\} \subseteq \mathrm{cl}_N(P_i)$. It is now easily checked that $\lambda_M(P_1 \cup P_i) = 2$, contradicting the fact that $\mathbf{P}$ satisfies (iii).

Thus $\{P_2, P_3, \dots, P_n\}$ is indeed a potato in N. The core of this potato is $C \cup \{p, q\}$. Repeating this process n times gives a 3-connected matroid M' with a potato having an empty set of eyes and a core consisting of C together with a disjoint set of n clonal pairs. If (X, Y) is a 3-separation then either X or Y is contained in an eye of this potato. This impossibility shows that M' is 4-connected. □

Sequential 3-separating sets Here we simply recall for convenience some elementary facts and prove an easy lemma. Let Z be a sequential 3-separating set of the 3-connected matroid M. Recall that a *sequential ordering* for Z is an ordering $(z_1, z_2, \dots, z_t)$ of Z such that, $\{z_1, z_2, \dots, z_i\}$ is 3-separating for all $i \in \{1, 2, \dots, t\}$. Recall also that subset Z' of Z *generates* Z if some ordering of Z' is an initial segment of a 3-sequence for Z. If $|Z'| \geqslant 2$, then it is easily seen that Z' generates Z if and only if $Z \subseteq \mathrm{fcl}(Z')$. The first three elements of a 3-sequence for Z either form a triangle or triad, so every sequential 3-separating set with at least three elements is generated by some triangle or triad T. It is then immediate that any 2-element subset of T generates Z.

Lemma 8.8. *Let Z be a sequential 3-separating set of the 3-connected matroid M and let X and Y be 3-separating subsets of Z where $Z \subseteq X \cup Y$. Then either X or Y generates Z.*

Proof. Note that either X or Y contains at least two elements of a generating triangle or triad for Z. Assume the former. Then $Z \subseteq \mathrm{fcl}(T \cap X)$ so that $Z \subseteq \mathrm{fcl}(X)$. By Lemma 2.33 X is sequential. Thus X has a sequential ordering. As $Z \subseteq \mathrm{fcl}(X)$ this extends to a sequential ordering of Z. □

Strong bag vertices Let T be a 3-tree of the 3-connected matroid M, and let v be a bag vertex of T. More highly connected parts of the matroids—in other words tangles of order 4—are identified by bag vertices, but not all bag vertices perform this role. If the bag vertex v has degree at least 4, then it is certainly the case that v identifies a more connected part of the matroid. For the main result of this section we are only interested in vertices of high degree, but later we will care about the low-degree case.

In the low-degree case, the issue is to identify whether there is substance in the bag. In general, having a lot of elements in B_v does not suffice as we could, for example, just be finding a large sequential 3-separating set. Say that (X, Y) is a non-sequential 3-separation. Then there is a path $(\mathrm{coh}(X), z_1, z_2, \dots, z_n, \mathrm{coh}(Y))$ of 3-separations of M and the internal elements of this path can all be in a bag of low degree. If M is a k-skeleton in $\mathcal{E}(q)$, then, by Theorem 7.1, this number is bounded by $f_{7.1}(k, q)$. In fact a much more modest bound is straightforwardly seen to hold, but we take the lazy way out.

Lemma 8.9. *Let M be a k-skeleton in $\mathcal{E}(q)$ and let (X, Y) be a 3-separation of M.*

(i) *If X is sequential, then X has at most $f_{7.1}(k, q)$ elements.*
(ii) *If (X, Y) is non-sequential, then $E(M) - (\mathrm{coh}(X) \cup \mathrm{coh}(Y))$ has at most $f_{7.1}(k, q)$ elements.*

Let M be a k-skeleton in $\mathcal{E}(q)$ and let T be a 3-tree for M. If v is a bag vertex of T, then we denote the subset of $E(M)$ that labels v by B_v. If we say that B is a *bag* of M, then we mean that $B = B_v$ for some bag vertex of a 3-tree for M. We now define what it means for a bag vertex to be *strong*. If v has degree at least 3, then v is strong. If v has degree at most 2, then v is strong if $|B_v| \geqslant 4 f_{7.1}(k, q)$.

Let v be a strong bag vertex of T. Define the collection $\mathcal{E}_v$ of 3-separating sets of M as follows. A 3-separating set X is in $\mathcal{E}_v$ if $|X| \geqslant 3$ and X is equivalent to a 3-separating set displayed by v or is equivalent to one contained in B_v. Note that as 3-trees do not attempt to display sequential 3-separating sets, there may well be many of the latter type. The maximal members of $\mathcal{E}_v$ are the *eyes* of v.

Lemma 8.10. *Let T be a 3-tree for the k-skeleton M and let v be a strong bag vertex of T. Then the eyes of v form a potato of M.*

Proof. Let $\mathbf{P} = \{P_1, P_2, \dots, P_n\}$ be the collection of eyes of v.

8.10.1. *The lemma holds if an eye of $\mathbf{P}$ is a k-wild triangle.*

Subproof. Say $P_1 = \{a, b, c\}$ is k-wild with associated partition $(A, B, C, \{a, b, c\})$ obtained from a k-wild display. Evidently $\{a, b, c\}$ is costandard. The sets A, B, C and $\{a, b, c\}$ are clearly the eyes of v and by Lemma 8.4 they form a potato. □

In what follows we may assume that no eye of $\mathbf{P}$ is a k-wild triangle or triad.

8.10.2. *$|E(M) - P_i| \geqslant 5$ for all $i \in \{1, 2, \dots, n\}$.*

Subproof. The claim is clear if $n \geqslant 3$. If $n \leqslant 2$, then the sublemma is a consequence of the definition of strong bag vertex and Lemma 8.9. □

8.10.3. *If P_i and P_j are distinct eyes of $\mathbf{P}$, then $\lambda(P_i \cup P_j) \geqslant 3$.*

Subproof. Consider the eyes P_1 and P_2 of $\mathbf{P}$. Assume that $\lambda(P_1 \cup P_2) \leqslant 2$. Consider the case that $n = 2$. Then $d(v) \leqslant 2$, so that $|B_v| \geqslant 4 f_{7.1}(k, q)$. Let $B'_v = B_v - (\mathrm{fcl}(P_1) \cup \mathrm{fcl}(P_2))$. Certainly $\lambda(B'_v) \leqslant 2$, and, by Lemma 8.9, $|B'_v| \geqslant 2 f_{7.1}(k, q)$. But then B'_v is a non-sequential 3-separating set that is not displayed by T, contradicting the definition of a 3-tree.

Assume that $n \geqslant 3$. Then $\lambda(P_1 \cup P_2) \geqslant 2$ so that $\lambda(E(M) - (P_1 \cup P_2)) = 2$. If this set is sequential, then $d(v) \leqslant 2$ and we again obtain the contradiction that $B_v - (\mathrm{fcl}(P_1) \cup \mathrm{fcl}(P_2))$ is a sequential 3-separating set with more than $f_{7.1}(k, q)$ elements. Hence $E(M) - (P_1 \cup P_2)$ is non-sequential. If $P_1 \cup P_2$ is sequential, then by Lemma 8.8, $P_1 \cup P_2$ is generated by either P_1 or P_2 contradicting the definition of eyes. So $P_1 \cup P_2$ is non-sequential. But, again, it is easily seen that $(P_1, P_2 - P_1, E(M) - (P_1 \cup P_2))$ is a flower in M. As this flower is not displayed in T, at least one petal, say P_1 must be sequential and v has degree at most 2, so that B_v has at least $4 f_{7.1}(k, q)$ elements. But, again by Lemma 8.9 we see that B_v is covered by three sets whose union has at most $3 f_{7.1}(k, q)$ elements. □

8.10.4. *The eyes of $\mathbf{P}$ are pairwise disjoint.*

Subproof. This follows from 8.10.3, Lemma 8.5 and the assumption that no eye of $\mathbf{P}$ is a k-wild triangle or triad. □

8.10.5. *If (X, Y) is a 3-separation of M, then either X or Y is contained in an eye of $\mathbf{P}$.*

Subproof. If (X, Y) is non-sequential then (X, Y) is equivalent to a 3-separation displayed by T and it is clear that either X or Y is contained in an eye of v. Say X is sequential. Then $X = \mathrm{fcl}(T)$ for some triangle or triad T. If T is k-wild, then $X = T$ contradicting the assumption that no eye of $\mathbf{P}$ is a k-wild triangle or triad. Thus T is a clonal triple in which case the argument that T (and hence $\mathrm{fcl}(T)$) is contained in P_i for some i is clear. □

The lemma follows from 8.10.2, 8.10.3, 8.10.4 and 8.10.5. □

We are now in a position to prove the main result of this section.

Proof of Theorem 8.2. Let $\mu(m, k, q) = \max\{m, f_{7.1}(k, q)\}$. Now let $f_{8.2}(m, k, q) = \mu(m, k, q)^{f_{7.1}(k,q)}$. Assume that T has at least $f_{8.2}(m, k, q)$ vertices.

We may assume that $m \geqslant 5$. Say T has a bag vertex v of degree m. Then v is strong and has at least m eyes. So, by Lemma 8.10, M has a potato with m eyes, and by Lemma 8.6, M has a 3-connected minor N with a potato with m eyes such that each eye contains a clonal pair. Then, by Lemma 8.7, M has a 4-connected minor with m clonal pairs. Thus the lemma holds if T has a bag vertex of degree m, so we may assume that each bag vertex has degree less than m.

Each flower vertex of degree l induces a path of 3-separations in M of length l, so by Theorem 7.1, T has no flower vertex of degree at least $f_{7.1}(k, q)$. Hence the degree of each vertex is less than

$\mu(m,k,q)$. But as T has at least $f_{8.2}(m,k,q)$ vertices, it has a path of length $f_{7.1}(k,q)$ contradicting the fact that $M \in \mathcal{E}(q)$. $\square$

3. Bounding fettered elements

An element of a matroid is *fettered* if it is either fixed or cofixed. Otherwise it is *unfettered*. A matroid is *unfettered* if it has no fettered elements.

In the previous section it was shown that if a 3-tree T for a k-skeleton M in $\mathcal{E}(q)$ has sufficiently many vertices, then M has a 4-connected minor with many clonal pairs. This turns the focus on to 3-trees of bounded size. The task of this section is to show that in this case we gain control over the number of fettered elements.

Theorem 8.11. *Let M be a k-skeleton in $\mathcal{E}(q)$ and let T be a 3-tree for M with ν vertices. Then there is a function $f_{8.11}(\nu,k,q)$ such that M has at most $f_{8.11}(\nu,k,q)$ fettered elements.*

There are no surprises in the next lemma.

Lemma 8.12. *Let (L,P,R,Q) be a swirl-like flower in the 3-connected matroid M. Let (L',R') be a partition of $L\cup P\cup R$ such that $L\subseteq L'\subseteq L\cup P$. Then $\lambda_{M\backslash Q}(L',R') = \lambda_M(L'\cup Q,R')-1$.*

Proof. As $L\subseteq L'\subseteq L\cup P$, we see that $\sqcap(L,Q)\leqslant \sqcap(L',Q)\leqslant \sqcap(L\cup P,Q)$, so that $\sqcap(L',Q)=1$. Also $\lambda_M(Q)=2$ so that $r(Q)-r(M)=2-r(L\cup P\cup R)$. Therefore

$$\begin{aligned}
\lambda_M\big(L'\cup Q,R'\big)-1 &= r\big(L'\cup Q\big)+r\big(R'\big)-r(M)-1\\
&= r\big(L'\big)+r(Q)+r\big(R'\big)-r(M)-2\\
&= r\big(L'\big)+r\big(R'\big)-r(L\cup P\cup R)\\
&= \lambda_{M\backslash Q}\big(L',R'\big)
\end{aligned}$$

as required. $\square$

The next lemma is essentially the dual of Lemma 5.13.

Lemma 8.13. *Let e be an unfixed element of the 3-connected matroid M and $l\geqslant 4$ be an integer. Let $\mathbf{P}=(P_1,P_2,\dots,P_l)$ be a swirl-like flower of M/e. Then, up to labels, $(P_1\cup P_2\cup\{e\},P_3,\dots,P_l)$ is a flower in M.*

The next corollary follows immediately from Lemma 8.13.

Corollary 8.14. *Let M be a k-coherent matroid and e be an element of M that is not fixed and has the property that M/e is 3-connected. Then M/e is $(k+1)$-coherent, and, if $\mathbf{P}$ is a flower in M that opens to a k-fracture in M/e, then $\mathbf{P}$ has order $k-1$.*

Lemma 8.15. *Let M be a k-skeleton in $\mathcal{E}(q)$ and let P be a petal of the swirl-like flower $\mathbf{P}$ of M. Let $\{a_1,a_2,\dots,a_n\}$ be a set of fettered elements in $\mathbf{P}$ such that the following hold for all $i\in\{1,2,\dots,n\}$.*

(i) *$M\backslash a_i$ is k-coherent.*
(ii) *M/a_i is 3-connected.*
(iii) *There is a k-fracture of M/a_i that opens $\mathbf{P}$.*

Then $n\leqslant q+1$.

Proof. Say $i \in \{1, 2, \ldots, n\}$. As $M \backslash a_i$ is k-coherent, a_i is not fixed in M. By Corollary 8.14, $\mathbf{P}$ is a swirl-like flower of order $k-1$ in M and the flower in M/a_i that opens $\mathbf{P}$ has order k. Say $\mathbf{P} = (P, P_2, \ldots, P_{k-1})$. Then for each $i \in \{1, 2, \ldots, n\}$, there is a partition (L_i, R_i) of P such that $\mathbf{P}_i = (L_i, R_i, P_2, \ldots, P_k)$ is a maximal swirl-like flower of M/a_i. We now show that $\mathbf{P}_i$ is canonical.

As $M \backslash a_i$ is k-coherent, a_i is not in a bogan couple, so, if p is a loose element of $\mathbf{P}$, then, by Theorem 5.35, $a_i \preccurlyeq p$. It is easily checked that p can only be a guts element in this flower, so p is fixed in M/a_i. This means that either a_i is fixed in M, or a_i and p are a clonal pair. As $M \backslash a_i$ is k-coherent, a_i is not fixed and the latter case contradicts the fact that a_i is cofixed. Hence $\mathbf{P}_i$ is indeed canonical. Let $R = P_3$, $Q = P_4$ and $L = P_5 \cup P_6 \cup \cdots \cup P_{k-1}$, so that (L, P, R, Q) is a (cyclically shifted) concatenation of $\mathbf{P}$.

8.15.1. *If $i \neq j$ and $a_j \in R_i$, then $L_i \subseteq L_j$.*

Subproof. We know that (L, L_i, R_i, R, Q) is a canonical flower in M/a_i. If $((L \cup L_j \cup \{a_j\}) - \{a_i\}, (R_j \cup R \cup Q) - \{a_i\})$ is a 3-separation in M/a_i, we obtain the contradiction that a_j is a loose element of the above flower. Hence $((L \cup L_j \cup \{a_j\}) - \{a_i\}, (R_j \cup R \cup Q) - \{a_i\})$ is an exact 4-separation in M/a_i. It now follows by Lemma 3.34 that there is a partition (C, R_j) of R_i such that $L_j - \{a_i\} = L_i \cup C$. But $a_i \in \mathrm{cl}_M(L \cup L_i)$, and it follows readily that $L_j = L_i \cup \{a_i\} \cup C$, so that $L_i \subseteq L_j$. $\square$

A consequence of 8.15.1 is that we may assume that indices are chosen for elements of $\{a_1, a_2, \ldots, a_n\}$ so that $L_1 \subseteq L_2 \subseteq \cdots \subseteq L_n$. Consider $M \backslash Q$. By Lemma 8.12, $\lambda_{M\backslash Q}(L \cup L_i \cup \{a_i\}, R_i \cup R) = \lambda_{M\backslash Q}(L \cup L_i, \{a_i\} \cup R_i \cup R) = 2$. Again, by Lemma 8.12, and the fact that $\mathbf{P}$ is maximal, we have $\kappa_{M\backslash Q}(L \cup L_1, R \cup R_n) = 2$. For $i \in \{2, 3, \ldots, n\}$, let $L'_i = L_i - (L_{i-1} \cup \{a_i\})$. Then

$$\left(L \cup L_1, a_1, L'_2, a_2, L'_3, \ldots, L'_n, a_n, R_n \cup R\right)$$

is a path of 3-separations in $M \backslash Q$.

Say $i \in \{1, 2, \ldots, n\}$. It is easily seen that a_i is not a coguts singleton in the path described above. Hence a_i is a guts singleton. Now $M \backslash a_i$ is k-coherent, so that a_i is not fixed in M and hence is not fixed $M \backslash Q$. Thus Lemma 7.13 applies and we deduce that M has a $U_{2,n}$-minor. As M is in $\mathcal{E}(q)$, we conclude that $n \leqslant q+1$. $\square$

Bounding feral elements The task of this subsection is to bound the number of feral elements in a bag B_v of a 3-tree of a k-skeleton as a function of the degree of v. We first develop some terminology specific to this task.

Let M be a 3-connected matroid and let $\mathbf{P} = (P_1, P_2, \ldots, P_m)$ be a tight swirl-like flower in M, where $m \geqslant 3$. An element $f \in P_1$ *expands* $\mathbf{P}$ *at* P_2 if $M \backslash f$ is 3-connected and there is a partition (P'_1, P'_{m+1}) of $P_1 - \{f\}$ such that

(i) $(P'_1, P_2, P_3, \ldots, P_m, P'_{m+1})$ is a tight swirl-like flower in $M \backslash f$, and
(ii) P'_1 is 3-separating in M.

The element f *expands* $\mathbf{P}$ if it expands $\mathbf{P}$ at either P_2 or P_m. Assume that f expands $\mathbf{P}$ at P_2 and that $(P'_1, P_2, \ldots, P_m, P'_{m+1})$ has properties (i) and (ii) above. Assume that, in addition, $(P'_1, P_2, \ldots, P_m, P'_{m+1})$ has the property that whenever $(Q'_1, P_2, \ldots, P_m, Q'_{m+1})$ is another flower in $M \backslash f$ obtained by expanding $\mathbf{P}$ at P_2 for which $P'_1 \subseteq Q'_1$, then $P'_1 = Q'_1$. In this case we will say that $(P'_1, P_2, \ldots, P_m, P'_{m+1})$ is an *expansion of* $\mathbf{P}$ *by* f *at* P_2. It is clear that if f expands $\mathbf{P}$, then an expansion of $\mathbf{P}$ by f at either P_2 or P_m exists.

Lemma 8.16. *Let M be a k-skeleton in $\mathcal{E}(q)$ and let v be a strong bag vertex of a 3-tree T for M. Let $\mathbf{P} = (P_1, P_2, \ldots, P_m)$ be a maximal swirl-like flower of M, where $m \geqslant 3$ and $B_v \subseteq P_1$. Assume that $\mathbf{P}$ is displayed by a vertex adjacent to v. Then there are at most $8(q^2+q+1)$ feral elements in B_v that expand $\mathbf{P}$.*

Proof. Assume that the flowers $\mathbf{P}' = (P_1', P_2, \ldots, P_m, P_{m+1}')$ and $\mathbf{Q}' = (Q_1', P_2, \ldots, P_m, Q_{m+1}')$ are expansions of $\mathbf{P}$ at P_2 by the distinct feral elements f and g in B_v. By the maximality of the choice of P_1' and Q_1' and the fact that M is a k-skeleton we deduce that P_1' and Q_1' are fully closed 3-separating sets of M.

Let $\mathbf{R}' = (R_1', P_2, \ldots, P_m, R_{m+1}')$ be an expansion of $\mathbf{P}$ at P_2 by an element in P_1. We will say that $\mathbf{R}'$ is *equivalent* to $\mathbf{P}'$ if $P_1' = R_1'$.

8.16.1. *There are at most $q^2 + q + 1$ distinct expansions of* $\mathbf{P}$ *that are equivalent to* $\mathbf{P}$*.*

Subproof. Note that $\lambda_M(P_{m+1}' \cup \{f\}) = 3$, as otherwise $\mathbf{P}$ is not a maximal flower in M. Thus f, and any other element that expands $\mathbf{P}$ to a flower equivalent to $\mathbf{P}'$, is in the coclosure in M of $P_1' \cup P_2 \cup \cdots \cup P_m$. As M is cosimple, the dual of Corollary 7.7 implies that this set has at most $q^2 + q + 1$ elements. □

8.16.2. *If $|P_1'| \geqslant 3$, then P_1' is an eye of v and $g \notin P_1'$.*

Subproof. Assume that P_1' is not sequential. Then a 3-separating set equivalent to P_1' is displayed in T. The maximality of the choice of P_1' ensures that P_1' is displayed by an edge of T. If this edge is not incident with v, then we again contradict the choice of P_1'. Thus P_1' is an eye of v. Elements in $P_1' \cap B_v$ are either in the guts or coguts of a 3-separation and are therefore not feral. Thus $g \notin P_1'$. Assume that P_1' is sequential. It is easily checked that P_1' is not a k-wild triangle or triad. Therefore $P_1' = \mathrm{fcl}(T)$ for some clonal triangle or triad of M and again it is evident that if P_1' is not an eye of v we contradict the maximality of the choice of P_1'. Again it is clear that $g \notin P_1'$. □

8.16.3. *If $\mathbf{P}'$ is not equivalent to $\mathbf{Q}'$ and $|P_1'| \geqslant 3$, then $|Q_1'| = 2$.*

Subproof. Assume that P_1' and Q_1' both have at least three elements. By 8.16.2 P_1' and Q_1' are eyes of v and by Lemma 8.10 the eyes of v form a potato. Thus P_1' and Q_1' are disjoint. But Q_1' does not contain a feral element in B_v. Hence $Q_1' \subseteq P_{m+1}'$. But then $\sqcap(Q_1', P_2) = 0$ contradicting the fact that these are adjacent petals in the swirl-like flower $\mathbf{Q}'$. □

8.16.4. *If $|P_1'| = |Q_1'| = 2$, and $P_1' \neq Q_1'$, then $f \in Q_1'$.*

Subproof. Say that $P_1' = \{f_1, f_2\}$ and $Q_1' = \{g_1, g_2\}$. As $\sqcap(Q_1', P_2) = 1$ we deduce that $Q_1' \nsubseteq P_{m+1}'$. Therefore we may assume that $g_1 \in \{f_1, f_2, f\}$. Assume for a contradiction that $g_1 = f_1$. Then $\{f_2, g_2\} \subseteq \mathrm{cl}(P_2 \cup \{f_1\})$, so that $r(P_2 \cup \{f_1, f_2, g_2\}) = r(P_2) + 1$. But $\sqcap(P_1, P_2) = 1$ as these are adjacent petals of swirl-like flower in M. Thus $\sqcap(P_2, \{f_1, f_2, g_2\}) = 1$ and it follows that $r(\{f_1, f_2, g_2\}) = 2$ meaning that this set is a triangle. This contradicts the maximality of the choice of P_1'. We conclude that $g_1 = f$. □

A consequence of 8.16.4 is that there are at most three inequivalent expansions of $\mathbf{P}$ at P_2 where the petal of the expansion adjacent to P_2 has two elements. Combining this with 8.16.3 we deduce that there are at most four inequivalent expansions of $\mathbf{P}$ at P_2 by a feral element in B_v. If we consider expansions at P_m we obtain at most eight inequivalent expansions altogether. The lemma follows from this fact and 8.16.1. □

Let f be a feral element of the k-coherent matroid M. By Theorem 4.25, f has a feral display in either M or M^*. If the former case holds we say that f is a *standard* feral element and if the later case holds we say that f is a *costandard* feral element.

Corollary 8.17. *Let v be bag vertex of a 3-tree for the k-skeleton M in $\mathcal{E}(q)$. If v has degree l, then there are at most $16l(q^2 + q + 1)$ feral elements in B_v.*

Proof. Say that v is not strong. If $d(v) \leqslant 1$, then the elements of B_v are peripheral and by Lemma 4.39, B_v contains no feral elements. Say $d(v) = 2$. Let X and Y be the 3-separating sets displayed by v. If $(X, B_v \cup Y)$ is equivalent to $(X \cup B_v, Y)$, then the elements of B_v are not feral as they are either in the guts or coguts of a 3-separation. On the other hand, if (X, B_v, Y) is a flower and B_v is not displayed in a 3-tree, then B_v is sequential and the elements of B_v are not feral unless possibly $|B_v| = 2$, in which case the bound of the lemma certainly holds.

Assume that v is strong. Let f be a feral element in v. Assume that f is standard. Let $\mathbf{P} = (P_1, P_2, \dots, P_l)$ be a maximal k-fracture of $M\backslash f$. Then, using the properties of a feral display associated with a feral element, there is an $i \in \{2, 3, \dots, l\}$ such that $(P_2, P_3, \dots, P_i, P_{i+1} \cup \cdots \cup P_l \cup P_1 \cup \{f\})$ and $(P_{i+1}, P_{i+2}, \dots, P_l, P_1 \cup P_2 \cup \cdots \cup P_i \cup \{f\})$ are maximal swirl-like flowers of M, one of which may have only two petals. We lose no generality in assuming that $i > 2$. In this case the flower $\mathbf{P}$ is an expansion of $(P_2, P_3, \dots, P_i, P_{i+1} \cup \cdots \cup P_l \cup P_1 \cup \{f\})$ by $\{f\}$. It is readily seen that this flower is displayed by a vertex adjacent to v. There are at most l such flowers and, by Lemma 8.16, each such flower is expanded by at most $8(q^2 + q + 1)$ feral elements. Hence there are at most $8l(q^2 + q + 1)$ standard feral elements. By duality there are at most $8l(q^2 + q + 1)$ costandard feral elements and the corollary follows. □

Lemma 8.18. *Let A be a sequential 3-separating set of the k-skeleton M. If $|A| \geqslant 4$, then no element of A is fettered.*

Proof. Let $B = E(M) - A$. Clearly we may assume that A is fully closed. Let $(a_1, a_2, \dots, a_n)$ be a sequential ordering of A. Up to duality we may assume that $\{a_1, a_2, a_3\}$ is a triangle.

8.18.1. *$\{a_1, a_2, a_3\}$ is a clonal triple and $M\backslash a_1$ is a k-skeleton.*

Subproof. As M is a k-skeleton $\{a_1, a_2, a_3\}$ is not in a 4-element fan, so by Tutte's Triangle Lemma, $M\backslash a_i$ is 3-connected for some $i \in \{1, 2, 3\}$. By Corollary 4.6, $M\backslash a_i$ is k-coherent. Hence $\{a_1, a_2, a_3\}$ is not k-wild and is therefore a clonal triple. By Corollary 5.34 either $M\backslash a_1$ or M/a_1 is a k-skeleton. But M/a_1 is not 3-connected so $M\backslash a_1$ is a k-skeleton. □

8.18.2. *The lemma holds if $|A| = 4$.*

Subproof. Say $A = \{a_1, a_2, a_3, a_4\}$. By 8.18.1, a_1, a_2 and a_3 are unfettered. Consider a_4. If A is a line, then the claim holds as A is a clonal set. In the other case $a_4 \in \mathrm{cl}^*(\{a_1, a_2, a_3\})$. Thus $\mathrm{co}(M\backslash a_4)$ is not 3-connected. By Bixby's Lemma and the fact that M has no 4-element fans, $\mathrm{si}(M/a_4)$ is 3-connected. By Corollary 4.6, M/a_4 is k-coherent. Thus a_4 is not cofixed in M.

It remains to show that a_4 is not fixed in M. Note that $\{a_2, a_3, a_4\}$ is a triad of $M\backslash a_1$. By Corollary 5.7(i), a_4 is not cofixed in $M\backslash a_1$. Hence $\{a_2, a_3, a_4\}$ is not a k-wild triad in $M\backslash a_1$. As $M\backslash a_1$ is a k-skeleton $\{a_2, a_3, a_4\}$ is a clonal triple of $M\backslash a_1$. Thus a_4 is not fixed in $M\backslash a_1$. By Lemma 5.17, a_4 is not fixed in M. □

Assume that $n > 4$ and, for induction, that the lemma holds if $|A| = n - 1$. Then all the elements of $A - \{a_1\}$ are unfettered in $M\backslash a_1$. By Lemma 5.17, they are unfettered in M. □

Proof of Theorem 8.11. Let T be a 3-tree for M. Assume that T has at most v vertices. It is possible that a fettered element z of M is in the guts or coguts of a 3-separation. Consider this case now. The element z may be in a k-wild triangle or triad. As a generous bound for the number of such elements we observe that

8.11.1. *There are at most $3v$ elements of M in k-wild triads or triangles.*

Assume that z is not in a k-wild triangle or triad. By Lemma 8.18, z is in the guts or coguts of a non-sequential 3-separation (X, Y). Note that there is a path $(\mathrm{coh}(X), z_1, z_2, \dots, z_m, \mathrm{coh}(Y))$ of

3-separations in M. By Theorem 7.1, there are at most $f_{7.1}(k,q)$ elements of M in such a path. Moreover, a 3-separation equivalent to (X,Y) is displayed by T. Non-sequential 3-separations are displayed by either flower vertices or edges of T. There are at most 2^ν 3-separations displayed in such a way. Putting this information together we get another generous bound.

8.11.2. *There are at most* $2^\nu f_{7.1}(k,q)$ *fettered elements in the guts or coguts of a non-sequential* 3*-separation in* M.

If z is a fettered element not covered by the previous cases, then $M\backslash z$ and M/z are both 3-connected. By Corollary 8.17 there are at most $16\nu(q^2+q+1)$ feral elements in any vertex bag of T. It follows that

8.11.3. *There are at most* $16\nu^2(q^2+q+1)$ *feral elements in* M.

Any fettered element not covered by the previous cases has the property that either (a) $M\backslash z$ is k-coherent and M/z is 3-connected and k-fractured, or (b) M/z is k-coherent and $M\backslash z$ is 3-connected and k-fractured. Assume that z has property (a). By Corollary 8.14, there is a swirl-like flower $(P_1, P_2, \dots, P_{k-1})$ of M, where $x \in P_1$ such that, for some partition (P_1', P_1'') of $P_1 - \{z\}$, the partition $(P_1', P_1'', P_2, \dots, P_{k-1})$ is a maximal k-fracture of M/z. By Lemma 8.15, there are at most $\nu(q+1)$ elements of this type. Dually, there are at most $\nu(q+1)$ fettered elements of type (b).

From the above fact, 8.11.1, 8.11.2 and 8.11.3, we deduce that the theorem holds by letting

$$f_{8.11}(\nu,k,q) = 3\nu + 2^\nu f_{7.1}(k,q) + 16\nu^2\big(q^2+q+1\big) + 2\nu(q+1). \qquad \square$$

4. Finding an unfettered minor

We are now in a position to simplify structure by obtaining a large 4-connected unfettered matroid from a sufficiently large k-skeleton.

Theorem 8.19. *There is a function* $f_{8.19}(m,\nu,k,q)$ *such that, if* M *is a* k*-skeleton in* $\mathcal{E}(q)$ *having a* 3*-tree with at most* ν *vertices, then* M *has a* 4*-connected unfettered minor with at least* m *elements.*

Our strategy will be to move to find a bounded size k' such that a sufficiently large k-skeleton has a large unfettered k'-coherent minor.

Lemma 8.20. *Let* M *be a* k*-coherent matroid that is not a wheel or a whirl. If* M *has* l *fettered elements, then there is an element* z *of* M *such that either* $M\backslash z$ *or* M/z *is* 2k*-coherent with at most* $(l-1)$ *fettered elements.*

Proof. Let x be a fixed element in M. Assume that the element z is fettered in $M\backslash x$. If z is fixed in $M\backslash x$, then z is clearly fixed in M. Say that z is cofixed in $M\backslash x$. Then, by Corollary 5.8 and the fact that is x is fixed in M, we see z is cofixed in M. From this we deduce

8.20.1. *If* x *is fixed in* M*, and* y *is fettered in* $M\backslash x$*, then* y *is fettered in* M.

Let x be a fettered element of M. If x is in a 4-element fan, then, as M is not a wheel or a whirl, this fan has an end z. Moreover, z is either fixed and $M\backslash z$ is k-coherent, or cofixed and M/z is k-coherent. By 8.20.1, the lemma is satisfied with this choice of z. We may thus assume that M has no 4-element fans.

Assume that x is in a triangle T. Certainly T is not a clonal triple so it has a fixed element z. If T has an unfixed element, then by Corollary 5.25, $M\backslash z$ is k-coherent and the lemma holds by 8.20.1. Assume that all elements of T are fixed. Then, as M has no 4-element fans, T has an element z such that $M\backslash z$ is 3-connected. It remains to show that $M\backslash z$ is 2k-coherent. Let $(P_1, P_2, \dots, P_n)$ be a maximal swirl-like flower in $M\backslash z$. Assume that the other elements of T are in P_1 and P_i. Then $(P_2, P_3, \dots, P_{i-1}, P_i \cup P_{i+1} \cup \dots \cup P_n \cup P_1 \cup \{z\})$ and $(P_1 \cup P_2 \cup \dots \cup P_i \cup \{z\}, P_{i+1}, P_{i+2}, \dots, P_n)$ are swirl-like flowers in M of order $i-1$ and $n-i+1$ respectively. Thus $i-1<k$ and $n-i+1<k$ so that $n<2k$ and $M\backslash z$ is $2k$-coherent.

We may now assume that x is not in a triangle or a triad. Assume that x is in the guts of a vertical 3-separation. Then $M\backslash x$ is 3-connected and there is a 3-separation (A, B) of $M\backslash x$ such that $x \in \mathrm{cl}(A)$ and $x \in \mathrm{cl}(B)$. Let $\mathbf{P} = (P_1, P_2, \dots, P_n)$ be a maximal swirl-like flower of $M\backslash x$. By Lemma 3.32, up to labels and flowers equivalent to $\mathbf{P}$ there is an $i \in \{1, 2, \dots, n\}$ such that either $A \subseteq P_i$ or $(A, B) = (P_1 \cup P_2 \cup \dots \cup P_i, P_{i+1} \cup P_{i+2} \cup \dots \cup P_n)$. As $x \in \mathrm{cl}(A), \mathrm{cl}(B)$ we again deduce that $M\backslash x$ is $2k$-coherent.

If x is fixed in M, then, by 8.20.1, $M\backslash x$ has at most $l-1$ fettered elements, but it may be the case that x is not fixed but cofixed. We consider this case now. Assume that the element z is fettered in $M\backslash x$, but not in M. Then, by Corollary 5.7, $x \succcurlyeq z$. As x is cofixed in M we see that $x \not\cong z$. Since $x \in \mathrm{cl}(A)$ and $x \in \mathrm{cl}(B)$, it is also the case that $z \in \mathrm{cl}(A)$ and $z \in \mathrm{cl}(B)$. Therefore z is in the guts of the 3-separation (A, B). Thus z is fixed in M. In this case $M\backslash z$ is $2k$-coherent with at most $l-1$ fettered elements.

We may now assume that both $M\backslash x$ and M/x are 3-connected. Assume that x is fixed in M. Then the lemma holds for $M\backslash x$ unless $M\backslash x$ is not $2k$-coherent. In this case, by Corollary 8.14, x is cofixed. If M/x is k-coherent, then the lemma holds for M/x. Otherwise x is a feral element and has a feral display in M or M^*. It follows immediately from the definition of a feral display that M/x is $2k$-coherent, and the lemma holds in this case too. □

As an immediate consequence of Lemma 8.20 we obtain

Corollary 8.21. *Let M be a k-coherent matroid with l fettered elements and a nonempty set of unfettered elements, then M has an unfettered $2^l k$-coherent minor with at least $|E(M) - l|$ elements.*

Lemma 8.22. *Let (A, B) be a 3-separation of the matroid M, where A is fully closed and let a_1 and a_2 be elements of A. Let N be a 3-connected minor of M on $B \cup \{a_1, a_2\}$. If the element b of B is unfettered in M, then b is unfettered in N.*

Proof. Say $a \in A$ and N is a minor of M/a. Then $a \notin \mathrm{cl}(B)$, as otherwise N is not 3-connected. As A is fully closed, $b \notin \mathrm{cl}^*(A)$, so there is a circuit $C \subseteq B$ containing b. As $a \notin \mathrm{cl}(B)$ we have $a \notin \mathrm{cl}(C)$. Therefore it is not the case that $a \preccurlyeq b$. Thus, by Corollary 5.7, b is not fixed in M/a. Also b is not cofixed in M/a as b is not cofixed in M. The lemma now follows by duality and induction. □

An unfettered k-coherent matroid is evidently a k-skeleton. We use this fact without comment from now on.

Lemma 8.23. *Let M be a k-coherent unfettered matroid in $\mathcal{E}(q)$, and let v be a strong bag vertex of a 3-tree for M. If $|B_v| \geqslant m f_{7.1}(k, q)$, then M has a 4-connected unfettered minor with at least m elements.*

Proof. By Lemma 8.10, the eyes of v form a potato $\mathbf{P}$ of M. Say $\mathbf{P}$ has l eyes. Note that M has no k-wild triangles or triads so that an obvious minor perturbation of Lemmas 8.6 and 8.7 proves that M has a 4-connected minor N containing the core of $\mathbf{P}$ and a set of $2l$ other elements. Moreover, by Lemma 8.22 this matroid is unfettered. If $l \geqslant m/2$, then the lemma follows. On the other hand say $l < m/2$. By Lemma 8.9, each eye contains at most $f_{7.1}(k, q)$ elements of B_v. As $f_{7.1}(k, q) \geqslant 2$, we see that in this case the core of $\mathbf{P}$ has at least m elements. Thus the lemma holds in either case. □

Proof of Theorem 8.19. Let $k' = 2^{f_{8.11}(v,k,q)} k$. Let $v' = f_{8.2}(m, k', q)$. Let $f_{8.19}(m, v, k, q) = (m f_{7.1}(k', q))^{v'} + f_{8.11}(v, k, q)$.

Assume that M as at least $f_{8.19}(m, v, k, q)$ elements. By Theorem 8.11, M has at most $f_{8.11}(v, k, q)$ fettered elements. By Corollary 8.21, M has an unfettered k'-coherent minor N with at least

$(mf_{7.1}(k', q))^{\nu'}$ elements. Let T' be a 3-tree for N. If T' has at least ν' vertices, then, by Theorem 8.2, N has a 4-connected minor with a set of m pairwise-disjoint clonal pairs. It is easily deduced from the construction of the minor and Lemma 8.22, that the minor is unfettered, so the theorem holds in this case. Thus we may assume that T' has at most ν' vertices. Then T has a vertex v for which $|B_v| \geqslant mf_{7.1}(k', q)$. We may assume that $m \geqslant 4$, so that v is strong. The theorem now follows from Lemma 8.23. □

5. Unfettered matroids

We now know that a sufficiently large k-skeleton in $\mathcal{E}(q)$ has a large 4-connected unfettered minor. The next task is to extract a 4-connected minor having many clonal pairs from such a matroid. While we only need the result for matroids in $\mathcal{E}(q)$, free spikes play no role in the arguments, so we focus on the class of matroids with no $U_{2,q+2}$- or $U_{q,q+2}$-minor, that is, the class $\mathcal{U}(q) \cap \mathcal{U}^*(q)$.

Theorem 8.24. *There exists a function $f_{8.24}(q, t)$ such that, if M is a 4-connected unfettered matroid in $\mathcal{U}(q) \cap \mathcal{U}^*(q)$ with at least $f_{8.24}(q, t)$ elements, then M has a 4-connected minor with at least t pairwise-disjoint clonal pairs.*

As usual we prepare for the proof with a series of lemmas. We begin with ones that focus on connectivity. A matroid M is 4-*connected up to rank-k 3-separators*, if it is 3-connected and, whenever (A, B) is a 3-separation of M, either $r(A) \leqslant k$ or $r(B) \leqslant k$. A vertically 4-connected matroid is one that is 4-connected up to rank-2 3-separators. If M is 4-connected up to rank-3 3-separators we will say that M is 4-*connected up to planes*.

If M is 3-connected and (A, B) is a 3-separation where $r(A) = 3$, then $\lambda(A - \mathrm{cl}(B)) = 2$, $r(A - \mathrm{cl}(B)) = 3$ and $A - \mathrm{cl}(B)$ is a cocircuit. It follows that, if M is 4-connected up to planes, then M is vertically 4-connected if and only if M has no 3-separating cocircuits. The next lemma is clear.

Lemma 8.25. *Let x be an element of the 3-connected matroid M. If $M\backslash x$ is 4-connected up to rank-k 3-separators, then M is also 4-connected up to rank-k 3-separators.*

Lemma 8.26. *Let M be an unfettered matroid that is 4-connected up to planes. If $r(M) \geqslant 6$, then the 3-separating cocircuits of M partition a subset of the ground set of M.*

Proof. Assume that C_1 and C_2 are 3-separating cocircuits of M. Say $|C_1 \cap C_2| \geqslant 2$. Then an uncrossing argument shows that $\lambda_M(C_1 \cup C_2) = 2$. We also have $r(C_1 \cup C_2) = 4$. As M has rank at least six, $r_M(E(M) - (C_1 \cup C_2)) \geqslant 4$ and we have contradicted the fact that M is 4-connected up to planes.

Assume that $|C_1 \cap C_2| = 1$; say $C_1 \cap C_2 = \{x\}$. As C_1 and C_2 are cocircuits, $x \in \mathrm{cl}^*(C_1 - \{x\})$ and $x \in \mathrm{cl}^*(C_2 - \{x\})$. Assume that $\sqcap^*(C_1 - \{x\}, C_2 - \{x\}) = 2$. Then $\sqcap^*(C_1, C_2 - \{x\}) = 2$. As $x \in \mathrm{cl}^*(C_1 - \{x\})$, we have $x \in \mathrm{cl}^*(E(M) - C_2)$. Using Lemma 2.12 we obtain the following.

$$\begin{aligned}
\lambda(C_1 \cup C_2) &= \lambda\big(C_1 \cup \big(C_2 - \{x\}\big)\big) \\
&= \lambda(C_1) + \lambda\big(C_2 - \{x\}\big) - \sqcap\big(C_1, C_2 - \{x\}\big) - \sqcap^*\big(C_1, C_2 - \{x\}\big) \\
&\leqslant 2,
\end{aligned}$$

and again we have contradicted the fact that M is 4-connected up to planes.

Hence $\sqcap^*(C_1 - \{x\}, C_2 - \{x\}) = 1$. In this case it follows from the dual of Lemma 5.11 that x is cofixed in M, contradicting the assumption that x is unfettered. □

The next lemma is a consequence of Lemma 5.16.

Lemma 8.27. *If T is a triangle of an unfettered matroid, then the members of T are clones.*

Lemma 8.28. *Let M be a vertically 4-connected unfettered matroid. If the element x of M does not have a clone, then both $M\backslash x$ and M/x are 3-connected and unfettered.*

Proof. Assume that x has no clone. Clearly $M\backslash x$ is 3-connected and, by Lemma 8.27 so too is M/x. Certainly no element of $M\backslash x$ is fixed. Assume that the element z is cofixed in $M\backslash x$. Then, by Corollary 5.7, either x is cofixed in M, contradicting the fact that M is unfettered, or x and z are clones, contradicting the fact that x does not have a clone. Thus $M\backslash x$ is unfettered and dually, so too is M/x. □

Lemma 8.29. *Let t be an element of a triangle of a vertically 4-connected unfettered matroid. Then $M\backslash t$ is vertically 4-connected and unfettered.*

Proof. Assume that t belongs to the triangle T of M. By Lemma 8.27, T is a clonal triple of M and it is easily seen that $M\backslash t$ is vertically 4-connected. No element of $M\backslash t$ is fixed in this matroid. It also follows routinely from Corollary 5.7 that $M\backslash t$ has no cofixed elements. □

Lemma 8.30. *Let M be an unfettered matroid that is 4-connected up to planes, and let A be a 3-separating cocircuit of M. Then the following hold.*

(i) *If l is a non-trivial line that meets A, then $l \subseteq A$.*
(ii) *If $a \in A$ and a has no clone, then M/a is 4-connected up to planes.*
(iii) *If the element a of A is in a non-trivial line l of M and $|A - l| \geqslant 2$, then $\mathrm{si}(M/a)$ is 4-connected up to planes.*

Proof. Let l be a non-trivial line of M that meets A. Assume that l is not contained in A. Then $|(E(M) - A) \cap l| = 1$; say $(E(M) - A) \cap l = \{x\}$. As x is in the guts of the 3-separation $(E(M) - A, A)$, we have $x \in \mathrm{cl}(E(M) - (A \cup \{x\}))$. But then x is fixed in M by Lemma 5.16, contradicting the fact that M is unfettered. Thus (i) holds.

Consider (ii). The result is routine if A is not a triad. Assume that A is a triad $\{a, b, c\}$. By the dual of Lemma 8.27, $\{a, b, c\}$ is a clonal triple so that $\{b, c\}$ is a clonal pair in M/a. Assume that (B, C) is a 3-separation of M/a. If $\{b, c\} \subseteq B$, then $(B \cup \{a\}, C)$ is a 3-separation of M, and it follows that either B or C has rank at most three in M/a. Assume that $b \in B$ and $c \in C$. If there is a circuit Z contained in B that contains c, then $c \in \mathrm{cl}_{M/a}(Z)$, and we may apply the previous argument to conclude that either B or C has rank at most 3 in M/a. Otherwise, both b and c are in the coguts of the 3-separation (B, C). Consider this case. Assume for a contradiction that $r_{M/a}(B) \geqslant 4$ and $r_{M/a}(C) \geqslant 4$. Then $(B - \{b\}, C \cup \{b\})$ is a 3-separation in M/a and $b \in \mathrm{cl}^*_{M/a}(B - \{b\})$. As $\{a, b, c\}$ is a triad of M we have $a \in \mathrm{cl}^*_M(C \cup \{b\})$ so that $(B - \{b\}, C \cup \{a, b\})$ is a 3-separation in M. But $b \in \mathrm{cl}^*_{M/a}(B - \{b\})$, so $b \in \mathrm{cl}^*_M(B - \{b\})$. Hence $(B, C \cup \{a\})$ is a 3-separation of M. But $r_M(B), r_M(C \cup \{a\}) \geqslant 4$ and we have contradicted the fact that M is 4-connected up to planes.

We omit the easy proof of (iii). □

A 3-separation (A, B) of a matroid M is *cyclic* if both A and B contain circuits of M. The matroid M is *cyclically 4-connected* if it is 3-connected and has no cyclic 3-separations. Note that M is cyclically 4-connected if and only if M^* is vertically 4-connected. Note also that cyclically 4-connected matroids do not contain triangles unless the matroid is degenerately small.

Lemma 8.31. *Let p be an element of the plane P of the cyclically 4-connected unfettered matroid M where $|P| \geqslant 5$. Assume that $M\backslash p$ is not cyclically 4-connected. Then $|P| = 5$ and $M\backslash z$ is cyclically 4-connected for all $z \in P - \{p\}$.*

Proof. Let (A, B) be a cyclic 3-separation of $M\backslash p$, where B is coclosed. If $|A \cap (P - \{p\})| \leqslant 1$, then $(A, B \cup \{p\})$ is a cyclic 3-separation of M. Thus $|A \cap (P - \{p\})| > 1$ and symmetrically $|B \cap (P - \{p\})| > 1$. But $r(A \cap (P - \{p\})) \leqslant 2$ and $r(B \cap (P - \{p\})) \leqslant 2$ as otherwise p is in the closure of either A

or B implying that M is not cyclically 4-connected. As M has no triangles we deduce that $|P| = 5$ and that $|A \cap (P - \{p\})| = |B \cap (P - \{p\})| = 2$. Let $\{a_1, a_2\} = A \cap (P - \{p\})$ and let $\{b_1, b_2\} = B \cap (P - \{p\})$. Observe that A and $\mathrm{cl}(P)$ are a modular pair of cyclic flats of M whose intersection does not span P. Thus any clone of a_1 or a_2 lies on the line $\{a_1, a_2\}$, and by symmetry any clone of b_1 or b_2 lies on the line $\{b_1, b_2\}$. It follows that $\{a_1, a_2\}$ and $\{b_1, b_2\}$ are clonal lines of M.

Assume that $M\backslash a_1$ is not cyclically 4-connected. Then, arguing as above, we deduce that, for some $x \in \{b_1, b_2, p\}$, the pair $\{a_2, x\}$ is a clonal line of M contradicting the fact that $\{a_1, a_2\}$ is a clonal line of M. □

The proof of the next lemma is entirely analogous to the proof of Bixby's Lemma.

Lemma 8.32. *Let x be an element of the vertically 4-connected matroid M that is not in a triangle. Assume that M/x is not 4-connected up to planes. Then $M\backslash x$ is 4-connected up to rank-4 3-separators. Moreover, if (D_1, D_2) is a vertical 4-separation of $M\backslash x$ such that $r(D_2) \leqslant 4$, then D_2 is covered by a pair of lines of M.*

Proof. Assume that M/x is not 4-connected up to planes. Then there is a 4-separation $(C_1 \cup \{x\}, C_2)$ of M with x in the guts such that $r(C_1), r(C_2) \geqslant 5$.

Let (D_1, D_2) be a 3-separation of $M\backslash x$. We need only consider the case that $r(D_1), r(D_2) \geqslant 3$. Then, as neither $(D_1 \cup \{x\}, D_2)$ nor $(D_1, D_2 \cup \{x\})$ is a 3-separation of M, we see that $(D_1 \cup \{x\}, D_2)$ is a 4-separation of M with x in the coguts.

Without loss of generality $r(C_1 \cap D_1) \geqslant 3$. Thus $\lambda(C_1 \cap D_1) \geqslant 3$. By uncrossing $\lambda((C_2 \cap D_2) \cup \{x\}) \leqslant 3$. But $x \in \mathrm{cl}(C_1 \cup D_1)$ and $x \in \mathrm{cl}^*(C_1 \cup D_1)$. Hence $r(C_1 \cup D_1 \cup \{x\}) = r(C_1 \cup D_1)$ and $r(C_2 \cap D_2) = r((C_2 \cap D_2) \cup \{x\}) - 1$. Therefore $\lambda(C_2 \cap D_2) \leqslant 2$. As M is vertically 4-connected, we have $r(C_2 \cap D_2) \leqslant 2$. But $r(C_2) \geqslant 5$ so that $r(C_2 \cap D_1) \geqslant 3$. Repeating the above argument shows that $r(C_1 \cap D_2) \leqslant 2$. Therefore $r(D_2) \leqslant 4$ and D_2 is covered by a pair of lines of M. □

Let S be a set of elements of the matroid M. Define $P_M(S)$ to be the set $\mathrm{cl}(\mathrm{cl}^*(S))$.

Lemma 8.33. *If $z \notin P_M(S)$, then $P_{M\backslash z}(S) \supseteq P_M(S)$.*

Proof. Observe that $\mathrm{cl}^*_{M\backslash z}(S) = \mathrm{cl}^*_M(S \cup \{z\}) - \{z\}$. It follows that $\mathrm{cl}^*_{M\backslash z}(S) \supseteq \mathrm{cl}^*_M(S)$. As $z \notin \mathrm{cl}_M(\mathrm{cl}^*_M(S))$, it follows that $\mathrm{cl}_M(\mathrm{cl}^*_M(S)) \subseteq \mathrm{cl}_{M\backslash z}(\mathrm{cl}^*_{M\backslash z}(S))$ as required. □

Recall that it follows from Lemma 7.6 that a simple rank-t matroid in $\mathcal{U}(q) \cap \mathcal{U}^*(q)$ has at most $(q^t - 1)/(t - 1)$ elements. Let $h(q, t) = (q^t - 1)/(t - 1)$. By duality, if M is a cosimple matroid in $\mathcal{U}(q) \cap \mathcal{U}^*(q)$ of corank t, then M has at most $h(q, t)$ elements.

Lemma 8.34. *Let M be a simple, cosimple matroid in $\mathcal{U}(q) \cap \mathcal{U}^*(q)$.*

(i) *If $S \subseteq E(M)$ and $|S| \leqslant t$, then $|P_M(S)| \leqslant h(q, h(q, t))$.*
(ii) *If $r^*(M) \geqslant h(q, t)$, then $r(M) \geqslant t$.*

Proof. Consider (i). Assume that $|S| \leqslant t$. As $r^*(S) \leqslant |S|$, and $r^*(\mathrm{cl}^*(S)) = r^*(S)$, we see that $|\mathrm{cl}^*(S)| \leqslant h(q, t)$. But $r(\mathrm{cl}(\mathrm{cl}^*(S))) = r(\mathrm{cl}^*(S))$ so that $|P_M(S)| \leqslant h(q, h(q, t))$.

Consider (ii). Assume that $r^*(M) \geqslant h(q, t)$. Then $|E(M)| \geqslant h(q, t)$ so that, by Lemma 7.6, $r(M) \geqslant t$. □

For an unfettered matroid M, let L_M denote the set of elements in non-trivial clonal classes of M. Not surprisingly, a *clonal line* of M is a line whose elements form a clonal set. Of course, if the line is non-trivial its elements will form a clonal class.

Lemma 8.35. *Let M be a vertically 4-connected unfettered matroid in $\mathcal{U}(q) \cap \mathcal{U}^*(q)$. Then there is a function $f_{8.35}(q, t)$ such that if $r^*(M) \geqslant f_{8.35}(q, t)$, then at least one of the following holds.*

(i) *M has a vertically 4-connected restriction N such that $|L_N| \geqslant t$.*
(ii) *There is an element $z \in E(M) - P_M(L_M)$ such that M/z is 4-connected up to planes.*

Proof. To simplify notation we inductively define $h^i(q,t)$ for $i \geqslant 2$ by $h^i(q,t) = h(q, h^{i-1}(q,t))$. Assume that $r^*(M) > h^6(q,t)$. For a set S of elements of a matroid N, set $Q_N(S) = P_N(P_N(S))$. Let $Z = (z_1, z_2, \dots, z_{l-1})$ be a maximal sequence of elements of M such that, for $i \in \{1, 2, \dots, l-1\}$, the following hold.

(i) $M \backslash z_1, z_2, \dots, z_i$ is vertically 4-connected.
(ii) $z_i \notin L_{N \backslash z_1, z_2, \dots, z_{i-1}}$.

Clearly $r(M \backslash Z) = r(M)$. By Lemma 8.28, $M \backslash Z$ is unfettered. If $|L_{M \backslash Z}| \geqslant t$, then the lemma holds. Assume that $|L_{M \backslash Z}| < t$. By Lemma 8.34(ii), $r(M) > h^5(q,t)$. As $r(M \backslash Z) = r(M)$, it also follows from Lemma 8.34(ii) that $r^*(M \backslash Z) > h^4(q,t)$. By Lemma 8.34(i), there is an element $z_l \in E(M \backslash Z) - Q_{M \backslash Z}(L_M)$.

If $M \backslash Z / z_l$ is 4-connected up to planes, then, by Lemma 8.25, M/z_l is 4-connected up to planes and the lemma holds. Assume otherwise. Then the conclusions of Lemma 8.32 hold for $M \backslash Z \backslash z_l$. Let $M' = M \backslash Z \backslash z_l$. It follows from the choice of Z that M' is not vertically 4-connected. Moreover, if A is a 3-separator of M' with $r(A) \in \{3, 4\}$, then $z_l \in \mathrm{cl}^*_{M \backslash Z}(A)$.

Assume that M' is not 4-connected up to planes. Let A be a rank-4 3-separator of M'. Then there are lines l_1 and l_2 such that $l_1 \cup l_2 = A$. As $r(A) = 4$, these lines are disjoint. If $A \subseteq P_{M \backslash Z}(L_{M \backslash Z})$, then $z_l \in Q_{M \backslash Z}(L_{M \backslash Z})$, contradicting the choice of z_l. Thus we may assume that there is an element $a \in l_1$ such that $a \notin P_{M \backslash Z}(L_{M \backslash Z})$. Certainly a is not in a triangle, so $|l_1| = 2$. Let L be a subset of l_2 with $|L| = |l_2| - 2$. By Lemma 8.29, $M \backslash Z \backslash L$ is vertically 4-connected, so that $(M \backslash Z \backslash L)^*$ is cyclically 4-connected. Moreover, $\{z\} \cup l_1 \cup (l_2 - L)$ is a 5-point plane of this matroid. It now follows by the dual of Lemma 8.31, that, if $z \in l_1$, then $M \backslash Z \backslash L / z$ is vertically 4-connected. By Lemma 8.25, M/z is vertically 4-connected and by Lemma 8.33, $z \notin P_M(L_M)$.

Assume that M' is 4-connected up to planes and let A be a rank-3, 3-separating cocircuit of M'. Arguing as above we deduce that there is an element $z \in A$ that is not in $P_{M \backslash Z}(L_{M \backslash Z})$. Now z is not in a triangle of M', as otherwise, by Lemma 8.27, $z \in L_M$. So, by Lemma 8.30, M'/z is 4-connected up to planes. Again it follows from Lemma 8.33 that $z \notin P_M(L_M)$.

We conclude that the lemma holds by letting $f_{8.35}(q,t) = h^6(q,t)$. $\square$

Lemma 8.36. *Let M be a vertically 4-connected unfettered matroid in $\mathcal{U}(q) \cap \mathcal{U}^*(q)$. Assume that $r^*(M) \geqslant f_{8.35}(q,t)$. Then at least one of the following holds.*

(i) *M has a vertically 4-connected restriction N such that $|L_N| \geqslant t$.*
(ii) *There is a nonempty set $J \subseteq (E(M) - L_M)$ such that M/J is unfettered and vertically 4-connected.*

Proof. Assume that the lemma fails. Then (i) does not hold, so, by Lemma 8.35, there is an element $z \in E(M) - P_M(L_M)$ such that M/z is 4-connected up to planes. If M/z is vertically 4-connected, then, by Lemma 8.28, M/z is unfettered so that the lemma is satisfied with $J = \{z\}$. It follows that M/z is not vertically 4-connected. Let $\{Z_1, Z_2, \dots, Z_m\}$ be the 3-separating cocircuits of M/z. Say $i \in \{1, 2, \dots, m\}$. If $Z_i \subseteq P_M(L_M)$, then $z \in P_M(L_M)$. Thus there is an element $z_i \in (Z_i - P_M(L_M))$.

8.36.1. *There is an $i \in \{1, 2, \dots, m\}$ such that every element of $Z_i - L_M$ is contained in a triangle in M/z.*

Subproof. Assume otherwise. For $i \in \{1, 2, \dots, m\}$, choose $z_i \in Z_i - L_M$ so that z_i is not in a triangle in M/z. It is straightforwardly seen that now the lemma holds with $J = \{z, z_1, z_2, \dots, z_m\}$, contradicting the assumption that the lemma fails. $\square$

Let Z be a 3-separating cocircuit of M/z satisfying 8.36.1 and let $H = E(M) - Z$.

8.36.2. *Z is coclosed in M and M/z.*

Subproof. If $z \in \mathrm{cl}^*(Z)$, then $r(H - \{z\}) = r(M) - 2$, so that $(H - \{z\}, Z \cup \{z\})$ is a vertical 3-separation of M, contradicting the fact that M is vertically 4-connected. Thus $z \notin \mathrm{cl}^*(Z)$. Say that $t \in H - \{z\}$, and $t \in \mathrm{cl}^*(Z)$. Then $t \in \mathrm{cl}^*_{M/z}(Z)$ and $Z \cup \{t\}$ is a rank-4 3-separation of M/z contradicting the fact that M/z is 4-connected up to planes. Therefore Z is coclosed in M and hence in M/z. $\square$

8.36.3. *Z partitions into two clonal lines of M/z. One of these, say l'_z, is a clonal line of M; the other, say l_z, spans a plane of M and contains at least two points u and v that are not in L_M.*

Subproof. If $Z \subseteq L_M$, then $z \in \mathrm{cl}(L_M)$ contradicting the choice of z. Thus there is a point $u \in Z - L_M$. By 8.36.1, u is in a line l_z of M/z, where $|l_z| \geqslant 3$. As M/z is unfettered, it follows from Lemma 8.30(i), that $l_z \subseteq Z$. As $u \notin L_M$, we see that l_z is not a line of M. Hence $r_M(l_z) = 3$, and $z \in \mathrm{cl}_M(l_z)$.

Say $y \in Z$. Assume that y is not in a triangle in M/z. By 8.36.1, $y \in L_m$ and hence has a clone in M and therefore in M/z. Such a clone must be in Z. Thus Z partitions into clonal lines of M/z.

Let l'_z be a clonal line of M/z in $Z - l_z$. Assume that $r_M(l'_z) = 3$. Then $z \in \mathrm{cl}_M(l'_z)$, and $z \in \mathrm{cl}_M(H - \{z\})$ (by 8.36.2). It follows routinely that z is fixed in M. Thus l'_z is a line of M. If this line is not a clonal line, then it has two points and is not contained in L_M, and we obtain a contradiction to 8.36.1. Thus $Z - l_z$ partitions into clonal lines of M.

If u is the only element of l_z not in L_M, the u is the only element of Z not in L_M. Thus $u \in \mathrm{cl}^*_{M/z}(L_M)$ so that $u \in \mathrm{cl}^*(L_M)$. This gives the contradiction that $z \in P_M(L_M)$. Hence there is another point $v \in l_z$ that is not in L_M.

Assume that there is more than one clonal line in $Z - l_z$. If these lines are skew in M, then $z \in \mathrm{cl}(L_M)$, contradicting the choice of z. Thus $Z - l_z$ spans a plane in M. If both u and v are on this plane, then any clone u' of u must lie on both this plane and the plane $l_z \cup \{z\}$ (certainly u is not a coloop of this plane). Thus $\{u', u, v\}$ is a triangle and we deduce that $\{u, v\}$ is a clonal line of M, contradicting the fact that these elements are not in L_M.

It follows that we may assume that $u \notin \mathrm{cl}_M(Z - l_z)$ and it is now easily checked that, in M/u, the set $Z \cup \{z\}$ contains a $U_{3,5}$ restriction (indeed a $U_{3,6}$ restriction, but we do not need this). Thus, if (P, Q) is a vertical 3-separation of M/u, we may assume that $Z \cup \{z\} \subseteq P$. But then $u \notin \mathrm{cl}_M(Q)$, so this 3-separation is not coblocked by u, so that $(P \cup \{u\}, Q)$ is a vertical 3-separation of M, contradicting the fact that M is vertically 4-connected. Therefore l'_z is the unique clonal line of $M \backslash l_z$, and the claim holds. $\square$

8.36.4. *M/u and M/v are 4-connected up to planes.*

Subproof. Let $P = (l_z - \{u\}) \cup \{z\}$. By Lemma 8.30, $\mathrm{si}(M/u, z)$ is 4-connected up to planes apart from the single parallel class P. If M/u is not 4-connected up to planes, then there is a 3-separation (X, Y) of $M/u, z$ with $r_{M/u,z}(X) = 3$, such that $z \in \mathrm{cl}_{M/u}(X)$ and $z \notin \mathrm{cl}_{M/u}(Y)$. We may assume that X is a cocircuit of $M/u, z$. If $P \subseteq \mathrm{cl}_{M/u,v}(Y)$, then $z \in \mathrm{cl}_{M/u}(Y)$, so $P \subseteq X$. But, by Lemma 8.26, the 3-separating cocircuits of M/z are all disjoint from P. $\square$

8.36.5. *We may assume that neither u nor v is in $P_M(L_M)$.*

Subproof. If all but one element t of l_z is in $P_M(L_M)$, then the remaining element is clearly in the closure of $Z - \{t\}$, so that $z \in \mathrm{cl}(Z - \{t\})$, and $z \in \mathrm{cl}(P_M(L_M))$, that is, $z \in P_M(L_M)$. $\square$

If M/u is vertically 4-connected, then the lemma holds with $J = \{u\}$. Thus we may assume that neither M/u nor M/v is vertically 4-connected. We now have symmetry between u, v and z in that there exist sets U and V with partitions $\{l_u, l'_u\}$ and $\{l_v, l'_v\}$ respectively such that the conclusions established above hold with (z, Z, l_z, l'_z) replaced by (u, U, l_u, l'_u) or (v, V, l_v, l'_v).

8.36.6. $l_z \cup \{z\} = l_u \cup \{u\} = l_v \cup \{v\}$.

Subproof. We first show that $l_u \cap l_z \neq \emptyset$. Assume otherwise. As l_u contains a circuit containing u, we have $|l_u \cap Z| \geqslant 2$. Thus there is an element of l'_z contained in l_u. As l'_z is a clonal line it follows that $l'_z \subseteq l_u$. Note that any other element of l_u is in H. Now $r(l'_z \cup \{u\}) = 3$, so $l'_z \cup \{u\}$ spans l_u. Let p and q be elements of $l_u - (l'_z \cup \{u\})$ and let M' be the matroid obtained by independently cloning p by p'. Then $\{p, p', q\} \subseteq \mathrm{cl}(l'_z \cup \{u\})$. Also, by 8.36.2, $p \in \mathrm{cl}(H - \{p\})$ so that $p' \in \mathrm{cl}(H)$. But $\sqcap(l'_z \cup \{u\}, H) = 2$, so $\{p, p', q\}$ is a triangle. This shows that $l_u - (l'_z \cup \{u\})$ is a clonal line of M contradicting the fact, established by 8.36.3 and symmetry, that this set has at least two elements not in L_M. Therefore $l_u \cap l_z \neq \emptyset$.

We may now assume that there is an element $p \in l_z \cap (l_u - \{u\})$. Then p is in U. But l_u contains a circuit containing p and U is a cocircuit. Hence there is another element of U contained in l_z. If such an element is contained in l_u, then, as u is not in a triangle, we see that $\mathrm{cl}(l_u \cup \{u\}) = \mathrm{cl}(l_z \cup \{z\})$ and it follows easily that $l_u \cap \{u\} = l_z \cap \{z\}$. Thus we may assume that the element is in l'_u, and as l'_u is a clonal line we deduce that $l'_u \subseteq l_z$. By symmetry we also have $l'_z \subseteq l_u$ and it follows easily that $U \subseteq \mathrm{cl}(Z)$. But then $\mathrm{cl}(Z)$ contains two distinct cocircuits, U and Z, so that $r(E(M) - \mathrm{cl}(Z)) \leqslant r - 2$, and hence $\lambda(Z) = 2$, contradicting the fact that the matroid M is vertically 4-connected. □

Let A denote the common set given by 8.36.6, that is, $A = l_z \cup \{z\}$.

8.36.7. *There is a pair $\{s, t\} \subseteq \{u, v, z\}$ such that $\sqcap(l'_s \cup l'_t, A) = 2$.*

Subproof. Assume that the sublemma fails. Then $\sqcap(l'_u \cup l'_v \cup l'_z, A) = 1$. Moreover, elementary rank calculations establish that $\sqcap(l'_u \cup l'_v, A) = 1$ and that $\sqcap_{M/l_z}(l'_u \cup l'_v, A) = 0$. From this latter fact we deduce that $\sqcap^*(l'_u \cup l'_v) \geqslant 3$. Using this and Lemma 2.12 we obtain

$$\begin{aligned}\lambda\big(l'_u \cup l'_v \cup A\big) &= \lambda\big(l'_u \cup l'_v\big) + \lambda(A) - \sqcap\big(l'_u \cup l'_v, A\big) - \sqcap^*\big(l'_u \cup l'_v, A\big)\\ &\geqslant 3 + 3 - 1 - 0 = 5.\end{aligned}$$

But by uncrossing the 4-separations $A \cup l'_u$ and $A \cup l'_v$, we see that $\lambda(l'_u \cup l'_v \cup A) \leqslant 3$. This contradiction establishes the sublemma. □

By 8.36.7, we may assume that $\sqcap(l'_u \cup l'_v, A) = 2$. Thus $\sqcap(l'_u \cup l'_v, l_z) = 2$. But $l'_u \cup l'_v \subseteq H - \{z\}$, and $\sqcap_{M/z}(H - \{z\}, l_z) \leqslant 1$, so that $\sqcap_{M/z}(l'_u \cup l'_v, l_z) < \sqcap_M(l'_u \cup l'_v, l_z)$. By Lemma 2.10, $z \in \mathrm{cl}_M(l'_u \cup l'_v)$ contradicting the fact that $z \notin L_M$. This contradiction at last completes the proof of the lemma. □

Corollary 8.37. *Let M be an unfettered vertically 4-connected matroid in $\mathcal{U}(q) \cap \mathcal{U}^*(q)$. If $r^*(M) \geqslant f_{8.35}(q, t)$, then M has a vertically 4-connected minor N with $|L_N| \geqslant t$.*

Proof. Assume that $r^*(M) \geqslant f_{8.35}(q, t)$. Let $Z = \{z_1, z_2, \dots, z_l\}$ be a maximal set of elements of M such that, for all $i \in \{1, 2, \dots, l\}$, the matroid $M/z_1, z_2, \dots, z_i$ is unfettered and vertically 4-connected and such that $z_i \notin L_{M/z_1, z_2, \dots, z_{i-1}}$. By Lemma 8.28 M/Z is unfettered. Certainly Z is independent. Hence $r^*(M/Z) = r^*(M) \geqslant f_{8.35}(q, t)$. By the definition of Z, part (ii) of Lemma 8.35 does not hold for M/Z. Thus part (i) of that lemma holds for M/Z and gives the required minor. □

Finally we can achieve the purpose of this section.

Proof of Theorem 8.24. Let $f_{8.24}(q, t) = h(q, f_{8.35}(q, (q + 2)t))$. Let M be an unfettered 4-connected matroid in $\mathcal{U}(q) \cap \mathcal{U}^*(q)$ with at least $f_{8.24}(q, t)$ elements. Then $r^*(M) \geqslant f_{8.35}(q, (q + 2)t)$. By Corollary 8.37, M has a vertically 4-connected minor N with the property that $|L_N| \geqslant (q + 2)t$. Let N' be the matroid obtained by deleting all but two elements from each non-trivial clonal line of N. Such lines have at most $q + 1$ points. By Lemma 8.29, N' is an unfettered vertically 4-connected matroid. But N' has no triangles, so that N' is 4-connected. Moreover $|L_{N'}| \geqslant 3t$. Hence N' has at least t pairwise-disjoint clonal pairs. □

Chapter 9. Unavoidable minors of large 3-connected clonal matroids

1. Introduction

Let M be a matroid and $\mathcal{C}$ be a partition of $E(M)$. Then $\mathcal{C}$ is a *clonal partition* of M if C is a partition into clonal pairs. The matroid M is a *clonal matroid* if $E(M)$ has a clonal partition.

Let N be a minor of M. Then a clonal subset A of N is an *M-clonal* subset if A is a clonal set in M. In particular, the clonal pair $\{a, a'\}$ is an *M-clonal pair* if $\{a, a'\}$ is a clonal pair in M. We say that N is a *clonal minor* of M if $E(N)$ has a partition into M-clonal pairs.

Let M be a clonal matroid with associated clonal partition $\mathcal{P}$. If $\{p, p'\}$ is a member of $\mathcal{P}$, then we say that p' is the *clonal mate* of p. We may do this at times without mentioning the underlying partition, but only when no danger of ambiguity arises. In any unexplained context, we indicate the clonal mate of an element by adding a prime symbol. Thus a' will denote the clonal mate of a. If the subset A of elements of M contains no member of $\mathcal{P}$, then A' will denote the set $A' = \{a'\colon a \in A\}$.

The goal of this chapter is to prove.

Theorem 9.1. *There exists a function $f_{9.1}(m, q)$ such that, if M is a 4-connected matroid in $\mathcal{E}(q)$ with at least $f_{9.1}(m, q)$ pairwise-disjoint clonal pairs, then M has a clonal Δ_m-minor.*

The next lemma gives the first step towards proving Theorem 9.1.

Lemma 9.2. *Let M be a 3-connected matroid with a nonempty set A of elements that has a partition into pairwise-disjoint clonal pairs. Then M has a 3-connected clonal minor on A.*

Proof. Certainly M is not a wheel or a whirl. Thus, if M has a maximal fan with at least four elements, then this fan has an end. The end of the fan gives an element f that is not in a clonal pair. The element f is not in A and can be either deleted or contracted to preserve 3-connectivity. Thus we may assume that M has no 4-element fans. Say $x \in E(M) - A$. Assume that x is in a triangle T. If T contains a member of A, then by Lemma 5.24, $M\backslash x$ is 3-connected. Assume that $T \cap A = \emptyset$. Then by Tutte's Triangle Lemma, there is an element of T that can be deleted to preserve 3-connectivity. Assume that M has no triangles or triads containing x. Then, by Bixby's Lemma, either $M\backslash x$ or M/x is 3-connected. The lemma follows by induction on $|E(M) - A|$. $\square$

Given that 4-connected matroids are 3-connected, Theorem 9.1 is an immediate corollary of Lemma 9.2 and the next theorem.

Theorem 9.3. *There is a function $f_{9.3}(m, q)$ such that if M is a 3-connected clonal matroid in $\mathcal{E}(q)$ whose ground set has at least $f_{9.3}(m, q)$ elements, then M has a clonal Δ_m-minor.*

Proving Theorem 9.3 is the task of this chapter. This theorem has a similar flavour to the next important theorem of Ding, Oporowski, Oxley and Vertigan [5]. A matroid is a *whorl* if it is either a whirl or the cycle matroid of a wheel.

Theorem 9.4. *There is a function $f_{9.4}(m)$ such that, if M is a 3-connected matroid with at least $f_{9.4}(m)$ elements, then M has one of the following as a minor: $U_{m,m+2}$, $U_{2,m+2}$, $M(K_{3,m})$, $M^*(K_{3,m})$, a rank-m whorl, or a rank-m spike.*

We use the following immediate corollary of Theorem 9.4.

Corollary 9.5. *There is a function $f_{9.5}(m)$ such that, if M is a 3-connected matroid in $\mathcal{E}(q)$ with at least $f_{9.5}(m)$ elements, then M has one of the following as a minor: $M(K_{3,m})$, $M^*(K_{3,m})$, a rank-m whorl, or a rank-m spike.*

A brief outline of our path to Theorem 9.3 follows. We begin by using Corollary 9.5 to find an $M(K_{3,m})$, $M^*(K_{3,m})$, rank-m whorl, or rank-m spike as a minor of our large 3-connected clonal matroid M in $\mathcal{E}(q)$. In doing so we have lost our clones and these need to be recovered. Up to duality we obtain a series extension of our minor which has many M-clonal series pairs. In Section 2 we show that we can find such a series extension where the series pairs are bridged in a particularly simple way. After that, it is a matter of inspecting each type of minor in turn and demonstrating that in each case the bridging process either produces a violation to the assumption that we are in $\mathcal{E}(q)$ or produces a large clonal free-swirl minor.

Theorem 9.1 serves the needs of this paper, but it is probably not the strongest possible result. Given a large 3-connected clonal matroid, we are happy to obtain a large line, coline or free-spike minor. The only minor we care about being clonal is the free swirl. We make the following conjecture.

Conjecture 9.6. *There is a function $f_{9.6}(m)$ such that if a 3-connected clonal matroid has at least $f_{9.6}(m)$ elements, then it has one of the following as a clonal minor: $U_{2,2m}$, $U_{2m-2,2m}$, Λ_m, or Δ_m.*

2. Bridging M-clonal series pairs

As noted above, our first objective is to produce a highly structured minor with many M-clonal series pairs that are bridged in a particularly simple way. Recall that a matroid M is 3-*connected up to series pairs* if, whenever (X, Y) is a 2-separation of M, either X or Y is a series pair. The next lemma is the goal of this section.

Lemma 9.7. *Let M be a 3-connected clonal matroid in $\mathcal{E}(q)$. Then there is a function $f_{9.7}(m,t,q)$ such that, if M has at least $f_{9.7}(m,t,q)$ elements, then either M has a clonal Δ_m-minor or, up to duality, M has a 3-connected minor N with a coindependent set J such that the following hold.*

(i) *$N\backslash J$ is 3-connected up to a set of at least t series pairs.*
(ii) *Each series pair of $N\backslash J$ is M-clonal.*
(iii) *$\mathrm{co}(N\backslash J)$ is either a spike, a whorl, or for some integer l, is isomorphic to $M(K_{3,l})$ or $M^*(K_{3,l})$.*

The reader should now recall material on bridging sequences from Chapter 2 Section 3. It is shown in [11,18] that a bridging sequence for a 2-separation has at most five elements. For an M-clonal series pair it is not hard to do better.

Lemma 9.8. *Let N be a connected minor of the matroid M, let $\{p,p'\}$ be an M-clonal parallel pair in N, and let V be a minimal bridging sequence for $\{p,p'\}$. Then $|V| \leqslant 2$.*

Proof. Say that $V = (v_0, v_1, \dots, v_t)$. Assume that $|V| > 2$, that is, assume that $\{p,p'\}$ is not bridged in $N[v_0,v_1]$. Let $Z = E(N) - \{p,p'\}$. Consider $N[v_0]$. If v_0 is an extension element of V, then $v_0 \in \mathrm{cl}_{N[v_0]}(\{p,p'\})$, so that $\{p,p',v_0\}$ is a parallel set in $N[v_0]$. But then $v_0 \in \mathrm{cl}_{N[v_0]}(Z)$, contradicting Lemma 2.27. Thus v_0 is a coextension element of V.

9.8.1. *$\{p,p',v_0\}$ is both a triangle and a triad in $N[v_0]$.*

Subproof. This seems clearer in the dual. Here $\{p,p'\}$ is a series pair of N^*. By Lemma 2.28, $v_0 \in \mathrm{cl}_{N^*[v_0]}(\{p,p'\})$. As $\{p,p'\}$ is a clonal pair, $\{p,p',v_0\}$ is a triangle in $N^*[v_0]$. If $v_0 \in \mathrm{cl}_{N^*[v_0]}(Z)$, then we violate Lemma 2.27. Therefore $\{p,p',v_0\}$ is also a triad. □

Consider $N[v_0,v_1]$. Then v_1 is an extension element of V. By this fact and Lemma 2.28, in $N[v_0,v_1]$ we have $v_1 \in \mathrm{cl}(\{p,p',v_0\})$, and v_1 is not parallel to p or p'. Thus $\{p,p',v_1\}$ is a triangle in $N[v_0,v_1]$. But $v_1 \notin \mathrm{cl}_{N[v_0,v_1]}(Z)$, so that $N[v_0,v_1]\backslash v_0 \cong N[v_0,v_1]\backslash v_1$. Therefore $N[V]/v_0$ has N as a minor, contradicting Lemma 2.26. □

It is perhaps surprising that we have not used the next easy fact about spikes and swirls earlier in this paper.

Lemma 9.9. *Let M be a spike or a swirl of rank at least four. If M contains a clonal pair, then M is a free spike or free swirl.*

Proof. Let $\{a, a'\}$ be a clonal pair of M. Then $\{a, a'\}$ is certainly a leg of the spike or swirl. Assume that M is not a free spike or free swirl. Then M contains a circuit-hyperplane H that is a transversal of the legs. But then H contains exactly one of $\{a, a'\}$, contradicting the assumption that $\{a, a'\}$ is a clonal pair. □

Proof of Lemma 9.7. We lose no generality in assuming that $m > q$ as otherwise, we can define $f_{9.7}(m, t, q)$ to be equal to $f_{9.7}(q + 1, t, q)$. Set $f_{9.7}(m, t, q) = f_{9.5}(m + 2t, q)$. Assume that $|E(M)| \geqslant f_{9.7}(m, t, q)$. By Corollary 9.5, we see that M has minor N that is either a rank-$(m + 2t)$ whorl, a rank-$(m + 2t)$ spike, or is isomorphic to $M(K_{3,m+2t})$ or $M^*(K_{3,m+2t})$.

9.7.1. *If $\{a, a'\}$ is a clonal pair of M and $a \in E(N)$, then $a' \notin E(N)$.*

Subproof. Elements of whorls, $M(K_{3,t+2m})$ and $M^*(K_{3,t+2m})$ are either fixed or cofixed so these matroids contain no clonal pairs. If a spike contains a clonal pair, then by Lemma 9.9, that spike is a free spike contradicting the assumptions that $M \in \mathcal{E}(q)$. □

Choose a partition $\mathcal{C}$ of $E(M)$ into clonal pairs. Assume that $N = M/I\backslash J$ where I is independent and J is coindependent in M. Each element e of N has a clonal mate $e' \in E(M) - E(N)$. As $|E(N)| \geqslant 2(m + 2t)$, we may assume, up to duality, that at least $m + 2t$ of these clonal mates are in J. Let K be the set of elements of J that are not clonal mates of elements of N. Consider $M/I\backslash K$. Say $z \in E(N)$ has a clonal mate $z' \in J$. Then $\{z, z'\}$ must be a parallel pair in $M\backslash I/K$ unless z is not fixed in N. The exceptional case can only happen if either (a) N is a whirl or (b) N is a spike. Assume that we are in one of these cases and that there are at least m members of $E(N)$ that have clonal mates in $M\backslash I/K$ that are not in parallel pairs. In case (a) we routinely see that $M\backslash I/K$ has a Δ_m-minor, the legs of which are M-clonal, so that the lemma holds. In case (b) we routinely see that $M\backslash I/K$ has a Λ_m-minor and we have contradicted the assumption that $M \in \mathcal{E}(q)$.

It follows from the argument of the previous paragraph that we may assume from now on that $M/I\backslash K$ is 3-connected up to a set of at least $2t$ parallel pairs and each of these parallel pairs is a member of $\mathcal{C}$. If $\{x, x'\}$ is such a parallel pair, then, by Lemma 9.8, its corresponding 2-separation has a bridging sequence of length at most 2.

Let X be the union of the clonal pairs in $M/I\backslash K$ that are bridged with a 1-element bridging sequence and let Y be the union of the clonal pairs that are bridged by a 2-element minimal bridging sequence. Let $A = X \cap E(N)$, $A' = X - A$, $B = Y \cap E(N)$ and $B' = Y - B$.

9.7.2. *The lemma holds if $|A| \geqslant t$.*

Subproof. For each $a \in A$, let a'' denote an element of I that bridges $\{a, a'\}$ and let $A'' = \{a'' \colon a \in A\}$. Note that A'' could be a small set, indeed it may only have one element. Consider $M/(I - A'')\backslash K$. Note that, if $b \in B$, then $\{b, b'\}$ is a parallel pair in this matroid as otherwise $\{b, b'\}$ has a 1-element bridging sequence. Hence each parallel pair of $M/I\backslash(J - A')$ is bridged in $M/(I - A'')\backslash(J - A')$ and the claim follows by taking the dual. □

We now consider the case that $|B| \geqslant t$. Say $B = \{b_1, b_2, \ldots, b_n\}$. Then $B' = \{b'_1, b'_2, \ldots, b'_n\}$. For $i \in \{1, 2, \ldots, n\}$, let b''_i and b'''_i be the first and second elements of a 2-element bridging sequence for $\{b_i, b'_i\}$. Set $B'' = \{b''_1, b''_2, \ldots, b''_n\}$ and $B''' = \{b'''_1, b'''_2, \ldots, b'''_n\}$. As $\{b_i, b'_i\}$ is a parallel pair in $M/I\backslash K$, we see that b''_i is a coextension element of the bridging sequence (b''_i, b'''_i) and hence that b'''_i is an extension element of this bridging sequence. Therefore $B'' \subseteq I$ and $B''' \subseteq J$.

9.7.3. *For each $i \in \{1, \dots, n\}$, the pair (b_i'', b_i''') is a minimal bridging sequence for $\{b_i, b_i'\}$ in $M/I\backslash(K \cup A)$.*

Subproof. Say that $a \in A$. Then $b_i'' \notin \mathrm{cl}_{(M/I\backslash K)[b_i'']}(\{a, a'\})$, as otherwise b_i'' coblocks $\{b_i, b_i'\}$ contradicting the definition of bridging sequences. Thus $\{a, a'\}$ is a parallel pair in $(M/I\backslash K)[b_i'']$ and also in $(M/I\backslash K)[b_i'', b_i''']$. The claim follows easily from this observation. □

The effect of 9.7.3 is that we can ignore A' and we have the following setup. Let $I'' = I - B''$ and $J'' = J - (B' \cup B''')$. Then $N = \mathrm{si}(M/(I'' \cup B'')\backslash(J'' \cup B'''))$. Moreover, for $i \in \{1, 2, \dots, n\}$, the pair $\{b_i, b_i'\}$ is parallel in $M/(I'' \cup B'')\backslash(J'' \cup B''')$ and is bridged in $M/I''\backslash J''$ by the bridging sequence (b_i'', b_i''').

By the definition of bridging sequence, $\{b_i, b_i', b_i''\}$ is 2-separating in $M/(I'' \cup (B'' - \{b_i''\}))\backslash(J'' \cup B''')$, and, in this matroid, $b_i'' \in \mathrm{cl}(\{b_i, b_i'\})$. This shows that $\{b_i, b_i', b_i''\}$ is a triangle in this matroid and, as $\{b_i, b_i'\}$ is a clonal pair, $\{b_i, b_i'\}$ is a series pair in $M/(I'' \cup (B'' - \{b_i''\}))\backslash(J'' \cup B''' \cup \{b_i''\})$.

Consider $M/I''\backslash(J'' \cup B''')$. If the 2-separating set $\{b_i, b_i', b_i''\}$ of $M/(I'' \cup (B'' - \{b_i''\}))\backslash(J'' \cup B''')$ is bridged in this matroid, then the parallel pair $\{b_i, b_i'\}$ of $M/(I'' \cup B'')\backslash(J'' \cup B''')$ is also bridged. But, as B'' is independent in $M/I''\backslash(J'' \cup B''')$, a minimal bridging sequence must have size one, contradicting the assumption that $\{b_i'', b_i'''\}$ is a minimal bridging sequence for $\{b_i, b_i'\}$. From this we deduce that $\{b_i, b_i', b_i''\}$ is a 2-separating triangle of $M/I''\backslash(J'' \cup B''')$, and that $\{b_i, b_i'\}$ is a series pair of $M/I''\backslash(J'' \cup B''' \cup B'')$.

Moreover, it is easily seen that $\{b_i, b_i', b_i''\}$ is a clonal triple of $M/I''\backslash(J'' \cup B''')$, so that $M/(I'' \cup B')\backslash(J'' \cup B''' \cup B'') \cong M/(I'' \cup B'')\backslash(J'' \cup B''' \cup B')$. But the latter matroid is equal to N and the former is $\mathrm{co}(M/I''\backslash(J \cup B''' \cup B''))$. Each triangle $\{b_i, b_i', b_i''\}$ of $M/I''\backslash(J'' \cup B''')$ is blocked by b_i''' in $M/I''\backslash J''$, and it follows that each series pair $\{b_i, b_i'\}$ of $M/I''\backslash(J'' \cup B''' \cup B'')$ is blocked by b_i''' in $M/I''\backslash(J'' \cup B'')$. Thus the lemma holds in this case too. □

3. The whorl case

We begin by examining the case when we have a large whorl minor. The next lemma is the goal of this section. Its proof is surprisingly lengthy.

Lemma 9.10. *Let M be a 3-connected matroid in $\mathcal{E}(q)$ with a coindependent set J such that the following hold.*

(i) *$M\backslash J$ is 3-connected up to a set of n series pairs that are M-clonal.*
(ii) *$\mathrm{co}(M\backslash J)$ is a whorl.*

Then there is a function $f_{9.10}(m, q)$ such that, if $n \geqslant f_{9.10}(m, q)$, then M has a Δ_m-minor, each leg of which is a series pair of $M\backslash J$.

Preliminary results The next three lemmas are easily proved and are certainly well known. Note that Lemmas 9.11 and 9.12 follow from Ramsey-theoretic results on matrices given in [4]. A vertex of a hypergraph is *isolated* if it is not incident with any edges. Edges of a hypergraph are *parallel* if they are incident with the same set of vertices.

Lemma 9.11. *Let l and n be integers and let $H = (V, E)$ be a hypergraph with no isolated vertices and no parallel pairs of edges. Assume that no edge of H is incident with more than l vertices and let U be an n-element set of vertices of H. Then there is a function $f_{9.11}(m, l)$ such that, if $n \geqslant f_{9.11}(m, l)$, then there is a subset $\{u_1, u_2, \dots, u_m\}$ of V and a set $\{e_1, e_2, \dots, e_m\}$ of edges of H such that, for $i \in \{1, 2, \dots, m\}$, u_i is incident with e_j if and only if $i = j$.*

The next lemma is a strengthening of Lemma 9.11 that gives a somewhat more specific outcome.

Lemma 9.12. *Let l be an integer and let $H = (V, E)$ be a hypergraph with no isolated vertices and no parallel pairs of edges. Assume that no edge of H is incident with more than l vertices. Let $\phi : V \to E$ be a function* *such that, for all $v \in V$, the vertex v is incident with $\phi(v)$. Then there is a function $f_{9.12}(m,l)$ such that, if $n \geqslant f_{9.12}(m,l)$, then there is a subset $\{u_1, u_2, \dots, u_m\}$ of V such that, for $i \in \{1, 2, \dots, m\}$, the vertex u_i is incident with $\phi(u_j)$ if and only if $i = j$.*

Lemma 9.13. *Let $G = (V, E)$ be a graph where V is cyclically ordered and E is a matching. Then there is a function $f_{9.13}(m)$ such that, if $|E| = n$ and $n \geqslant f_{9.13}(m)$, then, for some labelling $(v_1, v_2, \dots, v_l)$ of V that respects the cyclic order, there is a set of m edges in E that can be directed and ordered $((v_{i_1}, v_{j_1}), (v_{i_2}, v_{j_2}), \dots, (v_{i_m}, v_{j_m}))$ such that one of the following holds.*

(i) $i_1 < j_1 < i_2 < j_2 < \cdots < i_m < j_m$.
(ii) $i_1 < i_2 < \cdots < i_m < j_m < j_{m-1} < \cdots < j_1$.
(iii) $i_1 < i_2 < \cdots < i_m < j_1 < j_2 < \cdots < j_m$.

Recall that a flower $\mathbf{P}$ in a connected matroid M has the property that if (X, Y) is a 2-separation, then either X or Y is contained in a petal of $\mathbf{P}$. Let $\mathbf{P} = (P_1, P_2, \dots, P_m)$ be a flower in the connected matroid M. Recall that a clonal pair $\{p_i, q_i\}$ contained in the petal P_i is $\mathbf{P}$*-strong* if $\kappa(\{p_i, q_i\}, P_1 \cup P_2 \cup \cdots \cup P_{i-1} \cup P_{i+1} \cup \cdots \cup P_m) = 2$. Equivalently $\{p_i, q_i\}$ is $\mathbf{P}$-strong if there is no 2-separating set X of M with $\{p_i, q_i\} \subseteq X \subseteq P_i$. For convenience we restate here a special case of Lemma 6.11.

Lemma 9.14. *Let M be a connected matroid with a swirl-like flower $\mathbf{P} = (P_1, P_2, \dots, P_n)$ such that for all $i \in \{1, 2, \dots, n\}$ the petal P_i contains a $\mathbf{P}$-strong clonal pair $\{p_i, q_i\}$ Then M contains a Δ_n-minor with associated flower $(\{p_1, q_1\}, \{p_2, q_2\}, \dots, \{p_n, q_n\})$.*

We will also use the following technical but elementary lemma.

Lemma 9.15. *Let M be a matroid with an element z'' such that $M \backslash z''$ is connected with an M-clonal series pair $\{z, z'\}$. Assume that $M \backslash z''$ has an exact 3-separation (X, Y) where $\{z, z'\} \subseteq Y$. Assume further that (X, Y) is blocked in M and that $\lambda_{M \backslash z''}(X \cup \{z, z'\}) > 2$. Then the 3-separation $(X, Y - \{z'\})$ of $M \backslash z''/z'$ is blocked by z'' in M/z'.*

Proof. Evidently $(X, Y - \{z'\})$ is an exact 3-separation of $M \backslash z''/z'$. If this is not blocked by z'' in M/z', then either $z'' \in \mathrm{cl}_{M/z'}(Y - \{z'\})$ or $z'' \in \mathrm{cl}_{M/z'}(X)$. In other words, $z'' \in \mathrm{cl}_M(Y)$ or $z'' \in \mathrm{cl}_M(X \cup \{z'\})$. The former case does not occur. Consider the latter. In this case $z' \in \mathrm{cl}_M(X \cup \{z''\})$ and, as $\{z, z'\}$ are clones, $z \in \mathrm{cl}_M(X \cup \{z''\})$. Thus $r_M(X \cup \{z, z'\}) \leqslant r_M(X) + 1$. But $r_M(Y - \{z, z'\}) \leqslant r_M(Y) - 1$ as $\{z, z'\}$ is a series pair. Therefore $\lambda_{M \backslash z''}(X \cup \{z, z'\}) \leqslant 2$, contradicting a hypothesis of the lemma. □

It is perhaps surprising that we have not needed the following lemmas on freedom until now.

Lemma 9.16. *Let $\{a, b, c\}$ be a triad of the matroid M, where b and c are clones. Then a is freer than b in M/c.*

Proof. Let F be a cyclic flat of M/c that contains a. Then either F or $F \cup \{c\}$ is a cyclic flat of M. In the latter case F contains b as $\{b, c\}$ is a clonal pair. In the former case F contains either b or c as $\{a, b, c\}$ is a triad of M. But again, as b and c are clones we deduce that F contains b. □

Lemma 9.17. *Let a and b be elements of a matroid M, where a is freer than b. If N is a minor of M whose ground set contains both a and b, then a is freer than b in N.*

Proof. Say $c \in E(M) - \{a, b\}$. Consider $M \backslash c$. If F is a cyclic flat of $M \backslash c$ that contains a, then either F or $F \cup \{c\}$ is a cyclic flat of M. In either case we deduce that $b \in F$. Thus a is freer than b in $M \backslash c$.

As a is freer than b we have b is freer than a in M^*, so that b is freer than a in $M^* \backslash c$. Thus a is freer than b in M/c. The lemma follows from these observations. □

Cleanly-blocked coextended whorls The matroid M is a *coextended whorl* if it is 3-connected up to series pairs and $\mathrm{co}(M)$ is a whorl. We now consider the case where the series pairs of a coextended whorl are blocked one at a time by the blocking elements. We begin by developing some terminology for this case.

Let N be a rank-n whorl with rim elements labelled $R = \{r_1, r_2, \dots, r_n\}$ and spoke elements labelled $S = \{s_1, s_2, \dots, s_n\}$. We say this labelling is *standard* if, for all $i \in \{1, 2, \dots, n\}$, the sets $\{s_i, r_i, s_{i+1}\}$ and $\{r_i, s_{i+1}, r_{i+1}\}$ are respectively triangles and triads of N, where indices are taken modulo n. Let M be a matroid. Then M is a *cleanly-blocked* coextended whorl of *order* n with *distinguished* 5-*tuple* (R, S, T, T', T'') if we have labellings $R = \{r_1, r_2, \dots, r_n\}$, $S = \{s_1, s_2, \dots, s_n\}$, $T = \{t_1, t_2, \dots, t_n\}$, $T' = \{t'_1, t'_2, \dots, t'_n\}$ and $T'' = \{t''_1, t''_2, \dots, t''_n\}$ such that the following hold.

(i) $E(M)$ consists of the union of the disjoint sets R, S, T' and T''. The set T is contained in $R \cup S$.
(ii) $M\backslash T''/T'$ is a whorl for which R and S give a standard labelling.
(iii) $M\backslash T''$ is a coextended whorl with series pairs $\{\{t_i, t'_i\}\colon\ i \in \{1, 2, \dots, n\}\}$.
(iv) The series pairs of $M\backslash T''$ are M-clonal.
(v) For all $i \in \{1, 2, \dots, n\}$, the element t''_i blocks the series pair $\{t_i, t'_i\}$ of $M\backslash T''$, but blocks no other series pair.

Our goal for this case is to prove.

Lemma 9.18. *Let M be a cleanly-blocked coextended whorl in $\mathcal{E}(q)$ with distinguished* 5-*tuple* (R, S, T, T', T''). *Then there is a function $f_{9.18}(m, q)$ such that, if $|T| \geqslant f_{9.18}(m, q)$, then M has a Δ_m-minor, each leg of which is a series pair of $M\backslash T''$.*

Note that if M is a cleanly-blocked coextended whorl, then M is 3-connected, the set $\{t_i, t'_i, t''_i\}$ is a triad for all $i \in \{1, 2, \dots, n\}$ and $\mathrm{co}(M\backslash T'') = M\backslash T''/T'$. We call a triad of the form $\{t_i, t'_i, t''_i\}$ a *flap* of M. The next lemma is immediate.

Lemma 9.19. *If $\{t_i, t'_i, t''_i\}$ is a flap of M, then $M\backslash t''_i/t'_i$ is a cleanly-blocked coextended whorl of order $n-1$ with distinguished* 5-*tuple* $(R, S, T - \{t_i\}, T' - \{t'_i\}, T'' - \{t''_i\})$.

The minor N of M is obtained by *removing flaps* if it is obtained by a sequence of operations of the form described in Lemma 9.19. The matroid M is *rim based* if all the series pairs of $M\backslash T''$ are based at rim elements, that is, if $T \subseteq R$. It is *spoke based* if all the series pairs are based at spoke elements, that is, if $T \subseteq S$. We consider the two cases in turn.

The rim case Our goal in the rim case is to prove

Lemma 9.20. *Let M be a rim-based cleanly-blocked coextended whorl in $\mathcal{E}(q)$ with distinguished* 5-*tuple* (R, S, T, T', T''). *Then there is a function $f_{9.20}(m, q)$ such that, if $|T| \geqslant f_{9.20}(m, q)$, then M has a Δ_m-minor, each leg of which is a series pair of $M\backslash T''$.*

Throughout this subsection we assume that M is a rim-based cleanly-blocked coextended whorl with distinguished 5-tuple (R, S, T, T', T''). Note that a flap of M will have the form $\{r_i, r'_i, r''_i\}$ for some $r_i \in R$.

Say $|R| = n$. If $1 \leqslant k < n$, then the set $\{s_n, r_n, s_1, r_1, \dots, r_{k-1}, s_k\}$ is 3-separating in $M\backslash T''/T'$ and the corresponding 3-separation is clearly induced in $M\backslash T''$. We denote this induced 3-separation by (L_k, K_k). We say that the element r''_h of T'' *blocks* (L_k, K_k) *from the right* if $h \in \{k, k+1, \dots, n-1\}$ and r''_h blocks (L_k, K_k). On the other hand, r''_h *blocks* (L_k, K_k) *from the left* if $h \in \{1, 2, \dots, k-1\}$ and r''_h blocks (L_k, K_k).

Lemma 9.21. *If $M \in \mathcal{E}(q)$ and $2 \leqslant k < n$, then at most $\frac{q^k-1}{q-1}$ elements of T'' block (L_k, K_k) from the right.*

Proof. Let U'' be the set of elements of T'' that block (L_k, K_k) from the right. Let $U = \{r_i \in T\colon r_i'' \in U''\}$. Let $(L, K) = (\{s_n, r_n, s_1, r_1, \dots, r_{k-1}, s_k\}, \{r_k, s_{k+1}, \dots, s_{n-1}, r_{n-1}\})$. By Lemma 9.15, we have

9.21.1. *If $u'' \in U''$, then u'' blocks the 3-separation (L, K) of $M\backslash T''/T'$.*

Define the equivalence relation $\sim$ on $\{s_k, s_{k+1}, \dots, s_n\}$ as follows. If $k \leqslant i \leqslant j \leqslant n$, then $s_i \sim s_j$ if the set $\{r_i, r_{i+1}, \dots, r_{j-1}\}$ contains no member of U. Otherwise $s_i \sim s_j$ if $s_j \sim s_i$. Evidently $\sim$ has $|U| + 1$ equivalence classes. Let

$$V = U'' \cup \bigl\{r_i\colon i \in \{r_k, r_{k+1}, \dots, r_{n-1}\};\ r_i \notin U\bigr\}.$$

9.21.2. *V is independent in M/T'.*

Subproof. As $M\backslash T''/T'$ is a whorl, the set $\{r_k, r_{k+1}, \dots, r_{n-1}\}$ of rim elements is independent in M/T'. It is elementary that if x is an element of an independent set X of a matroid and x'' is freer than x, then $(X - \{x\}) \cup \{x''\}$ is also independent. The claim now follows from the above facts, Lemma 9.16 and Lemma 9.17. □

We now focus on the rank-k matroid $M/T'/V$, the goal being to show that it has at least $|U''|$ parallel classes.

9.21.3. *If $i \in \{1, 2, \dots, k\}$, then $\{s_i\}$ is independent in $M/T'/V$.*

Subproof. As $M\backslash T''/T'$ is a whorl, $\{s_i, r_k, r_{k+1}, \dots, r_{n-1}\}$ is independent in $M\backslash T''/T'$. By Lemma 9.17 we see that $\{s_i\} \cup V$ is independent in M/T', proving the claim. □

9.21.4. *If $i, j \in \{k, k+1, \dots, n\}$, and $s_i \nsim s_j$, then $\{s_i, s_j\}$ is independent in $M/T'/V$.*

Subproof. Assume otherwise. Then, by 9.21.3, $\{s_i, s_j\}$ is a circuit of $M/T'/V$ and there is a subset W of V such that $\{s_i, s_j\} \cup W$ is a circuit of M/T'. Let X be the set consisting of those elements r_t of $\{r_k, r_{k+1}, \dots, r_{n-1}\}$ such that either $r_t \in W$ or $r_t'' \in W$. Note that, as $s_i \nsim s_j$, there is at least one element r_α'' in $W \cap U''$.

Say $r_t'' \in W$. Then $r_t'' \in \mathrm{cl}_{M/T'}((W - \{r_t''\}) \cup \{s_i, s_j\})$ and, as r_t'' is freer than r_t in this matroid, we see that $r_t \in \mathrm{cl}_{M/T}(W \cup \{s_i, s_j\})$. We conclude that $\mathrm{cl}_{M/T}(W \cup \{s_i, s_j\})$ contains $X \cup \{s_i, s_j\}$. But, in $M\backslash T''/T'$, this is a set of at most $r - 2$ rim elements together with two spoke elements and contains at most one circuit. Thus the rank of $X \cup \{s_i, s_j\}$ in M/T' is equal to the rank of $W \cup \{s_i, s_j\}$ in this matroid. Therefore $X \cup \{s_i, s_j\}$ spans $\mathrm{cl}_{M/T'}(W \cup \{s_i, s_j\})$, so that $r_\alpha'' \in \mathrm{cl}(X \cup \{s_i, s_j\})$ contradicting the fact that r_α'' blocks this set. □

As $r(M/T'/V) = k$, and the relation $\sim$ has $|U| + 1$ equivalence classes, it follows from 9.21.4 that this matroid has at least $|U| = |U''|$ parallel classes. The elements of U'' block (L_u, K_u) from the right. As $M \in \mathcal{E}(q)$, by Lemma 7.6, there are at most $\frac{q^k-1}{q-1}$ of them. □

It may be that an element $r_i \in R - T$ has the property that $M/r_i\backslash s_i$ is a cleanly-blocked coextended whorl with distinguished sets $(R - \{r_i\}, S - \{s_i\}, T, T', T'')$. If this is the case we say that $M/r_i\backslash s_i$ is a *reduction* of M, and that M is *reduced* if it has no reductions. The next lemma is clear.

Lemma 9.22. *Assume that M is reduced. If $r_i \in R - T$, then there is a flap $\{r_j, r_j', r_j''\}$ such that $\{r_i, r_j, r_j', r_j''\}$ is a circuit.*

In the reduced case we can bound the number of elements that are in $R - T$.

Lemma 9.23. *Assume that $M \in \mathcal{E}(q)$ is reduced. Then there is a function $f_{9.23}(m,q)$ such that, if $|R-T| \geqslant f_{9.23}(m,q)$, then M has a Δ_m-minor all of whose legs are series pairs of $M\backslash T''$.*

Proof. Let $f_{9.23}(m,q) = (q+2)f_{9.13}(2f_{7.37}(m,q)+2)$.

Define a graph G with vertex set R such that, for $r_i \in R-T$ and $r_j \in T$, the pair $\{r_i, r_j\}$ is an edge if $\{r_i, r_j, r'_j, r''_j\}$ is a circuit. Note that, if $r_j \in T$, then the degree of r_j is at most one in this graph so that it decomposes into stars.

9.23.1. *If $r_i \in R$, then $d(r_i) \leqslant q+2$.*

Subproof. Up to labels we may assume that $i=1$, that is $r_i = r_1$. Assume that $d(r_1) > 1$. Then $r_i \in R-T$. By Lemma 9.21 at most $q+1$ elements block (L_2, K_2) from the right. If $\alpha \neq s$, and $\{r_1, r_\alpha\}$ is an edge of G, then $r_\alpha \in T$, and r''_α blocks (L_2, K_2) from the right. There are at most $q+1$ such elements. It is possible that $\{r_1, r_n\}$ is an edge. Altogether we have $d(r_i) \leqslant q+2$. □

Thus G has a matching of size $f_{9.13}(2f_{7.37}(m,q)+2)$. By Lemma 9.13, G has a collection A of $2f_{7.37}(m,q)+2$ edges whose indices can be labelled to satisfy one of the cases of Lemma 9.13. Let N be the matroid obtained from M by removing all flaps $\{r_i, r'_i, r''_i\}$ for which $r_i \notin A$. If either case (i) or (ii) of Lemma 9.13 holds, then it is easily seen that N has a path of 3-separations of length at least $f_{7.37}(m,q)$ each step of which contains a series pair of $M\backslash T''$. Thus each step of $\mathbf{P}$ contains a $\mathbf{P}$-strong clonal pair, so that in these cases the lemma follows from Corollary 7.37. Note that, if case (i) holds this minor is found more directly via Lemma 9.14.

Consider case (iii). In this case we have a sequence of indices $i_1 < i_2 < \cdots < j_1 < j_2 < \cdots$ such that, for $1 \leqslant l \leqslant 2f_{7.37}(m,q)+2$, either $r_{i_l} \in T$ and $\{r_{i_l}, r'_{i_l}, r''_{i_l}, r_{j_l}\}$ is a circuit, or $r_{j_l} \in T$ and $\{r_{j_l}, r'_{j_l}, r''_{j_l}, r_{i_l}\}$ is a circuit. Up to labels we may assume that the former case occurs at least $t = f_{7.37}(m,q)+1$ times. (We do this simply for notational convenience; not for any structural reason.) After another round of flap removals, reductions and label resetting, we obtain a cleanly-bridged coextended whorl N' with distinguished 5-tuple (R', S', U, U', U'') such that $|R'| = 2t$, $U = \{r_1, r_2, \dots, r_t\}$ and, for all $i \in \{1, 2, \dots, t\}$, the set $\{r_i, r'_i, r''_i, r_{t+i}\}$ is a circuit. Let $N'' = N'\backslash\{r_1, r'_1, r''_1, r_{t+1}\}$. For $k \in \{2, 3, \dots, t\}$, let $P_k = \{s_k, s_{t+k}, r_k, r'_k, r'_{t+k}\}$ and set $\mathbf{P} = (P_2, P_3, \dots, P_{t-1}, P_t \cup \{s_1, s_{2t}\})$.

9.23.2. *N'' is 3-connected, and $\mathbf{P}$ is a path of 3-separations in this matroid.*

Subproof. Consider the matroid $N''\backslash U''/U'$. As this matroid is obtained by deleting two rim elements of a whorl, it is a matter of elementary graph theory to verify the following facts. Let $S_1 = (s_2, r_2, s_3, \dots, r_t, s_{t+1})$ and $S_2 = (s_{t+2}, r_{t+2}, s_{t+3}, \dots, s_{2t}, r_{2t}, s_1)$.

(i) The unique separation of $N''\backslash U''/U'$ is (S_1, S_2).
(ii) If (X, Y) is a 2-separation of $N''\backslash U''/U'$ then either X or Y is an initial or terminal segment of S_1 or S_2.
(iii) For $i \in \{2, 3, \dots, t-1\}$, the set $Z_i = \{s_2, r_2, \dots, s_i, r_i, s_{t+2}, r_{t+2}, \dots, s_{t+i}, r_{t+i}\}$ is 3-separating in $N''\backslash U''/U'$.

We omit the routine verification of the following claims. The separation (S_1, S_2) of $N''\backslash U''/U'$ is not induced in N''/U'. Indeed $\kappa_{N''}(S_1, S_2) \geqslant 3$. Also all 2-separations of $N''\backslash U''/U'$ are bridged in N''. From these claims we deduce that N'' is 3-connected.

Consider a 3-separating set Z_i as described in (iii). Say $j \in \{1, 2, \dots, i\}$. Then $r_j \in \mathrm{cl}^*_{N''\backslash U''}(\{r_j\})$ as $\{r_j, r'_j\}$ is a series pair of $N''\backslash U''$. Thus $\lambda_{N''\backslash U''}(Z_i \cup \{r'_2, r'_3, \dots, r'_i\}) = 2$. Also $\{r_j, r'_j, r''_j, r_{t+j}\}$ is a circuit of N''. Hence $Z_i \cup \{r'_2, r'_3, \dots, r'_i\} \cup \{r''_2, r''+3, \dots, r''_i\}$ is 3-separating in N''. But this set is equal to $P_2 \cup P_3 \cup \cdots \cup P_i$. □

The lemma now follows from 9.23.2, the fact that $\mathbf{P}$ has length $t-1 = f_{7.37}(m,q)$, and Corollary 7.37. □

We develop some notation for the next lemma. If M_t is a minor of M obtained by flap removal and reductions, we denote its distinguished 5-tuple by $(R_t, S_t, T_t, T'_t, T''_t)$, where for some integer l, we have $R_t = \{r^t_1, r^t_2, \dots, r^t_l\}$ and $S_t = \{s^t_1, s^t_2, \dots, s^t_l\}$. For an integer i, we define the 3-separation (L^t_i, K^t_i) in a way that is precisely analogous to the way that we defined (L_i, K_i) in M.

Lemma 9.24. *Assume that $M \in \mathcal{E}(q)$ and M has no Δ_m-minor each leg of which is a clonal pair of $M\backslash T''$. Then there is a function $f_{9.24}(h, m, q)$ with the property that, if $|T| \geqslant f_{9.24}(h, m, q)$, then M has a minor M_h obtained by flap removals and reductions such that the following hold.*

(i) *There is a sequence of indices $(h_1, h_2, \dots, h_h)$ such that, for $i \in \{1, 2, \dots, h\}$, the 3-separation $(L^h_{h_i}, K^h_{h_i})$ is not blocked from the right.*
(ii) *$L^h_{h_1}$ contains exactly one member of T_h and, if $1 < i \leqslant h$, then $L^h_{h_i} - L^h_{h_{i-1}}$ contains exactly one member of T_h.*

Proof. For $0 \leqslant u \leqslant h$, let $\mu(u, m, q) = (q^{(u+1+f_{9.23}(m,q))} - 1)/(q - 1)$, and let $f_{9.24}(h, m, q) = \sum_{u=0}^{h} \mu(u, m, q)$.

Let $M_0 = M$ and assume that, for some $u \in \{0, 1, \dots, h-1\}$, the matroid M_u, obtained from M by flap removals and reductions, satisfies the conclusion of the lemma with h replaced by u.

9.24.1. *If $|K^u_{u_u} \cap T_u| \geqslant 1$, then M has a minor M_{u+1}, obtained from M by flap removals and reductions, that satisfies the conclusions for the lemma with m replaced by $u + 1$, and such that*

$$\left|K^{u+1}_{(u+1)_{u+1}} \cap T_{u+1}\right| \geqslant \left|K^u_{u_u}\right| - \mu(u, m, q).$$

Subproof. It is clear that reductions preserve the desired properties of M_u, so that we may assume that M_u is reduced. Let i be the least integer such that $r^u_i \notin L^u_i$, and $r^u_i \in T_u$. Consider L^u_i. This set contains $u + 1$ elements of $R_u \cap T_u$ and, by Lemma 9.23, at most $f_{9.23}(m, q)$ elements of $R_u - T_u$. Hence $i \leqslant u + 1 + f_{9.23}(m, q)$. It now follows from Lemma 9.21 that at most $\mu(u, m, q)$ elements of T''_u block (L^u_i, K^u_i) from the right. If we remove the flaps associated with the blocking elements, then we obtain the desired minor. □

The lemma follows from 9.24.1 and induction. □

Proof of Lemma 9.20. Let

$$f_{9.20}(m, q) = f_{9.24}\big(f_{9.24}\big(\big(f_{7.37}(m, q), m, q\big), m, q\big)\big).$$

Assume that $|T| \geqslant f_{9.24}(m, q)$. Let $h = f_{9.24}(f_{7.37}(m, q), m, q)$. Assume that the lemma fails. Then by Lemma 9.24 and a reversal of the indices, we obtain a minor M_h of M with a sequence $\{h_1, h_2, \dots, h_h\}$ of indices such that, for $i \in \{1, 2, \dots, h\}$, the 3-separation $(L^h_{h_i}, R^h_{h_i})$ is not blocked from the left and such that, after possibly removing some extra flaps, has the property that $|L^h_1 \cap T_h| = 1$, and, for $i \in \{2, 3, \dots, h\}$, the set $L^h_i - L^h_{i-1}$ contains one element of T_h. We would like to apply Lemma 9.24 again, but we are not quite in a position to do this as we have distinguished indices to worry about. We omit the details of the obvious upgrade of Lemma 9.24 that covers this and conclude that, for some $l \geqslant f_{7.37}(m, q)$, we have a minor M_l of M, obtained by flap removal and reductions that has the property that there is a sequence of indices, $\{l_1, l_2, \dots, l_l\}$ such that, for $i \in \{1, 2, \dots, l\}$, the separation (L^l_i, K^l_i) is neither blocked from the left nor the right, that is, is 3-separating in M_l.

Let $P_1 = L^l_1$, let $P_i = L^l_i - L^l_{i-1}$ for $i \in \{2, 3, \dots, h-1\}$, and let $P_h = K^l_{h-1}$. Then the path $(P_1, P_2, \dots, P_h)$ of 3-separations of M_l has the property that each step contains a clonal pair that is a series pair of $M\backslash T''$. By Corollary 7.37 M_l has a Δ_m-minor each leg of which is a clonal series pair of $M\backslash T''$ and the lemma follows. □

The spoke case The goal here is to prove

Lemma 9.25. *Let M be a spoke-based cleanly-blocked coextended whorl in $\mathcal{E}(q)$ with distinguished 5-tuple (R, S, T, T', T''). Then there is a function $f_{9.25}(m,q)$ such that, if $|T| \geqslant f_{9.25}(m,q)$, then M has a Δ_m-minor, each leg of which is a series pair of $M\backslash T''$.*

Now for the series of lemmas that lead to Lemma 9.25.

Lemma 9.26. *Let M be a matroid whose ground set contains disjoint sets $T = \{t_1, t_2, \dots, t_n\}$, $T' = \{t'_1, t'_2, \dots, t'_n\}$, and $T'' = \{t''_1, t''_2, \dots, t''_n\}$ such that the following hold.*

(i) *For all $i \in \{1, 2, \dots, n\}$, the pair $\{t_i, t'_i\}$ is an M-clonal series class of $M\backslash T''$, and these are the only non-trivial series classes of $M\backslash T''$.*
(ii) *T is a parallel class of $\mathrm{co}(M\backslash T'') = M\backslash T''/T'$.*
(iii) *Up to the parallel class T, the matroid $M\backslash T''/T'$ is a whorl with R as its set of rim elements, and T is at a spoke of this whorl.*
(iv) *There is an injective function $\phi : \{1, 2, \dots, n\} \to R$ such that, for all $i \in \{1, 2, \dots, n\}$, the set $\{t_i, t'_i, t''_i, \phi(i)\}$ is a circuit of M.*

Then there is a function $f_{9.26}(q)$ such that, if $n \geqslant f_{9.26}(q)$, then $M \notin \mathcal{E}(q)$.

Proof. Let $f_{9.26}(q) = f_{7.1}(5, q) + 3$.

Let $(s_1, r_1, s_2, r_2, \dots, s_t, r_t)$ be a labelling of $\mathrm{si}(M\backslash T''/T') = M\backslash T''/T'\backslash(T - \{t_1\})$, where $R = \{r_1, r_2, \dots, r_t\}$, $S = \{s_1, s_2, \dots, s_t\}$, $t_1 = s_1$ and triples of the form (s_i, r_i, s_{i+1}) and (r_i, s_{i+1}, r_{i+1}) are triangles and triads respectively. We may assume that, if $i > j$, then $\phi(i) > \phi(j)$. Define a partition $(P_1, P_2, \dots, P_n)$ of $E(M\backslash T''/T')$ as follows: let $P_1 = \{t_1, r_1, s_2, r_2, \dots, r_{\phi(1)}\}$; let $P_k = \{t_k, s_{\phi(k-1)+1}, r_{\phi(k-1)+1}, \dots, s_{\phi(k)}, r_{\phi(k)}\}$ for $k \in \{2, \dots, n-1\}$; and let $P_n = \{t_n, s_{\phi(n-1)+1}, r_{\phi(n-1)+1}, \dots, s_t, r_t\}$. This is a path of 3-separations in $M\backslash T''/T$. Moreover, $(P_1 \cup \{t'_1, t''_1\}, P_2 \cup \{t'_2, t''_2\}, \dots, P_n \cup \{t'_n, t''_n\})$ is a path of 3-separations in M of length $n-1$.

Furthermore, it is readily checked that if (A, B) is a 3-separation of M, where $|A|, |B| > 4$, then either A or B is a fan of M. Thus M is 5-coherent. It now follows from Corollary 7.2 that, if $n \geqslant f_{9.26}(q)$, then M is not in $\mathcal{E}(q)$. $\square$

Lemma 9.27. *Let M be a spoke-based cleanly-blocked coextended whorl in $\mathcal{E}(q)$ with distinguished 5-tuple (R, S, T, T', T''). Assume that there is an injection ϕ from the indices of members of T to the indices of members of R such that, for all $s_i \in T$, the set $\{s_i, s'_i, s''_i, r_{\phi(i)}\}$ is a circuit of M. Then there is a function $f_{9.27}(m,q)$ such that, if $|T| \geqslant f_{9.27}(m,q)$, then M has a Δ_m-minor, each leg of which is a series pair of $M\backslash T''$.*

Proof. Let $f_{9.27}(m,q) = \max\{f_{9.13}(2m), f_{9.13}(2f_{9.26}(q))\}$. Assume that $n \geqslant f_{9.27}(m,q)$. Then $n \geqslant f_{9.13}(2t)$, where $t \geqslant \max\{m, f_{9.26}(q)\}$.

By Lemma 9.13, a majority argument, and an appropriate cyclic ordering of the indices, we see that there is a minor N of M, obtained by flap removal, with t flaps,

$$\left\{\left\{s_{i_1}, s'_{i_1}, s''_{i_1}, r_{\phi(i_1)}\right\}, \left\{s_{i_2}, s'_{i_2}, s''_{i_2}, r_{\phi(i_2)}\right\}, \dots, \left\{s_{i_t}, s'_{i_t}, s''_{i_t}, r_{\phi(i_t)}\right\}\right\}$$

where either

(i) $i_1 < \phi(i_1) < i_2 < \phi(i_2) < \cdots < i_t < \phi(i_t)$,
(ii) $i_1 < i_2 < \cdots < i_t < \phi(i_t) < \cdots < \phi(i_1)$, or
(iii) $i_1 < i_2 < \cdots < i_t < \phi(i_1) < \phi(i_2) < \phi(i_t)$.

Assume that case (i) holds. For $k \in \{1, 2, \dots, t-1\}$, let $P_k = \{s_{i_k}, r_{i_k}, s_{i_k+1}, r_{i_k+1}, \dots, s_{i_{k+1}-1}, r_{i_{k+1}-1}, s'_{i_k}, s''_{i_k}\}$ and let $P_t = E(N) - (P_1 \cup P_2 \cup \cdots \cup P_{t-1})$. Note that, for $k \in \{1, 2, \dots, t\}$, the set P_k contains the clonal pair $\{s_{i_k}, s'_{i_k}\}$ and that $(P_1, P_2, \dots, P_t)$ is a swirl-like flower in N. It now follows from Lemma 9.14 that in this case, as $t \geqslant m$, the lemma is satisfied by producing a Δ_m-minor each of whose legs are clonal pairs of the form $\{s_i, s'_i\}$.

Assume that either (ii) or (iii) holds. Let $R' = \{r_j \in R\colon\ i_1 \leqslant j \leqslant i_k\}$. Consider N/R'. Apart from a single parallel class at the common basepoint of the series pairs $\{\{s_{i_1}, s'_{i_1}\}, \{s_{i_2}, s'_{i_2}\}, \dots, \{s_{i_t}, s'_{i_t}\}\}$ in $N/R' \backslash \{s''_{i_1}, s''_{i_2}, \dots, s''_{i_t}\}$ that does not affect the argument, the hypotheses of Lemma 9.26 hold for N/R' so that, as $t \geqslant f_{9.26}(m)$, we obtain the contradiction that $M \notin \mathcal{E}(q)$. $\square$

Proof of Lemma 9.25. Define $f_{9.25}(m, q)$ by

$$f_{9.25}(m, q) = \frac{m^{f_{9.26}(q) f_{9.27}(m,q)} - 1}{2(m-1)}.$$

Let Z be a maximal subset of R with the property that each series pair of $M\backslash T''/Z$ is blocked in M/Z. Note that, up to parallel classes at spokes, $M\backslash T''/(T' \cup Z)$ is a whorl.

Let λ denote the maximum number of elements contained in a single parallel class of $M\backslash T''/(T' \cup Z)$. The next claim follows from an elementary bookkeeping argument and Lemma 9.26.

9.25.1. *If $\lambda \geqslant f_{9.26}(q)$, then $M \notin \mathcal{E}(q)$.*

Thus $\lambda < f_{9.26}(m)$. Let μ denote the rank of $M\backslash T''/(T' \cup Z)$.

9.25.2. *The lemma holds if $\mu \geqslant f_{9.26}(q) f_{9.27}(m, q)$.*

Proof. Let R' denote the rim elements of $M\backslash T''/(T' \cup Z)$. If $r \in R'$, then, by the definition of Z, there is an element t_i of T such that $\{t_i, t'_i, t''_i, r\}$ is a circuit. There may be more than one. Arbitrarily choose one for each rim element to define a function $\rho : R' \to T$. Note that this function is injective. As $\lambda < f_{9.26}(q)$ and $\mu \geqslant f_{9.26}(q) f_{9.27}(m, q)$, we may choose a subset R'' of R' with $|R''| \geqslant f_{9.27}(m, q)$ such that, if r_i and r_j are elements of R'', then $\rho(r_i)$ is not parallel to $\rho(r_j)$ in $M\backslash T''/(T' \cup Z)$. It is now straightforward to take an appropriate minor of M and apply Lemma 9.27 to prove the sublemma. $\square$

We may now assume that $\mu < f_{9.26}(q) f_{9.27}(m, q)$. Assume that $n \geqslant f_{9.25}(m, q)$. Then $n \geqslant \frac{m^{\mu}-1}{2(m-1)}$. Consider $M/(Z \cup T'')$. This matroid has rank μ. Moreover, each clonal pair $\{t_i, t'_i\}$ is independent in this matroid. Hence it has at least $2n$ parallel classes, that is, it has at least $\frac{m^{\mu}-1}{m-1}$ parallel classes. By Lemma 7.6 we obtain the contradiction that $M \notin \mathcal{E}(q)$. $\square$

The general case First observe that Lemma 9.18 follows routinely from Lemmas 9.20 and 9.25. We omit the ritual incantation that establishes it.

Lemmas 9.20 and 9.25 deal with one case that arises in Lemma 9.10. We now consider a complementary case.

Lemma 9.28. *Let M be a 3-connected matroid with an element x such that $M\backslash x$ is 3-connected up to an n-element set of M-clonal series pairs and such that $\mathrm{co}(M\backslash x)$ is a whorl. Then there is a function $f_{9.28}(q)$ such that, if $n \geqslant f_{9.28}(q)$, then $M \notin \mathcal{E}(q)$.*

Proof. Let $f_{9.28}(q) = f_{7.1}(5, q) + 4$ and let $t = f_{7.1}(5, q) + 2$.

Let r_0 be a rim element of $\mathrm{co}(M\backslash x)$. It may be that r_0 is in a series pair $\{r_0, r'_0\}$ of $M\backslash x$. In this case let $N = M\backslash r_0/r'_0$. Otherwise let $N = M\backslash r_0$. Observe that $N\backslash x$ has a path $(P_0, P_1, \dots, P_t)$ of 2-separations each step of which contains a series pair of $M\backslash x$. We omit the routine verification of the following claims. The path $(P_0 \cup \{x\}, P_1, \dots, P_t)$ is a well-defined path of 3-separations in N. If (A, B) is a 2-separation of N, then either A or B is contained in either P_0 or P_t. The underlying

3-connected matroid M' obtained by appropriately removing all but one element of each maximal 2-separating subset of M contained in P_0 or P_t is 5-coherent. Each step of the path in M induced by $\mathbf{P}$ contains a clonal pair of M'. By Corollary 7.2, $M \notin \mathcal{E}(q)$. □

At last we can achieve the purpose of this section.

Proof of Lemma 9.10. Recall the hypotheses of the lemma. Let $l = f_{9.18}(m,q)$ and let $f_{9.10}(m,q) = f_{9.11}(l, f_{9.28}(q))$. Assume that $n \geqslant f_{9.10}(m,q)$.

Let H be the hypergraph whose vertex set is the collection of series pairs of $M\backslash J$ and whose edge set is J where an element $x \in J$ is incident with a series pair if x blocks that series pair. If H has an edge containing at least $f_{9.28}(q)$ vertices, then it is easily seen that M has a minor satisfying the hypotheses of Lemma 9.28 giving the contradiction that $M \notin \mathcal{E}(q)$. Thus, by Lemma 9.11 and the definition of $f_{9.10}(m,q)$, there is a set $S = \{\{s_1, s_1'\}, \{s_2, s_2'\}, \ldots, \{s_l, s_l'\}\}$ of series pairs and a collection $T = \{t_1, t_2, \ldots, t_l\}$ of elements of J such that, if $i, j \in \{1, 2, \ldots, l\}$, then t_i blocks $\{s_i, s_j\}$ if and only if $i = j$.

We ignore the elements of J not in T and focus on $M\backslash(J-T)$. Let $\{u, u'\}$ be a series pair of $M\backslash J$ that is not in S. Consider $M\backslash(J-T)/u'$. Assume, for a contradiction, that, for some $i \in \{1, 2, \ldots, l\}$, the series pair $\{s_i, s_i'\}$ is not blocked in this matroid. Then we have $t_i \in \mathrm{cl}_{M\backslash(J-T)\backslash u'}(\{s_i, s_i'\})$, that is, $t_i \in \mathrm{cl}_M(\{s_i, s_i', u'\})$, so that $u' \in \mathrm{cl}_M(\{s_i, s_i', t_i\})$. But $\{u, u'\}$ is a clonal pair of M, so that $\{u, u'\} \subseteq \mathrm{cl}_M(\{s_i, s_i', t_i\})$. However, this implies that $\lambda_{M\backslash J}(\{u, u', s_i, s_i'\}) = 1$, contradicting the fact that $M\backslash J$ is 3-connected up to series pairs.

Therefore each series pair of S is blocked in $M\backslash(J-T)/u'$. It follows from this fact and an obvious induction that M has a minor N with a set T of elements that is a cleanly-blocked coextended whorl, where the series pairs of $N\backslash T$ are series pairs of $M\backslash J$. Moreover, $N\backslash T$ has at least $l = f_{9.18}(m,q)$ series pairs. By Lemma 9.18 N, and hence M, has a Δ_m-minor each leg of which is a series pair of $N\backslash T$, and hence of $M\backslash T$. □

4. The $M(K_{3,n})$ case

We now consider the case where the underlying matroid is $M(K_{3,n})$. The goal of this section is to prove

Lemma 9.29. *Let M be a 3-connected matroid with a coindependent set J such that the following hold.*

(i) *$M\backslash J$ is 3-connected up to a set of n series pairs that are M-clonal.*
(ii) *$\mathrm{co}(M\backslash J) \cong M(K_{3,t})$ for some integer t.*

Then there is a function $f_{9.29}(q)$ such that, if $n \geqslant f_{9.29}(q)$, then $M \notin \mathcal{E}(q)$.

As usual we develop a series of lemmas. The next lemma is just a special case of Lemma 6.11 that we state here for convenience.

Lemma 9.30. *Let M be a connected matroid with a paddle $\mathbf{P} = (P_1, P_2, \ldots, P_n)$ such that, for all $i \in \{1, 2, \ldots, n\}$ the petal P_i contains a $\mathbf{P}$-strong clonal pair. Then M contains a $U_{2,2n}$-minor.*

To make life easier in this section we develop some local terminology. Let N be a connected matroid such that $\mathrm{co}(N) \cong M(K_{3,t})$ for some $t \geqslant 3$. Just as there is a unique flower of order t in $\mathrm{co}(N)$, so too is there a unique flower of order t in N and, extending existing terminology, we say that this is the *canonical flower associated with N*.

A matroid M is a *blocked coextended $M(K_{3,t})$* of *order n* with *blocking set J* if the following hold.

(i) $M\backslash J$ is 3-connected up to a set of n series pairs that are M-clonal.
(ii) $\mathrm{co}(M\backslash J) \cong M(K_{3,t})$.
(iii) If $j \in J$, then there is a series pair of $M\backslash J$ that is blocked by j.

Let M be a blocked coextended $M(K_{3,t})$ with blocking set J and let $\mathbf{P} = (P_1, P_2, \dots, P_t)$ be the canonical flower associated with $M\backslash J$. For $x \in J$, and $i \in \{1, 2, \dots, t\}$, we say that x is *incident* with P_i if either x blocks a series pair in P_i, or x blocks P_i. Note that the former case is redundant unless $x \in \mathrm{cl}(P_i)$. The above incidence relation defines a hypergraph whose vertex set is the set of petals of $\mathbf{P}$ and whose edge set is J. We denote this hypergraph by $H(\mathbf{P}, J)$.

Lemma 9.31. *Let M be a blocked coextended $M(K_{3,t})$ with blocking set J and canonical associated flower $(P_1, P_2, \dots, P_t)$. Let $\{e, e'\}$ be a series pair of $M\backslash J$ that is contained in P_i. Say that the element x of J blocks P_j, where $P_j \neq P_i$. Then x blocks P_j in M/e.*

Proof. Assume otherwise. Then $x \in \mathrm{cl}_{M/e}(P_j)$ so $x \in \mathrm{cl}_M(P_j \cup \{e\}) - \mathrm{cl}_M(P_j)$. Thus $r(P_j \cup \{x, e\}) = r(P_j \cup \{e\}) = r(P_j) + 1 = r(P_j \cup \{x\})$. As e and e' are clones, we deduce that $r(P_j \cup \{e, e'\}) = r(P_j) + 1$, so $e' \in \mathrm{cl}_{M/e}(P_j)$, contradicting the fact that $\mathrm{co}(M\backslash J) \cong M(K_{3,t})$. □

Lemma 9.32. *Let M be a blocked coextended $M(K_{3,t})$ with blocking set J and canonical flower $\mathbf{P} = (P_1, P_2, \dots, P_t)$. If $H(\mathbf{P}, V)$ has an edge that is incident with at least q vertices, then M has a Λ_q-minor.*

Proof. Assume that x is incident with at least q vertices. We may assume that $J = \{x\}$ and, by Lemma 9.31, we may assume that $M\backslash x$ has a single series pair $\{a_1, a_1'\}$ that is contained in P_1 and is blocked by x. We may further assume that x blocks every petal of $\mathbf{P}$.

Let $A = \{a_1, a_2, \dots, a_t\}$, $B = \{b_1, b_2, \dots, b_t\}$ and $C = \{c_1, c_2, \dots, c_t\}$ be a partition of $(P_1 - \{a_1'\}) \cup P_2 \cup \dots \cup P_t$ such that the following hold: $P_1 = \{a_1, a_1', b_1, c_1\}$; for each $i \in \{2, 3, \dots, t\}$, we have $P_i = \{a_i, b_i, c_i\}$; and A, B, and C are cocircuits of $M\backslash x/a_1'$. In other words, A, B and C are stars of the underlying $K_{3,t}$. Consider $N = \mathrm{si}(M/b_1, c_1)$. The parallel pairs of $M/b_1, c_1$ are $\{b_2, c_2\}, \{b_3, c_3\}, \dots, \{a_t, b_t\}$, so we may assume that $E(N) = A \cup (B - \{b\}) \cup \{a_1', x\}$. Note that $N\backslash x \cong M(K_{2,t})$, that for all $i \in \{2, 3, \dots, t\}$, the set $\{a_1, a_1', a_i, b_i\}$ is a circuit of N, and that each series pair of $N\backslash x$ is blocked by x. It follows that N/x is a spike. But the clonal pair $\{a_1, a_1'\}$ is a leg of this spike. By Lemma 9.9, the only spikes with clonal pairs are free spikes. Hence $N/x \cong \Lambda_t$. As t is the degree of x and $t \geqslant q$, the lemma follows. □

Lemma 9.33. *Let M be a blocked coextended $M(K_{3,t})$ of order n with blocking set J and canonical flower $\mathbf{P} = (P_1, P_2, \dots, P_t)$ and let l be an integer. Then there is a function $f_{9.33}(l, q)$ such that, if each edge of $H(\mathbf{P}, J)$ has at most l vertices and $n \geqslant f_{9.33}(l, q)$, then M has a $U_{2,q+2}$-minor.*

Proof. Let $w = \lceil \frac{q+2}{2} \rceil$ and let $f_{9.33}(1, q) = w$. For $l > 1$ let $f_{9.33}(l, q) = f_{9.12}(s, l)$ where $s = w(q + 2)(f_{9.33}(l - 1, q)) + w(q + 3)$.

Assume that $n \geqslant f_{9.33}(l, q)$. Say that $l = 1$. Up to labels we may assume that $P_1, P_2, \dots, P_w$ all contain series pairs of $M\backslash J$. But then, for each $i \in \{1, 2, \dots, w\}$, there is an element $x_i \in J$ such that x_i blocks a series pair in P_i. By the fact that $l = 1$ we have $x_i \in \mathrm{cl}(P_i)$. Thus $(P_1 \cup \{x_1\}, P_2 \cup \{x_2\}, \dots, P_w \cup \{x_w\})$ is a paddle in $M|(P_1 \cup \dots \cup P_n \cup \{x_1, \dots, x_w\})$ each petal of which contains a clonal pair that is strong relative to this flower. By Lemma 9.30 M has a $U_{2,q+2}$-minor.

Assume that $l > 1$ and, for induction, assume that the lemma holds for smaller values of l. Up to labels of the petals of $\mathbf{P}$, there is, by Lemma 9.12, a subset $J' = \{x_1, x_2, \dots, x_s\}$ of J and a set $\{\{a_1, a_1'\}, \{a_2, a_2'\}, \dots, \{a_s, a_s'\}\}$ of series pairs of $M\backslash J$ such that $\{a_i, a_i'\} \subseteq P_i$ for $i \in \{1, 2, \dots, s\}$, with the properties that

(i) x_i blocks $\{a_i, a_i'\}$ for all $i \in \{1, 2, \dots, s\}$; and
(ii) if $i \in \{1, 2, \dots, s\}$, $j \in \{1, 2, \dots, t\}$, and x_i blocks P_j, then either $j = i$ or $j > s$.

It may be that there exist $i \in \{1, 2, \dots, s\}$ and $j \in \{s + 1, s + 2, \dots, t\}$ such that, for some $z \in P_j$, the set $\{a_i, a_i', x_i, z\}$ is a circuit. For terminology restricted to this proof, we say in this case that x_i is a *fragile* element of J' and z is a *fragile* element of P_j.

Let y be an element of P_j for some $j \in \{s+1, s+2, \dots, t\}$ and assume that y is not fragile. Then, for all $i \in \{1, 2, \dots, s\}$, the series pair $\{a_i, a_i'\}$ of $[M/z\backslash(P_j - \{z\})]\backslash J$ is blocked by x_i. It follows from this that if the lemma holds in the case that every element of $P_{s+1} \cup P_{s+2} \cup \dots \cup P_t$ is fragile, then it holds in general. Thus we may assume that every element of this set is fragile.

Let K be the set of fragile elements of J'.

9.33.1. *The lemma holds if* $|K| > w(q+2)$.

Proof. Note that the elements of K are 2-element edges of $H(\mathbf{P}, J)$. Indeed the subhypergraph induced by K is a union of stars. Thus, it either has a vertex of degree $q+2$ or a matching of size w. The latter case implies that M has a 3-connected minor with a paddle containing w petals, each petal of which contains a clonal pair. In this case, by Lemma 9.30, M has a $U_{2,q+2}$-minor. Consider the former case. Assume that P_j has degree $q+2$ in $H(\mathbf{P}, J)$. Up to labels we may assume that the members of $\{x_1, x_2, \dots, x_{q+2}\}$ are fragile elements incident with P_j in $H(\mathbf{P}, J)$. Say $i \in \{1, 2, \dots, q+2\}$. Then, as $\sqcap(\{a_i, a_i'\}, P_j) = 0$, there is at most one element of P_j in the closure of $\{a_i, a_i', x_i\}$. Thus, again up to labels, we may assume that there is an element z in P_j such that $z \notin \mathrm{cl}(\{a_i, a_i', x_i\})$ for $i \in \{1, 2, \dots, w\}$. This means that, if $i \in \{1, \dots, w\}$, the series pair $\{a_i, a_i'\}$ of $M/z\backslash J$ is blocked by x_i. But $x_i \in \mathrm{cl}(P_i \cup P_j)$ and it follows from properties of $M(K_{3,n})$ that $x_i \in \mathrm{cl}_{M/z}(P_i)$. Indeed $(M/z)|(P_1 \cup P_2 \cup \dots \cup P_w \cup \{x_1, \dots, x_w\})$ is a 3-connected matroid with a paddle $(P_1 \cup \{x_1\}, P_2 \cup \{x_2\}, \dots, P_w \cup \{x_w\})$, each petal of which contains a clonal pair. Again, by Lemma 9.30 M has a $U_{2,q+2}$-minor. $\square$

We may now assume that J' has at most $w(q+2)$ fragile elements. If J' contains at least w elements of degree 1 in $H(\mathbf{P}, J)$, then we again find a $U_{2,q+2}$-minor, so we may assume that J' has at least $s - (w(q+2) + w) = s - w(q+3)$ elements that are not fragile with degree at least 2. But, each element of $\{P_{s+1}, P_{s+2}, \dots, P_t\}$ is incident with a fragile element and, somewhat crudely, we see that $t - s \leqslant w(q+2)$. From these facts and the definition of s, we deduce that, for some $j \in \{s+1, s+2, \dots, t\}$, at least $f_{9.33}(l-1, q)$ members of J' are incident with P_j. Let J'' be the set of members of J' that are incident with P_j. Then $|J''| \geqslant f_{9.33}(l-1, q)$. Up to labels we may assume that $J'' = \{x_1, x_2, \dots, x_p\}$, where $p \geqslant f_{9.33}(l-1, q)$. Say $z \in P_j$. Consider $M/z|(P_1 \cup P_2 \cup \dots \cup P_p \cup J'')$. This is a blocked coextended $M(K_{3,t})$ of order $f_{9.33}(l-1, q)$ with blocking set J'' such that each edge of $H((P_1, P_2, \dots, P_p), J'')$ is incident with at most $l-1$ vertices. By the induction assumption this matroid, and therefore M, has a $U_{2,q+2}$-minor. $\square$

Proof of Lemma 9.29. Let $f_{9.29}(q) = f_{9.33}(q, q)$. Let $\mathbf{P}$ be the canonical flower associated with $M\backslash V$. If $H(\mathbf{P}, V)$ has an edge incident with q vertices, then, by Lemma 9.32, $M \notin \mathcal{E}(q)$. Otherwise $M \notin \mathcal{E}(q)$ by Lemma 9.33. $\square$

5. The spike case

We now turn to spikes. In this case we prove

Lemma 9.34. *Let M be a 3-connected matroid with a coindependent set J such that the following hold.*

(i) *$M\backslash J$ is 3-connected up to a set of n series pairs that are M-clonal.*
(ii) *$\mathrm{co}(M\backslash J)$ is a spike.*

Then there is a function $f_{9.34}(q)$ such that, if $n \geqslant f_{9.34}(q)$, then M has a Λ_q-minor.

Lemma 9.34 is a routine consequence of the next lemma, which is indeed somewhat stronger. Let x be an element of the matroid M and $\mathbf{P}$ be a flower in $M\backslash x$. Then $\mathbf{P}$ is *well blocked* by x if x blocks (P, Q) whenever (P, Q) is a 3-separation of $M\backslash x$ such that both P and Q are unions of at least two petals of $\mathbf{P}$. Note that while we have used “well blocked” with a different meaning—as in a well-blocked 3-separation—here we are using it as an adjective that applies to flowers, so there is no danger of ambiguity.

Lemma 9.35. *Let M be a 3-connected matroid with a triad $\{x, a_1, a_1'\}$ such that $M\backslash x/a_1'$ is a rank-n spike and $\{a_1, a_1'\}$ is an M-clonal series pair in $M\backslash x$. Then, if $n \geqslant q^2 - 1$, the matroid M has a Λ_q-minor.*

Proof. Let $\mathbf{P} = (P_1, P_2, \dots, P_n)$ be the spike-like flower associated with $M\backslash x$, where $\{a_1, a_1'\} \subseteq P_1$ and $(P_1 - \{a_1'\}, P_2, \dots, P_n)$ is a spike in $M\backslash x/a_1'$. Say $P_1 = \{a_1, a_1', b_1\}$ and, for $i \in \{2, 3, \dots, n\}$, let $P_i = \{a_i, b_i\}$.

9.35.1. *If $\mathbf{P}$ is well blocked by x and $n \geqslant q + 1$, then M has a Λ_q-minor.*

Subproof. First observe that $M\backslash x/b_1\backslash P_n \cong M(K_{2,n-1})$. We omit the easy rank calculations that establish this fact. Let $M' = M/b_1\backslash P_n$. We now show that every petal of $(P_1 - \{b_1\}, P_2, \dots, P_{n-1})$ in $M'\backslash x$ is blocked by x in M'.

Consider $P_1 - \{b_1\} = \{a_1, a_1'\}$. If x does not block $P_1 - \{b_1\}$, then $x \in \mathrm{cl}_{M'}(P_2 \cup P_3 \cup \dots \cup P_{n-1})$, so that $x \in \mathrm{cl}_M(E(M) - \{a_1, a_1'\})$, contradicting the fact that x blocks $\{a_1, a_1'\}$ in M. Say $i \in \{2, 3, \dots, n-1\}$ and assume that x does not block P_i. Then $x \in \mathrm{cl}_{M'}((P_1 - \{a_1'\}) \cup P_2 \cup \dots \cup P_{i-1} \cup P_{i+1} \cup \dots \cup P_{n-1})$ so that $x \in \mathrm{cl}_{M\backslash P_n}(P_1 \cup P_2 \cup \dots \cup P_{i-1} \cup P_{i+1} \cup \dots \cup P_{n-1})$ and therefore x does not block $P_i \cup P_n$ in M, contradicting the assumption that $\mathbf{P}$ is well blocked by x in M. Therefore every member of $(P_1 - \{b_1\}, P_2, \dots, P_{n-1})$ in $M'\backslash x$ is indeed blocked by x in M'.

We now have an anemone in M'. But, if P and P' are petals of this flower, then $\sqcap(P, P') = 1$. It follows that the flower is spike-like, that is, M' is a spike with cotip x. Thus M'/x is a spike. But $\{a_1, a_1'\}$ is a clonal pair, so $M'/x \cong \Lambda_{n-1}$ by Lemma 9.9. As $n - 1 \geqslant q$, the claim follows. □

Let $\mathcal{P}$ and $\mathcal{Q}$ be maximal sets of petals of $\mathbf{P}$ such that $x \in \mathrm{cl}(E(M) - \bigcup_{P\in\mathcal{P}}(P))$ and $x \in \mathrm{cl}(E(M) - \bigcup_{Q\in\mathcal{Q}}(Q))$. If $\mathcal{P} \cap \mathcal{Q} \neq \emptyset$, then, by Lemma 3.31, we deduce that $x \in \mathrm{cl}(E(M) - (\bigcup_{P\in\mathcal{P}}(P) \cup \bigcup_{Q\in\mathcal{Q}}(Q)))$ and we have contradicted the assumption that $\mathcal{P}$ and $\mathcal{Q}$ were maximal with the given property. Observe that $x \notin \mathrm{cl}(E(M) - (P_1 \cup \{x\}))$, but, if $i \in \{2, 3, \dots, n\}$, then $x \in \mathrm{cl}(E(M) - (P_i \cup \{x\}))$. This establishes the next claim.

9.35.2. *There is a partition $(\{P_1\}, \mathcal{P}_2, \dots, \mathcal{P}_s)$ of the petals of $\mathbf{P}$ such that, for all $i \in \{2, 3, \dots, s\}$, the set $\mathcal{P}_i$ is a maximal collection of petals with the property that $x \in \mathrm{cl}_M(E(M) - \bigcup_{P\in\mathcal{P}_i}(P))$.*

Consider the partition given by 9.35.2.

9.35.3. *If $s \geqslant q + 1$, then M has a Λ_q-minor.*

Subproof. Up to labels we may assume that $(P_2, P_3, \dots, P_s)$ is a transversal of $(\mathcal{P}_2, \mathcal{P}_3, \dots, \mathcal{P}_s)$. Let $M' = M\backslash\{a_{s+1}, a_{s+2}, \dots, a_n\}/\{b_{s+1}, b_{s+2}, \dots, b_n\}$. Then $(P_1, P_2, \dots, P_s)$ is a spike-like flower in $M'\backslash x$. Elementary rank calculations show that $(P_1, P_2, \dots, P_s)$ is well blocked by x in M' and that $\{a_1, a_1'\}$ is a series pair in $M'\backslash x$ that is blocked by x. By 9.35.1, M' has a Λ_{s-1}-minor and the sublemma follows. □

9.35.4. *If $i \in \{2, 3, \dots, s\}$ and $|\mathcal{P}_i| \geqslant q - 1$, then M has a Λ_q-minor.*

Subproof. Say $|\mathcal{P}_i| = l$. Up to labels we may assume that $\mathcal{P}_i = P_2 \cup P_3 \cup \dots \cup P_{l+1}$. Let $P_1' = P_1 \cup P_{l+2} \cup P_{l+3} \cup \dots \cup P_n$. Observe that $\mathbf{P}' = (P_1', P_2, \dots, P_{l+1})$ is a spike-like flower in M and that $\{a_1, a_1'\}$ is a $\mathbf{P}'$-strong clonal pair in P_1'. It follows easily from Tutte's Linking Lemma and the fact that $\{a_1, a_1'\}$ is a clonal pair, that M has a 3-connected minor on $\{a_1, a_1'\} \cup P_2 \cup P_3 \cup \dots \cup P_{l+1}$ such that $(\{a_1, a_1'\}, P_2, \dots, P_{l+1})$ is a spike in M''. Now $\{a_1, a_1'\}$ is a clonal pair and $M'' \cong \Lambda_q$ by Lemma 9.9. □

As $n \geqslant (q+1)(q-1)$, either 9.35.3 or 9.35.4 applies and the lemma follows. □

Corollary 9.36. *Let M be a 3-connected matroid with an element x such that $M\backslash x$ is 3-connected up to a nonempty set of M-clonal series pairs and that* $\mathrm{co}(M\backslash x)$ *is a rank-n spike. If $n \geqslant q^2 - 1$, then M has a Λ_q-minor.*

Proof. The goal is to contract elements from M that are in series pairs of $M\backslash x$ keeping a matroid satisfying the hypotheses of the lemma until we have only one series pair remaining, in which case we can apply Lemma 9.35. No difficulties arise with this strategy unless we have two clonal pairs $\{s_i, s_i'\}$ and $\{s_j, s_j'\}$ such that $\{s_j, s_j'\}$ is not blocked in M/s_i, that is, if $x \in \mathrm{cl}_{M/s_i}(\{s_j, s_j'\})$. In this case $s_i \in \mathrm{cl}_M(\{s_j, s_j', x\})$ and, as $\{s_i, s_i'\}$ is a clonal pair, $\{s_i, s_i'\} \subseteq \mathrm{cl}_M\{s_j, s_j', x\}$. Therefore $r_M(\{s_i, s_i', s_j, s_j'\}) = 3$. But this means that $\{s_i', s_j'\}$ is dependent in $M/s_i, s_j$ contradicting the fact that $\mathrm{co}(M)$ is 3-connected. It follows that our strategy is fine and that the corollary indeed follows from Lemma 9.35. □

Finally we observe that Lemma 9.34 is an almost immediate consequence of Corollary 9.36.

6. The $M^*(K_{3,n})$ case

We now come to the final case. This seems to be more difficult than the previous cases—although we may be missing an easy argument. Rather than solve it independently of the other cases, we combine it with the fact that the previous cases have been resolved. We begin by developing some specialist terminology.

Let M be a matroid and l be an integer. A set X of elements of M is a *star* of *order l* if:

(i) X is closed;
(ii) X is the union of l 2-element series classes of M; and
(iii) $X \cap E(M)$ is a parallel set in $\mathrm{co}(M)$.

Note that we have also used "star" to describe a graph that is a certain type of tree. However no danger of confusion arises as here the adjective "star" qualifies structures in matroids, not graphs.

Evidently a star is *maximal* if it is not contained in a larger star. Insisting that X be closed is just a non-triviality condition that we do not really need, but it helps to make certain arguments cleaner. The matroid M is 3-connected *up to stars*, if whenever (X, Y) is a 2-separation of M, either X or Y is a star.

A matroid M is a *star extended* $M^*(K_{3,t})$ if M is 3-connected up to stars, and $\mathrm{si}(\mathrm{co}(M)) \cong M^*(K_{3,t})$. Let M be a star extended $M^*(K_{3,t})$. Then there is a flower $\mathbf{P} = (P_1, P_2, \dots, P_t)$ such that $(P_1 \cap E(\mathrm{si}(\mathrm{co}(M))), P_2 \cap E(\mathrm{si}(\mathrm{co}(M))), \dots, P_t \cap E(\mathrm{si}(\mathrm{co}(M))))$ is the canonical flower associated with $M^*(K_{3,t})$ in $\mathrm{si}(\mathrm{co}(M))$. We call this flower the copaddle *associated* with M.

Let M be a star extended $M^*(K_{3,t})$, then M is an (n,l)*-copaddle* if $n \leqslant t$ and the flower $\mathbf{P} = (P_1, P_2, \dots, P_t)$ associated with M has the property that P_i contains a unique star of order l for each $i \in \{1, 2, \dots, n\}$.

Let M be a connected matroid. The triple $(M, \mathbf{P}, J)$ is a *blocked* (n,l)-copaddle, with *blocking set* J and *associated flower* $\mathbf{P} = (P_1, P_2, \dots, P_t)$ if J is a coindependent set of M such that $M\backslash J$ is an (n,l)-copaddle with associated flower $\mathbf{P}$ and the following properties hold.

(i) Every series pair of $M\backslash J$ in $P_1 \cup P_2 \cup \cdots \cup P_n$ is blocked in M.
(ii) All series pairs of $M\backslash J$ are M-clonal.

If we say that a matroid M is a blocked (n,l)-copaddle, we mean that there exist $\mathbf{P}$ and J such that $(M, \mathbf{P}, J)$ is a blocked (n,l)-copaddle.

Let M be a blocked (n,l)-copaddle with blocking set J and associated copaddle $\mathbf{P} = (P_1, P_2, \dots, P_t)$. If we say that a set $S \subseteq P_1 \cup P_2 \cup \cdots \cup P_t$ is a series pair or a star of $\mathbf{P}$, we will always mean that it has this property in $M\backslash J$ and if we say that an element x of J blocks S, we will always mean that this set is blocked by x in $M\backslash(J - \{x\})$.

For the time being we focus on blocked $(n,1)$-copaddles so that our stars are series pairs. The more general structures will eventually play a role. We begin by showing that, given a sufficiently large 3-connected clonal matroid, we can always either achieve a desirable outcome or produce a blocked $(n, 1)$-copaddle.

Lemma 9.37. *Let M be a 3-connected clonal matroid in $\mathcal{E}(q)$. Then there is a function $f_{9.37}(s, m, q)$ such that, if $|E(M)| \geqslant f_{9.37}(s, m, q)$, then at least one of the following holds.*

(i) *M has a clonal Δ_m-minor.*
(ii) *Either M or M^* has a minor N with an associated triple $(N, \mathbf{P}, J)$ such that $(N, \mathbf{P}, J)$ is a blocked $(s, 1)$-copaddle. Moreover, all of the series pairs of $N\backslash J$ are M-clonal.*

Proof. Let $\mu = \max\{3s, f_{9.10}(m, q), f_{9.29}(m, q), f_{9.34}(m, q)\}$. Let $f_{9.37}(m, q) = f_{9.7}(\mu, q)$.

By Lemma 9.7, M or M^* has a 3-connected minor N with a coindependent set J such that one of the following holds.

(i) $N\backslash J$ is 3-connected up to a set of at least μ series pairs.
(ii) Each series pair of $N\backslash J$ is M-clonal.
(iii) $\mathrm{co}(N\backslash J)$ is either a spike, a whorl, or for some integer t, is isomorphic to $M(K_{3,t})$ or $M^*(K_{3,t})$.

If $\mathrm{co}(N\backslash J)$ is a whorl, then, by Lemma 9.10, M has a clonal Δ_m-minor and the lemma holds. If $\mathrm{co}(N\backslash J)$ is a spike or is isomorphic to $M(K_{3,t})$, then, by Lemma 9.34 or Lemma 9.29, we contradict the assumption that $M \in \mathcal{E}(q)$. Hence $\mathrm{co}(N\backslash J) \cong M^*(K_{3,t})$. In this case it is easily seen that N has a minor N' that has a representation as a blocked $(\mu/3, 1)$-copaddle where all of the series pairs in the copaddle associated with N' are M-clonal. By the definition of μ, we have $\mu/3 \geqslant s$, and the lemma follows. $\square$

We now define a more highly structured type of blocked $(n, 1)$-copaddle. Let $(M, \mathbf{P}, J)$ be a blocked $(n, 1)$-copaddle where $\mathbf{P} = (P_1, P_2, \ldots, P_t)$. For all $i \in \{1, 2, \ldots, n\}$ there is a unique series pair in P_i which we denote by $\{a_i, a_i'\}$. The triple $(M, \mathbf{P}, J)$ is an *n-blockage* if there is a function $\phi : \{1, 2, \ldots, n\} \to J$ such that, for all $i \in \{1, 2, \ldots, n\}$, the element $\phi(i)$ blocks $\{a_i, a_i'\}$ and $\phi(i)$ blocks $P_{n+1} \cup P_{n+2} \cup \cdots \cup P_t$ but $\phi(i)$ does not block P_j for any $j \in \{1, 2, \ldots, i-1, i+1, \ldots, n\}$. We call the function ϕ the *blocking function* of $(M, \mathbf{P}, J)$. The remainder of this section is devoted to producing a large m-blockage from a very large blocked $(n, 1)$-copaddle.

Theorem 9.38. *Let $(M, \mathbf{P}, J)$ be a blocked $(n, 1)$-copaddle in $\mathcal{E}(q)$. Then there is a function $f_{9.38}(m, q)$ such that, if $n \geqslant f_{9.38}(m, q)$, then there is a permutation $\mathbf{Q}$ of the petals of $\mathbf{P}$ and a partition (J', J'') of J such that $(M\backslash J', \mathbf{Q}, J'')$ is an m-blockage.*

This is a technical result and probably does not merit the honour of being called a theorem, but doing so helps clarify the organisation of this section. Before proving Theorem 9.38, we note a corollary of it.

Corollary 9.39. *Let M be a 3-connected clonal matroid in $\mathcal{E}(q)$. Then there is a function $f_{9.39}(s, m, q)$ such that, if $|E(M)| \geqslant f_{9.39}(s, m, q)$, then either*

(i) *M has a clonal Δ_m-minor, or*
(ii) *M or M^* has a minor M' with an associated triple $(M', \mathbf{P}, J)$ such that $(M', \mathbf{P}, J)$ is an s-blockage. Moreover, all of the series pairs in $M'\backslash J$ are M-clonal.*

Proof. Let $f_{9.39}(s, m, q) = f_{9.37}(f_{9.38}(s, q), m, q)$. With this function, the corollary follows immediately from Lemma 9.37 and Theorem 9.38. $\square$

Let $\mathcal{P}$ be a collection of subsets of the ground set of a matroid. To simplify life we make some convenient abbreviations. We will often say $\mathrm{cl}(\mathcal{P})$ to refer to $\mathrm{cl}(\bigcup_{P\in\mathcal{P}} P)$. This convention is extended to $r(\mathcal{P})$, $\lambda(\mathcal{P})$, and so on.

Lemma 9.40. *Let* $\mathbf{P}$ *be a copaddle in the matroid* M *and let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be disjoint subsets of petals of* $\mathbf{P}$ *whose union does not contain all the petals of* $\mathbf{P}$*. Then* $r(\mathcal{P}) + r(\mathcal{Q}) = r(\mathcal{P} \cup \mathcal{Q})$*.*

Part (i) of the next lemma is true for any petal of a copaddle that has no elements in the coguts of a petal. This is clearly true in our case.

Lemma 9.41. *Let* M *be an* (n, l)*-copaddle with associated flower* $(P_1, P_2, \dots, P_t)$*. Then the following hold.*

(i) $M|P_i$ *is connected for all* $i \in \{1, 2, \dots, t\}$*.*
(ii) *If* $i \in \{1, 2, \dots, n\}$ *then* M/P_i *is an* $(n-1, l)$*-copaddle with associated flower* $(P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_t)$*.*
(iii) *If* $i \in \{n+1, n+2, \dots, t\}$*, then* M/P_i *is an* (n, l)*-copaddle with associated flower* $(P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_t)$*.*

The next lemma is also elementary.

Lemma 9.42. *Let* M *be a matroid with an element* x *such that* $M\backslash x$ *is connected with a copaddle* $(P_1, P_2, \dots, P_n)$*. Say that* x *blocks* P_2 *in* M*, but not in* M/P_1*. Then the following hold.*

(i) $x \in \mathrm{cl}(P_1 \cup P_2)$*.*
(ii) *If* $\mathcal{P}$ *is any set of petals that contains* P_1 *but not* P_2*, then* x *blocks* $\bigcup_{P \in \mathcal{P}}(P)$*.*

The straightforward properties described above enable us to prove the next useful lemma.

Lemma 9.43. *Let* M *be a matroid with an element* x *such that* $M\backslash x$ *is an* (n, l)*-copaddle with associated flower* $\mathbf{P} = (P_1, P_2, \dots, P_t)$*. Let* $\{p, p'\} \subseteq P_1$ *be an* M*-clonal series pair of* $M\backslash x$ *and let* $\mathbf{P}' = (P_1 - \{p'\}, P_2, \dots, P_t)$*. Let* $(\mathcal{P}, \mathcal{Q})$ *be a partition of the petals of* $\mathbf{P}'$ *where* $|\mathcal{P}| \geqslant 2$ *and* $(P_1 - \{p'\}) \in \mathcal{P}$*. Then* x *blocks* $(\mathcal{P} \cup \{p'\}, \mathcal{Q})$ *in* M *if and only if* x *blocks* $(\mathcal{P}, \mathcal{Q})$ *in* M/p'*.*

Proof. For the not completely trivial direction assume that x blocks $(\mathcal{P} \cup \{p'\}, \mathcal{Q})$ in M and, for a contradiction, that x does not block $(\mathcal{P}, \mathcal{Q})$ in M/p'. In this case it is clear that $x \in \mathrm{cl}_{M/p'}(\mathcal{Q})$, so that $x \in \mathrm{cl}_M(\mathcal{Q} \cup \{p'\})$. But $x \notin \mathrm{cl}_M(\mathcal{Q})$. Hence $p' \in \mathrm{cl}_M(\mathcal{Q} \cup \{x\})$. By Lemma 9.41(i), $p' \in \mathrm{cl}_M(P_1 - \{p'\})$. By Lemma 9.40, $\sqcap_M(\mathcal{Q} \cup \{x\}, P_1 - \{p'\}) \leqslant 1$. Now, by Lemma 5.11, p' is fixed in M, contradicting the fact that the pair $\{p, p'\}$ is M-clonal. $\square$

The previous four lemmas will be used freely without reference throughout this section.

The well-blocked case Let M be a matroid with an element x such that $M\backslash x$ is connected with a copaddle $\mathbf{P}$. Then $\mathbf{P}$ is *well blocked* by x if, $P_i \cup P_j$ is blocked in M for all distinct petals P_i and P_j of $\mathbf{P}$. This situation corresponds to a straightforward case in our analysis.

Lemma 9.44. *Let* M *be a connected matroid with an element* x *such that* $M\backslash x$ *is* 3*-connected up to* M*-clonal series pairs and* $\mathrm{co}(M\backslash x) \cong M^*(K_{3,n})$*. Assume that the canonical copaddle associated with* $M\backslash x$ *is well blocked by* x*. Then there is a function* $f_{9.44}(q)$ *such that, if* $n \geqslant f_{9.44}(q)$*, the matroid* M *has a* $U_{q,q+2}$*-minor.*

Lemma 9.44 is a consequence of the next lemma. We prove the dual version as it seems intuitively clearer.

Lemma 9.45. *Let* M *be a matroid with an element* x *such that* $M/x \cong M(K_{3,n})$ *for some* $n \geqslant 4$*. Assume that* x *coblocks every pair of triads of* $M(K_{3,n})$*. Then there is a function* $f_{9.45}(q)$ *such that, if* $n \geqslant f_{9.45}(q)$*, then* M *has a* $U_{2,q+2}$*-minor.*

Proof. Let $\mathbf{P} = (P_1, P_2, \dots, P_n)$ be the canonical maximal flower in $M(K_{3,n})$ and let $A = \{a_1, a_2, \dots, a_n\}$ be a transversal of the petals of $\mathbf{P}$. Let $P_i' = P_i - \{a_i\}$. By assumption x coblocks $P_i \cup P_j$.

Therefore $r_M(P_i \cup P_j) = 5$ for all distinct $i, j \in \{1, 2, \dots, n\}$. Consider $M/x/A$. Note that $r(M/x/A) = 2$ and, for all $i \in \{1, 2, \dots, n\}$, we have $r_{M/x/A}(P'_i) = r_{M/A}(P'_i) = 2$. As x coblocks $P_i \cup P_j$, we see that $x \in \mathrm{cl}_M(P_i \cup P_j)$, so $x \in \mathrm{cl}_{M/A}(P'_i \cup P'_j)$. Hence, for all distinct $i, j \in \{1, 2, \dots, n\}$, we have $r_{M/A}(P'_i \cup P'_j) = 3$. Thus P'_i and P'_j span distinct lines of M/A. Hence M/A has at least n distinct lines. By Lemma 7.6, a rank-3 matroid with no $U_{2,q+2}$-minor has at most $q^2 + q + 1$ parallel classes and hence at most $\binom{q^2+q+1}{2}$ lines. The lemma holds by letting $f_{9.45}(q) = \binom{q^2+q+1}{2}$. □

Proof of Lemma 9.44. Let $f_{9.44}(q) = f_{9.45}(q)$ and assume that $n \geqslant f_{9.44}(q)$. It follows from Lemmas 9.42 and 9.43 that the flower associated with $M^*(K_{3,t})$ in $\mathrm{co}(M\backslash x)$ is well blocked by x. Now M has a $U_{q,q+2}$-minor by the dual of Lemma 9.45. □

For a blocked $(n, 1)$-copaddle $(M, \mathbf{P}, J)$ we would not expect to find elements giving us the easy win of Lemma 9.44. Nonetheless, for $x \in J$, we can find a partition which effectively gives us a concatenation of $\mathbf{P}$ which is well blocked by x.

Lemma 9.46. *Let M be a connected matroid with an element x such that $M\backslash x$ is connected with a copaddle $\mathbf{P} = (P_1, P_2, \dots, P_t)$. Then there is a partition $(\mathcal{Q}, \mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_s)$ of the petals of $\mathbf{P}$, where possibly $\mathcal{Q} = \emptyset$ such that the following property holds: if $\mathcal{P}$ is a set of petals of $\mathbf{P}$ such that $x \in \mathrm{cl}(\mathcal{P})$, then $(\{P_1, P_2, \dots, P_t\} - \mathcal{P}) \subseteq \mathcal{P}_i$ for some $i \in \{1, 2, \dots, s\}$.*

Proof. Assume that $\{\mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_s\}$ are the distinct minimal subsets of petals whose closure spans x. For $i \in \{1, 2, \dots, s\}$, let $\mathcal{P}_i = \{P_1, P_2, \dots, P_t\} - \mathcal{Q}_i$, and let $\mathcal{Q} = \mathcal{Q}_1 \cap \mathcal{Q}_2 \cup \dots \cup \mathcal{Q}_s$. By Lemma 3.31, if i and j are distinct elements of $\{1, 2, \dots, s\}$, then $\mathcal{Q}_i \cup \mathcal{Q}_j$ contains all of the petals of $\mathbf{P}$, that is, $\mathcal{P}_i$ and $\mathcal{P}_j$ are disjoint. It follows from this fact that $(\mathcal{Q}, \mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_s)$ satisfies the lemma. □

Note that, in the partition given by Lemma 9.46, $\mathcal{Q}$ consists of the petals of $\mathbf{P}$ that are blocked by x. Let $(M, \mathbf{P}, J)$ be a blocked $(n, 1)$-copaddle. For $x \in J$, let $\pi(x)$ denote the partition of the petals of $\mathbf{P}$ given by Lemma 9.46.

Lemma 9.47. *Let $(M, \mathbf{P}, J)$ be a blocked $(n, 1)$-copaddle where $M \in \mathcal{E}(q)$, and let x be an element of J with $\pi(x) = (\mathcal{Q}, \mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_k)$. Then $|\mathcal{Q}| + k \leqslant f_{9.44}(q)$.*

Proof. Let $\{Q_1, Q_2, \dots, Q_k\}$ be a transversal of $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_k)$, and let Q be the union of the set of petals of $\mathbf{P}$ not in $\mathcal{Q} \cup \{Q_1, Q_2, \dots, Q_k\}$. Then $\mathcal{Q} \cup \{Q_1, Q_2, \dots, Q_k\}$ is the set of petals of a copaddle in $M\backslash J/Q$. It is easily checked that this flower is well blocked by x. The lemma now follows from Lemma 9.44. □

As immediate consequences of Lemma 9.47 we get the next two results.

Corollary 9.48. *Let $(M, \mathbf{P}, J)$ be a blocked $(n, 1)$-copaddle where $M \in \mathcal{E}(q)$. If $x \in J$, then x blocks at most $f_{9.44}(q)$ petals of $\mathbf{P}$.*

Corollary 9.49. *Let $(M, \mathbf{P}, J)$ be a blocked $(n, 1)$-copaddle where $M \in \mathcal{E}(q)$, and let $\mathcal{P}$ be a subset of $\mathbf{P}$ with $|\mathcal{P}| \geqslant 2 f_{9.44}(q)$. If $x \in J$, then there is a subset $\mathcal{Q}$ of $\mathcal{P}$ such that:*

(i) *$|\mathcal{Q}| \geqslant |\mathcal{P}|/f_{9.44}(q)$, and*
(ii) *x is spanned by the union of the petals of $\mathcal{P}$ that are not in $\mathcal{Q}$.*

Good 3-paths Let $(M, \mathbf{P}, J)$ be a blocked $(n, 1)$-copaddle where $\mathbf{P} = (P_1, P_2, \dots, P_t)$. A *good 3-path of length s* in $(M, \mathbf{P}, J)$ is a partition $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_s)$ of $\{P_{n+1}, P_{n+2}, \dots, P_t\}$, and a subset $\{x_1, x_2, \dots, x_t\}$ of J such that

(i) $x_i \in \mathrm{cl}(\mathcal{P}_i)$ for all $i \in \{1, 2, \dots, s\}$, and
(ii) x_i blocks a series pair of a member of $\mathcal{P}_i$ for all $i \in \{1, 2, \dots, s\}$.

It is not hard to see that having a sufficiently long good 3-path is a clear win.

Lemma 9.50. *Let $(M, \mathbf{P}, J)$ be a blocked $(n, 1)$-copaddle. Then there is a function $f_{9.50}(q)$ such that, if M has a good 3-path of length s, where $s > f_{9.50}(q)$, then $M \notin \mathcal{E}(q)$.*

Proof. Let $f_{9.50}(q) = f_{7.36}(3, q)$. Assume that $s > f_{9.50}(q)$. Assume that the partition $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_s)$ of $\{P_{n+1}, P_{n+2}, \dots, P_t\}$, and the subset $\{x_1, x_2, \dots, x_t\}$ of J define a good 3-path in $(M, \mathbf{P}, J)$. Let Q_1 be the union of the sets in $\{P_1, P_2, \dots, P_n, \{x_1\}\} \cup \mathcal{P}_1$, and for $i \in \{2, 3, \dots, s\}$, let $Q_i = \bigcup_{P \in \mathcal{P}_i}(P) \cup \{x_i\}$. Then $\mathbf{Q} = (Q_1, Q_2, \dots, Q_s)$ is a path of 3-separations in $M \backslash (J - \{x_1, x_2, \dots, x_s\})$, each step of which contains a $\mathbf{Q}$-strong clonal pair. Assume that $\mathbf{Q}$ displays a swirl-like flower (R_1, R_2, R_3). Let $E' = E(\mathrm{co}(M \backslash J))$. By Lemma 3.19 $(R_1 \cap E', R_2 \cap E', R_3 \cap E')$ is a swirl-like flower in $\mathrm{co}(M \backslash J)$. But, also by Lemma 3.19 $(R_1 \cap E', R_2 \cap E', R_3 \cap E')$ is a copaddle in $\mathrm{co}(M \backslash J)$. Thus $\mathbf{Q}$ does not display a 3-petal swirl-like flower. By Lemma 7.36, $M \notin \mathcal{E}(q)$. $\square$

Near *n*-blockages Let $(M, \mathbf{P}, J)$ be a blocked $(n, 1)$-copaddle. For $i \in \{1, 2, \dots, n\}$ there is an element of J that blocks the series pair $\{a_i, a_i'\}$ in P_i. Of course there may be more than one such element. Let $\phi : \{1, 2, \dots, n\} \to J$ be a function with the property that $\phi(i)$ blocks $\{a_i, a_i'\}$. If $(M, \mathbf{P}, J)$ is endowed with such a function we will say that $(M, \mathbf{P}, J)$ is a blocked $(n, 1)$-copaddle with *blocking function* ϕ. Evidently deleting the members of J that are not in the range of ϕ preserves the property of being a blocked $(n, 1)$-copaddle.

Let $(M, \mathbf{P}, J)$ be a blocked $(n, 1)$-copaddle with blocking function ϕ where $\mathbf{P} = (P_1, P_2, \dots, P_t)$. Then $(M, \mathbf{P}, J)$ is a *near* n-blockage if $P_{n+1} \cup P_{n+2} \cup \dots \cup P_t$ blocks $\phi(i)$ for all $i \in \{1, 2, \dots, n\}$.

The next task is to find a near n-blockage. Note that, if $(M, \mathbf{P}, J)$ is a blocked $(n, 1)$-copaddle of order n, then it is also a blocked $(n, 1)$-copaddle of any order $m \leqslant n$.

Lemma 9.51. *Let $(M, \mathbf{P}, J)$ be a blocked $(n, 1)$-copaddle with blocking function ϕ, where $\mathbf{P} = (P_1, P_2, \dots, P_t)$. Assume that $(M, \mathbf{P}, J)$ has a good 3-path of length s and that $(M, \mathbf{P}, J)$ is not a near n-blockage. If $n \geqslant m f_{9.44}(q)$, then there is an ordering of the elements of $\{P_1, P_2, \dots, P_n\}$ such that, relative to this ordering, $(M, \mathbf{P}, J)$ is a blocked $(m, 1)$-copaddle with a good 3-path of length $s + 1$.*

Proof. Assume that $n \geqslant m f_{9.44}(q)$. Assume that the partition $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_s)$ of $\{P_{n+1}, \dots, P_t\}$ together with the subset $\{x_1, x_2, \dots, x_s\}$ of J give a good 3-path of length s. As $(M, \mathbf{P}, J)$ is not a near n-blockage, there is an $i \in \{1, 2, \dots, t\}$ such that $\phi(i)$ is not blocked by $P_{n+1} \cup P_{n+2} \cup \dots \cup P_t$. By Corollary 9.49, there is a subset $\mathcal{P}$ of $\{P_1, P_2, \dots, P_n\}$ of size m such that x is in the closure of the union of the petals of $\mathbf{P}$ that are not in $\mathcal{P}$. Up to labels we may assume that this subset is $\{P_1, P_2, \dots, P_m\}$. As $\phi(i)$ blocks $\{a_i, a_i'\}$, we see that $i \notin \{1, 2, \dots, m\}$. Let $\mathcal{P}_{s+1} = \{P_{m+1}, P_{m+2}, \dots, P_n\}$ and let $x_{s+1} = \phi(i)$. It is now clear that $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_{s+1})$ and $(x_1, x_2, \dots, x_{s+1})$ define a good 3-path of length $s + 1$ in the blocked $(m, 1)$-copaddle $(M, \mathbf{P}, J)$. $\square$

Corollary 9.52. *Let $(M, \mathbf{P}, J)$ be a blocked $(n, 1)$-copaddle with blocking function ϕ where $M \in \mathcal{E}(q)$. Then there is a function $f_{9.52}(m, q)$ such that, if $n \geqslant f_{9.52}(m, q)$, then there is an ordering of the petals of $\mathbf{P}$ such that, with respect to this ordering, $(M, \mathbf{P}, J)$ is a near m-blockage whose blocking function is ϕ restricted to $\{1, 2, \dots, m\}$.*

Proof. Let $f_{9.52}(m, q) = m f_{9.44}(q)^{f_{9.50}(q)}$. With this function, the corollary follows immediately from Lemmas 9.50 and 9.51. $\square$

Cleaning a near *n*-blockage At last we are able to achieve the goal of this section.

Lemma 9.53. *Let $(M, \mathbf{P}, J)$ be a near n-blockage with blocking function ϕ, where $\mathbf{P} = (P_1, P_2, \dots, P_t)$. Assume that $M \in \mathcal{E}(q)$. Then there is a function $f_{9.53}(m, q)$ such that if $n \geqslant f_{9.53}(m, q)$, then there is an ordering of the petals of $\mathbf{P}$ such that, with respect to this ordering, $(M, \mathbf{P}, J)$ is an m-blockage.*

Proof. Let $f_{9.53}(m,q) = f_{9.12}(f_{9.44}(q), m)$.

Define a hypergraph whose vertices are $\{P_1, P_2, \dots, P_n\}$ and whose edges are $\{\phi(1), \phi(2), \dots, \phi(n)\}$ as follows: for $i \in \{1, 2, \dots, n\}$, the vertices incident with $\phi(i)$ are the members of $\{P_1, P_2, \dots, P_n\}$ that ϕ blocks. It follows from Corollary 9.48 that each edge is incident with at most $f_{9.44}(q)$ vertices. It follows from the definition of ϕ that P_i is incident with $\phi(i)$ for all $i \in \{1, 2, \dots, n\}$. It now follows from Lemma 9.12 that, up to an appropriate permutation of $\{P_1, P_2, \dots, P_n\}$, we may assume, for all $i, j \in \{1, 2, \dots, m\}$, that $\phi(i)$ blocks P_j if and only if $i = j$. The lemma now follows by observing that the members of $\{\phi(i), \phi(2), \dots, \phi(m)\}$ have the properties required for an m-blockage. $\square$

Proof of Theorem 9.38. Let $f_{9.38}(m,q) = f_{9.52}(f_{9.53}(m,q), q)$. Let ϕ be a blocking function for $(M, \mathbf{P}, J)$. By Corollary 9.52, there is an ordering of the petals of $\mathbf{P}$ such that, with respect to this ordering, $(M, \mathbf{P}, J)$ is a near $f_{9.52}(m,q)$-blockage whose blocking function is ϕ restricted to $\{1, 2, \dots, m\}$. Life is now good as we can observe that, by Lemma 9.53, there is an ordering of the petals of $\mathbf{P}$, relative to which, $(M, \mathbf{P}, J)$ is an m-blockage. $\square$

7. Building a blocked (n, l)-copaddle

In this section we prove that given a very large 3-connected clonal matroid in $\mathcal{E}(q)$ then we can either find a large free-swirl minor or we can build blocked (n, l)-copaddles, where l is large.

Theorem 9.54. *Let M be a 3-connected clonal matroid in $\mathcal{E}(q)$. Then there is a function $f_{9.54}(s, l, m, q)$ such that, if $|E(N)| \geqslant f_{9.54}(s, l, m, q)$, then at least one of the following holds.*

(i) *M has a clonal Δ_m-minor.*
(ii) *Either M or M^* has a minor N with an associated triple $(N, \mathbf{P}, J)$ such that $(N, \mathbf{P}, J)$ is a blocked (s, l)-copaddle where all of the series pairs of $N \backslash J$ are M-clonal.*

Of course we will need some preliminary lemmas. The next technical lemma is straightforward but crucial.

Lemma 9.55. *Let M be a connected matroid with an element x such that $M \backslash x$ is 3-connected up to stars and let S be a maximal star of $M \backslash x$. Assume that $x \in \mathrm{cl}(S) \cap \mathrm{cl}(E(M) - S)$ and that x is cofixed in $M \backslash S$. Let M' be a matroid obtained from M by cocloning x by x'. If $\{x, x'\}$ is not a series pair of M' then there is a partition (S_1, S_2) of S such that the following hold.*

(i) *$\{x, x'\}$ is a parallel pair in $M' \backslash S_1 / S_2$.*
(ii) *$M' \backslash S_1 / S_2 \backslash x' = M \backslash S$.*

Proof. Let $T = E(M) - (S \cup \{x\})$. We first observe,

9.55.1. *T is not coblocked by x'.*

Proof. Otherwise $x' \in \mathrm{cl}_{M'}(T)$ so that $\{x, x'\} \subseteq \mathrm{cl}_{M'}(T)$. This means that $\{x, x'\}$ is a coindependent clonal pair in $M' | (T \cup \{x, x'\})$. But $M' | (T \cup \{x, x'\}) / x' \cong M \backslash S$, and we have contradicted the fact that x is cofixed in $M \backslash S$. $\square$

Note that $\{x, x'\}$ is independent in M', as otherwise x is a loop of $M = M'/x$. Let $S' \subseteq S$ be a transversal of the series pairs of S and let $S'' = S - S'$. Each series pair in S is also a series pair in M' and it follows that $\{x, x'\}$ is independent in M'/S'. As S is a star of M, we have $r_{M'/S'/x'}(S'' \cup \{x\}) = 1$. Thus $r_{M'/S'}(S'' \cup \{x, x'\}) = 2$. If S'' consists of a single parallel set in the closure of T, then $\{x, x'\}$ is a series pair in M'/S' and hence, also in M'. Thus there is an element $s \in S''$ such that $s \notin \mathrm{cl}_{M'/S'}(T)$. As $\{x, x'\}$ is a clonal pair, $\{s, x, x'\}$ is a triangle. Let $S_1 = S'' - \{s\}$ and $S_2 = S' \cup \{s\}$. It is now clear that the lemma is satisfied by this choice of S_1 and S_2. $\square$

Let $(M, \mathbf{P}, J)$ be an n-blockage with blocking function ϕ, where $\mathbf{P} = (P_1, P_2, \ldots, P_t)$. Then $(M, \mathbf{P}, J)$ is a *minimal* n-blockage if $J = \{\phi(i)\colon\ i \in \{1, 2, \ldots, n\}\}$, and M is 3-connected, that is, if there are no unblocked series pairs in $P_{n+1} \cup P_{n+2} \cup \cdots \cup P_t$. Clearly there is no loss of generality in focussing on minimal n-blockages. Note that, for any n-blockage, ϕ is injective, so that for a minimal n-blockage ϕ is a bijection.

In what follows, in the n-blockage $(M, \mathbf{P}, J)$, we denote the series pair in the petal P_i of $\mathbf{P}$ by $\{a_i, a'_i\}$ for $i \in \{1, 2, \ldots, n\}$. We denote the other two elements of P_i by $\{b_i, c_i\}$. Let $B = \{b_1, b_2, \ldots, b_n\}$ and $C = \{c_1, c_2, \ldots, c_n\}$.

Lemma 9.56. *Let $(M, \mathbf{P}, J)$ be a minimal n-blockage with blocking function ϕ. Then the following hold.*

(i) *M/C is 3-connected.*
(ii) *If $i \in \{1, 2, \ldots, n\}$, then $(b_i, a_i, a'_i, \phi(i))$ is a maximal fan in M/C where $\{b_i, a_i, a'_i\}$ is a triangle and $\{a_i, a'_i, \phi(i)\}$ is a triad.*

Proof. Consider part (i). Evidently M/C is connected. Assume that M/C is not 3-connected. Let (X', Y') be a 2-separation of M/C. Let $(X, Y) = (X' - J, Y' - J)$. Assume that $|X| \leqslant 1$. As J is coindependent in M, and hence in M/C, we see that $r_{M/C}(X') \in \{1, 2\}$. In either case we deduce that there is an element $z \in E(M)$ and an $\{i, j\} \subseteq \{1, 2, \ldots, n\}$ such that $C \cup \{\phi(i), \phi(j), z\}$ contains a circuit. A routine check shows that in all possible cases we have a contradiction to the definition of minimal n-blockage. Therefore (X, Y) is a 2-separation in $M/C\backslash J$. It is straightforwardly verified that, up to labels, we may assume that either

(a) $X = \{a_1, a'_1\}$ or
(b) $X = \{\{a_1, a'_1, b_1\}, \{a_2, a'_2, b_2\}, \ldots, \{a_i, a'_i, b_i\}\}$ for some $i \in \{1, 2, \ldots, n\}$.

The verification of this is particularly routine if one considers the dual. Recall that $\phi(1)$ blocks both $\{a_1, a'_1\}$ and $P_{n+1} \cup P_{n+2} \cup \cdots \cup P_t$ in M. Thus, in either of the above cases, $\phi(1)$ blocks X in M/C. Therefore M/C is 3-connected so that (i) holds.

Consider (ii). Say $i \in \{1, 2, \ldots, n\}$. Evidently $\{a_i, a'_i, \phi(i)\}$ is a triad in M/C. As $\{a_i, a'_i, b_i, c_i\}$ is a circuit in M and M/C is 3-connected we see that $\{a_i, a'_i, b_i\}$ is a triangle in M/C. Thus (ii) holds. □

Let C and D be disjoint subsets of a matroid and let $N = M\backslash D/C$. Say $d \in D$ and $c \in C$. In the remainder of this section we will at times refer to the matroid $M\backslash(D - \{d\})/C$ as being obtained from N by *undeleting* d and the matroid $M\backslash D/(C - \{c\})$ as being obtained from M by *uncontracting* c.

Lemma 9.57. *Let M be a 3-connected clonal matroid in $\mathcal{E}(q)$ and let N be a 3-connected minor of M with the following properties.*

(i) *There is a set $B = \{b_1, b_2, \ldots, b_n\}$ of elements of N such that $N\backslash B$ is an (n, l)-copaddle, with associated flower $(P_1, P_2, \ldots, P_t)$.*
(ii) *The series pairs of $N\backslash B$ are M-clonal.*
(iii) *For $i \in \{1, 2, \ldots, n\}$, the maximal star $S_i \in P_i$ of $N\backslash B$ has the property that $b_i \in \mathrm{cl}_N(S_i) \cap \mathrm{cl}_N(E(N) - S_i)$.*

Then there is a function $f_{9.57}(m, q)$ such that, if $n \geqslant f_{9.57}(m, q)$, then M has an $(m, l+1)$-copaddle minor all of whose series pairs are M-clonal.

Proof. Let $f_{9.57}(m, q) = f_{9.29}(q) + m$. Assume that $n \geqslant f_{9.57}(m, q)$. As M is clonal, each member of $\{b_1, b_2, \ldots, b_n\}$ has a clone in M. For $i \in \{1, 2, \ldots, n\}$, denote the clone of b_i by b'_i. Let C and D be disjoint subsets of $E(M)$ such that $N = M\backslash D/C$.

Consider P_1. There are two cases to consider. For the first assume that $b'_1 \in D$. Let N' denote the matroid obtained by undeleting b'_1. As b_1 is fixed in N, we see that $\{b_1, b'_1\}$ is a parallel pair in N'. In this case let $P'_1 = (P_1 - S_1) \cup \{b'_1\}$ and let $N_1 = N'\backslash S_1$. Note that we have replaced the star S_1 of $N\backslash B$ by an M-clonal parallel pair at the basepoint of S_1.

For the second case, assume that $b_1' \in C$. In this case let N' be the matroid obtained from N by uncontracting b_1'. Assume that $\{b_1, b_1'\}$ is not a series pair in N'. Note that b_1 is cofixed in $N\backslash S_1$ so that Lemma 9.55 applies. By this lemma there is a partition (X, Y) of S_1 such that $\{b_1, b_1'\}$ is a parallel pair in $N'\backslash X/Y$ and $N'\backslash X/Y\backslash b_1' = N$. In this case, let $N_1 = N'\backslash X/Y$ and let $P_1' = (P_1 - S_1) \cup \{b_1'\}$. Observe that we have again replaced the star S_1 by an M-clonal parallel pair at the basepoint of the star. On the other hand, if $\{b_1, b_1'\}$ is a series pair, then let $N' = N_1$ and let $P_1' = P_1 \cup \{b_1'\}$. Observe that $S_1 \cup \{s_1, s_1'\}$ is a star of order $l+1$ in P_1'.

Repeat the above process for the remaining elements of B to obtain a matroid N_n with associated flower $\mathbf{P}' = (P_1', P_2', \dots, P_t')$, where, for $i \in \{n+1, n+2, \dots, t\}$ we have $P_i' = P_i$. Assume that for some $s \geqslant f_{9.29}(q)$ there are s members of $\{P_1', P_2', \dots, P_n'\}$ that contain M-clonal parallel pairs. Up to labels we may assume that these are $\{P_1', P_2', \dots, P_s'\}$. Observe that $(N_n/(P_{s+1}' \cup P_{s+2}' \cup \dots \cup P_t'))^*$ satisfies the hypotheses of Lemma 9.29, giving the contradiction that $N_n \notin \mathcal{E}(q)$. Otherwise, by the definition of $f_{9.57}(m, q)$, at least m petals in $\{P_1', P_2', \dots, P_n'\}$ have stars of order $l+1$. The series pairs in these stars are M-clonal. The lemma follows from these observations. $\square$

Lemma 9.58. *Let M be a 3-connected matroid in $\mathcal{E}(q)$ with a minor N that is an (n, l)-copaddle. Assume that all series pairs in N are M-clonal. Then there is a function $f_{9.58}(m, q)$ such that, if $n \geqslant f_{9.58}(m, q)$, then M has a minor M' that is a blocked (n, l)-copaddle $(M', \mathbf{P}', J)$. Moreover, all of the series pairs of $\mathbf{P}'$ are M-clonal.*

Proof. Let $f_{9.58}(m, q) = f_{9.29}(q) + m$. Assume that $n \geqslant f_{9.58}(m, q)$. Let $\mathbf{P} = (P_1, P_2, \dots, P_t)$ be the flower associated with N. For $i \in \{1, 2, \dots, n\}$, let S_i denote the maximal star in P_i. By Lemma 9.8, if $\{s_i, s_i'\}$ is a series pair in S_i, then $\{s_i, s_i'\}$ has a 1- or 2-element bridging sequence. We will say that P_i has *type*-1 if every series pair in S_i has a 1-element bridging sequence and has *type*-2 otherwise.

Note that the first element of a bridging sequence for a series pair consists of a delete element. It follows routinely that if there are m members of $\{P_1, P_2, \dots, P_n\}$ that are of type-1, then the lemma holds. Otherwise, we may assume that for some $\mu \geqslant f_{9.29}(q)$, there are μ members of P_i that are of type-2. Up to labels we may assume that $\{P_1, P_2, \dots, P_\mu\}$ are all of type-2. For $i \in \{1, 2, \dots, \mu\}$, let $\{s_i, s_i'\}$ be a series pair in S_i with a 2-element bridging sequence (t_i, u_i). Let $T = \{t_1, t_2, \dots, t_\mu\}$ and let N' be the matroid obtained from N by undeleting the elements of T. Say $t_i \in T$. As (t_i, u_i) is a bridging sequence and t_i is a delete element of this bridging sequence, $t_i \in \mathrm{cl}_{N'}(\{s_i, s_i'\})$. Note that this means that $\{s_i, s_i'\}$ is a series pair in $N'\backslash t_i$ and that $(P_1 \cup \{t_1\}, P_2 \cup \{t_2\}, \dots, P_n \cup \{t_n\}, P_{n+1}, \dots, P_t)$ is a copaddle in N'. As $\{s_i, s_i'\}$ is M-clonal, $\{t_i, s_i, s_i'\}$ is a triangle in N'. If $\{s_i, s_i'\}$ is a series pair in N', then u_i cannot coblock $\{s_i, s_i'\}$. Hence $\{s_i, s_i'\}$ is not a series pair in N' so that $\{t_i, s_i, s_i'\}$ is also a triad in N'.

For $i \in \{1, 2, \dots, n\}$ let $P_i' = P_i - (S_i - \{s_i, s_i'\})$. Let $N'' = N'/T\backslash((S_1 - \{s_1, s_1'\}) \cup \dots \cup (S_\mu - \{s_\mu, s_\mu'\}))/(P_{n+1} \cup P_{n+2} \cup \dots \cup P_t)$. Then $\mathbf{P}' = (P_1', P_2', \dots, P_n')$ is a copaddle in N''. Moreover, each petal of $\mathbf{P}'$ contains a single M-clonal parallel pair and $\mathrm{si}(N'') \cong M^*(K_{3,n})$. Thus $\mathbf{P}'$ satisfies the hypotheses of Lemma 9.29 in M^* and, by that lemma we obtain the contradiction that $M \notin \mathcal{E}(q)$. $\square$

Proof of Theorem 9.54. The proof is by induction on l. If $l > 1$, assume that $f_{9.54}(s, l-1, m, q)$ has been defined and, for induction, assume that the theorem holds with this definition of $f_{9.54}(s, l-1, m, q)$. Let $n_4 = f_{9.58}(s, q)$. Let $n_3 = f_{9.57}(n_4, q)$. Let $n_2 = n_3 + f_{9.29}(q)$. Let $n_1 = f_{9.37}(n_2, m, q)$ if $l = 1$ and let $n_1 = f_{9.54}(n_2, l-1, m, q)$ if $n > 1$. Finally let $f_{9.54}(s, l, m, q) = f_{9.39}(n_1, m, q)$.

Assume that $n \geqslant f_{9.54}(s, l, m, q)$. Assume that M does not have a clonal Δ_m-minor. By Corollary 9.39, M or M^* has a minor M_1 with an associated triple $(M_1, \mathbf{P}, J)$ such that $(M_1, \mathbf{P}, J)$ is an n_1-blockage. We now apply Lemma 9.56. Using the notation of that lemma we have a set C such that M_1/C is 3-connected, and every set in $\{\{a_1, a_1'\}, \{a_2, a_2'\}, \dots, \{a_{n_1}, a_{n_1}'\}\}$ is contained in both a triangle and a triad. By Lemma 9.2 M_1/C has a 3-connected minor N whose ground set is $\{a_1, a_1'\} \cup \{a_2, a_2'\} \cup \dots \cup \{a_{n_1}, a_{n_1}'\}$.

If $l = 1$ we apply Lemma 9.37 and if $l > 1$ we apply the induction assumption to deduce that either M_1/C or $(M_1/C)^*$ has a minor M_2 with an associated triple $(M_2, \mathbf{P}_2, J_2)$ such that $(M_2, \mathbf{P}_2, J_2)$ is a blocked (n_2, l)-copaddle. Moreover, all of the series pairs of $M_2\backslash J_2$ are M_1-clonal and hence are M-clonal. Let $M_1' = M_1/C$ if M_2 is a minor of M_1 and otherwise let $M_1' = (M_1/C)^*$. Every series pair of $M_2\backslash J_2$ is in a 4-element fan in M_1'. Thus every series pair in $M_2\backslash J_2$ is in a triangle of M_1'. By a slight abuse of notation assume that $\mathbf{P}_2 = (P_1, P_2, \dots, P_t)$. For $i \in \{1, 2, \dots, n_2\}$, let S_i denote the maximal star in P_i and let $\{s_i, s'_i\}$ be a series pair in S_i. Let $B' = \{b_1, b_2, \dots, b_{n_2}\}$ be elements of $E(M'_1) - E(M_2)$ such that $\{s_i, s'_i, b_i\}$ is a triangle in M'_1 for $i \in \{1, 2, \dots, n_2\}$.

Observe that the members of B' were all deleted from M'_1 as otherwise, for some $i \in \{1, 2, \dots, n_2\}$, we contradict the fact that $\{s_i, s'_i\}$ is independent in M_2. Let M_3 be the matroid obtained by undeleting the elements of B' from M_2. Observe that $\{s_i, s'_i, b_i\}$ is a triangle in M_3 for all $i \in \{1, 2, \dots, n_2\}$. Applying Lemma 9.29 and arguing just as in the proof of Lemma 9.58, we deduce, up to the labels of petals in $\{P_1, P_2, \dots, P_{n_2}\}$, that we may assume that $\{s_i, s'_i\}$ is a series pair in M_3 for all $i \in \{1, 2, \dots, n_3\}$. It follows from this that, for $i \in \{1, 2, \dots, n_3\}$, the element b_i is in the guts of a 2-separation of M_3 one side of which is S_i. Let $B = \{b_1, b_2, \dots, b_{n_3}\}$ and let M_4 be the matroid obtained by undeleting the elements of B from M_2.

We may now apply Lemma 9.57 and deduce that M or M^* has an $(n_4, l+1)$-copaddle minor all of whose series pairs are M-clonal. Finally, by Lemma 9.58, M or M^* has a minor M' with an associated triple $(M', \mathbf{P}', J')$ such that $(M', \mathbf{P}', J')$ is a blocked $(s, l+1)$-copaddle such that all series pair of $M' \backslash J'$ are M-clonal, completing the proof of the theorem. $\square$

8. Excluding a blocked (n, l)-copaddle

We now show that a matroid in $\mathcal{E}(q)$ cannot contain a blocked (n, l)-copaddle for large values of l. In striking contrast to Theorem 9.54, the bound here is quite modest. We will use the following theorem of Lemos and Oxley [17].

Theorem 9.59. *Let M be a connected matroid in which a largest circuit and cocircuit have c and c^* elements respectively. Then M has at most $\frac{1}{2}cc^*$ elements.*

Lemma 9.60. *Let M be a 3-connected matroid with a set S such that $M|S \cong M(K_{2,l})$. Assume that all of the series pairs in $M|S$ are M-clonal. Then there is a function $f_{9.60}(q)$ such that, if $l \geqslant f_{9.60}(q)$, then M has a Λ_q- or $U_{2,q+2}$-minor.*

Proof. Let $f_{9.60}(q) = (\frac{1}{2}q^2 + 1)\lfloor \frac{q+2}{2} \rfloor$. Assume that $l \geqslant f_{9.60}(q)$.

Say that the series pairs in $M|S$ are $\mathcal{S} = \{\{s_1, s'_1\}, \{s_2, s'_2\}, \dots, \{s_l, s'_l\}\}$. Then, by Lemma 9.2, M has a 3-connected minor M_1 on S. It follows from Lemma 2.10 that $\sqcap_{M_1}(\{s_i, s'_i\}, \{s_j, s'_j\}) \geqslant 1$ for all distinct i and j in $\{1, 2, \dots, l\}$. Consider the lines of M_1 that are spanned by members of $\mathcal{S}$. If any of these lines has at least $q + 2$ points, then the lemma holds, so we may assume that each line contains at most $\lfloor \frac{q+2}{2} \rfloor$ members of $\mathcal{S}$. Thus there is an integer n_1, where $n_1 + 1 \geqslant \frac{1}{2}q^2 + 1$ such that, up to labels, $\sqcap_{M_1}(\{s_i, s'_i\}, \{s_j, s'_j\}) = 1$ for all $i \in \{1, 2, \dots, n_1, n_1 + 1\}$. Let $M_2 = M_1|(\{s_1, s'_1\} \cup \{s_2, s'_2\} \cup \dots \cup \{s_{n_1+1}, s'_{n_1+1}\})$. Evidently M_2 is 3-connected.

9.60.1. *There is an $i \in \{1, 2, \dots, n_1 + 1\}$ such that M_2/s_i is 3-connected.*

Subproof. Assume that the sublemma fails. Then, for all $i \in \{1, 2, \dots, n_1 + 1\}$ there is a path $(X_i, \{s_i, s'_i\}, Y_i)$ of 3-separations in M_2, where $\{s_i, s'_i\} \subseteq \mathrm{cl}(X_i)$ and, of course, $\{s_i, s'_i\} \subseteq \mathrm{cl}(Y_i)$. Amongst all such paths assume that we have chosen s_i and the associated path so that $|X_i|$ is minimal. Note that we may assume that X_i is a union of clonal pairs. If $|X_i| = 2$, then $\sqcap_{M_1}(X_i, \{s_i, s'_i\}) = 2$ and X_i is a clonal pair so that we have contradicted the fact that $\sqcap_{M_1}(\{s_i, s'_i\}, \{s_j, s'_j\}) = 1$ for all $i \in \{1, 2, \dots, n_1, n_1 + 1\}$. Thus $|X_i| \geqslant 4$.

Choose $\{s_j, s'_j\} \subseteq X_i$ and let $(X_j, \{s_j, s'_j\}, Y_j)$ be an associated path of 3-separations where $\{s_i, s'_i\} \subseteq Y_j$. Assume that $X_j \cap Y_i \neq \emptyset$. An easy uncrossing argument shows that $\lambda(X_j \cap Y_i) = \lambda((X_j \cap Y_i) \cup \{s_j\}) = 2$. Thus s_j is in the guts or coguts of $(X_j \cap Y_i, E(M) - (X_j \cap Y_i))$. But $s_j \in \mathrm{cl}(Y_j)$ and $Y_j \subseteq E(M) - (X_j \cap Y_i)$, so that s_j, and hence also s'_j is in $\mathrm{cl}(X_j \cap Y_i)$. Therefore $\{s_i, s'_i, s_j, s'_j\} \subseteq \mathrm{cl}(Y_i)$ and $M|\{s_i, s'_i, s_j, s'_j\} \cong U_{2,4}$, contradicting the fact that $\sqcap_{M_1}(\{s_i, s'_i\}, \{s_j, s'_j\}) = 1$. We conclude that $X_j \cap Y_i$ is empty, so that X_j is a subset of X_i. As $\{s_j, s'_j\} \subseteq X_i - X_j$, we have contradicted the minimality assumption and the sublemma follows. $\square$

By 9.60.1 we may assume that M/s_{n_1+1} is 3-connected. Relabel s'_{n_1+1} by t and let $M_3 = M_2/s_{n_1+1}$. Observe that $\{s_i, s'_i, t\}$ is a triangle in M_3 for all $i \in \{1, 2, \dots, n_1\}$. Consider M_3/t. This matroid is connected, so $\mathrm{si}(M_3/t)$ is connected. By Theorem 9.59 $\mathrm{si}(M_3/t)$ either has a circuit of size at least q or a cocircuit of size at least q. Assume that C is such a set. Up to labels we may assume that $C = \{s_1, s_2, \dots, s_\mu\}$, where $\mu \geqslant q$.

Assume that C is a circuit of $\mathrm{si}(M_3/t)$ of size at least q. As $\{s_1, s'_1\}$ is a clonal pair in M_3, we have $s'_1 \in \mathrm{cl}_{M_3}(C)$ so that $t \in \mathrm{cl}_{M_3}(C)$. Thus $C \cup \{t\}$ is a circuit in M_3. It now follows that $M_3|(\{s_1, s'_1\} \cup \{s_2, s'_2\} \cup \dots \cup \{s_\mu, s'_\mu\}) \cong \Lambda_\mu$. Hence M has a Λ_q-minor.

Assume that C is a cocircuit of $\mathrm{si}(M_3/t)$ of size at least q. Then $C' = \{s_1, s'_1\} \cup \{s_2, s'_2\} \cup \dots \cup \{s_\mu, s'_\mu\}$ is a cocircuit of M_3. Let $H = E(M_3) - C'$ and let B be a basis of $M_3|H$ containing t. Consider $\mathrm{si}(M_3/(B - \{t\}))$. We may assume that $t \in \mathrm{si}(M_3/(B - \{t\}))$. Evidently $r(\mathrm{si}(M_3/(B - \{t\}))) = 2$. Moreover, if $i \in \{1, 2, \dots, \mu\}$, the pair $\{s_i, s'_i\}$ is independent in $\mathrm{si}(M_3/(B - \{t\}))$, as otherwise we have, at some stage, contracted a point on a line spanned by $\{s_i, s'_i\}$. As C' is a cocircuit, such a point must have been parallel with t and we contradict the fact that t is not a loop of $\mathrm{si}(M_3/(B - \{t\}))$. Hence $\mathrm{si}(M_3/(B - \{t\})) \cong U_{2,2\mu+1}$ and in this, the final case, we also conclude that M has a $U_{2,q+2}$-minor. $\square$

9. Summing up

At last we are able to achieve the goals of this chapter. We first consider Theorem 9.3.

Proof of Theorem 9.3. Let $f_{9.3}(m,q) = f_{9.54}(1, f_{9.60}(q), m, q)$. Assume that $n \geqslant f_{9.3}(m,q)$ and that the 3-connected matroid M has a partition into n pairwise-disjoint clonal pairs. Assume that M does not have a clonal Δ_m-minor. By Theorem 9.54, either M or M^* has a minor N with an associated triple $(N, \mathbf{P}, J)$ such that $(N, \mathbf{P}, J)$ is a blocked $(1, f_{9.60}(m,q))$-copaddle where all of the series pairs of $N \backslash J$ are M-clonal. Say $\mathbf{P} = (P_1, P_2, \dots, P_t)$. Let S be the maximal star of order $f_{9.60}(m,q)$ in P_1. Evidently $N|S \cong M(K_{2, f_{9.60}}(m,q))$. The hypotheses of Lemma 9.60 are satisfied and by that lemma $N \notin \mathcal{E}(q)$. This contradicts the assumption that $M \in \mathcal{E}(q)$. It follows from this contradiction that M has a clonal Δ_m-minor. $\square$

We conclude by observing again that Theorem 9.1 is an immediate corollary of Theorem 9.3 and Lemma 9.2.

Chapter 10. Strict paths of 2-separations

1. Introduction

Let M be a connected matroid. A *path of* 2*-separations* in M is a partition $\mathbf{P} = (P_0, P_1, \dots, P_l)$ of $E(M)$ into subsets such that $\lambda(P_0 \cup P_1 \cup \dots \cup P_i) = 1$ for all $i \in \{0, 1, \dots, l-1\}$. Adapting terminology from Chapter 7, we say, for $i \in \{0, 1, \dots, l\}$, that P_i is a *step* of $\mathbf{P}$, that P_0 and P_l are *end* steps and otherwise P_i is an *internal* step. A 2-separation (X, Y) of M is *displayed* by $\mathbf{P}$ if $(X, Y) = (P_0 \cup P_1 \cup \dots \cup P_i, P_{i+1} \cup P_{i+2} \cup \dots \cup P_l)$ for some $i \in \{0, 1, \dots, l-1\}$. Let $\mathbf{Q}$ be a path of 2-separations in M. Then $\mathbf{Q}$ is a *concatenation* of $\mathbf{P}$ if every 2-separation displayed by $\mathbf{Q}$ is also displayed by $\mathbf{P}$. If $\mathbf{P}$ has $l+1$ nonempty steps, then $\mathbf{P}$ has *length* l.

So far we have simply extended terminology from paths of 3-separations. The next two definitions are of specific importance to this chapter. Let $\mathbf{P}$ be a path of 2-separations in the connected matroid M. A 2-separation (X, Y) of M is $\mathbf{P}$*-relevant* if there is a $j \in \{1, 2, \dots, n-1\}$ such that either X or Y has the form $P_0 \cup P_1 \cup \dots \cup P_{j-1} \cup P'_j$ for some subset P'_j of P_j. The path $\mathbf{P}$ is *strict* if

(i) $\lambda(P_i) = 2$ for all $i \in \{1, 2, \dots, l-1\}$, and
(ii) if (X, Y) is a 2-separation of M, that is not $\mathbf{P}$-relevant, then either X or Y is contained in a step of $\mathbf{P}$.

We are particularly interested in paths whose steps contain clonal pairs. A clonal pair $\{p_i, q_i\}$ contained in the internal step P_i is $\mathbf{P}$*-strong* if $\kappa_M(\{p_i, q_i\}, E(M) - P_i) = 2$.

The results of this chapter focus on bridging strict paths of 2-separations whose steps contain strong clonal pairs. It is clearly possible to have a matroid in $\mathcal{E}(q)$ in which all the steps of such a path are bridged. An example is a free swirl. But if constraints are placed on the way that the 2-separations are bridged, then we do obtain certificates that prove that a matroid is not in $\mathcal{E}(q)$. In this chapter we obtain a sequence of such results leading to Lemma 10.22. This lemma will, in turn, be used in Chapter 11 to prove that large free swirls that are bridged in certain ways cannot be in $\mathcal{E}(q)$.

2. Basic facts

We begin by obtaining some basic facts on strict paths of 2-separations.

Lemma 10.1. *Let $\mathbf{P} = (P_0, P_1, \ldots, P_l)$ be a strict path of 2-separations in the connected matroid M. If $i \in \{1, 2, \ldots, l\}$, then the following hold.*

(i) $\sqcap(P_0 \cup P_1 \cup \cdots \cup P_{i-1}, P_{i+1} \cup P_{i+2} \cup \cdots \cup P_l) = 0$.
(ii) $\sqcap^*(P_0 \cup P_1 \cup \cdots \cup P_{i-1}, P_{i+1} \cup P_{i+2} \cup \cdots \cup P_l) = 0$.

Proof. Let $Q_0 = P_0 \cup P_1 \cup \cdots \cup P_{i-1}$ and $Q_l = P_{i+1} \cup P_{i+2} \cup \cdots \cup P_l$. By Lemma 2.12 $\lambda(Q_0 \cup Q_l) = \lambda(Q_0) + \lambda(Q_l) - \sqcap(Q_0, Q_l) - \sqcap^*(Q_0, Q_l)$. The lemma follows from this and the fact that $\lambda(Q_0) = \lambda(Q_l) = 1$. □

We omit the easy arguments establish the properties of the next lemma.

Lemma 10.2. *Let $\mathbf{P} = (P_0, P_1, \ldots, P_l)$ be a strict path of 2-separations in the connected matroid M. Then the following hold.*

(i) *$\mathbf{P}$ is a strict path of 2-separations in M^*.*
(ii) *If $0 < i \leqslant j < n$, then $\lambda(P_i \cup P_{i+1} \cup \cdots \cup P_j) = 2$.*
(iii) *If $\mathbf{Q}$ is a concatenation of $\mathbf{P}$, then $\mathbf{Q}$ is a strict path of 2-separations in M.*

Presumably the next lemma is well known. We omit the easy proof. It is a generalisation of the fact that if x is an element of the connected matroid M, then either $M\backslash x$ or M/x is connected.

Lemma 10.3. *Let (A, B) be a 2-separation of the connected matroid M. Then either $M\backslash A$ or M/A is connected.*

The next lemma follows from Lemma 10.3.

Lemma 10.4. *If $\mathbf{P} = (P_0, P_1, \ldots, P_n)$ is a strict path of 2-separations in the connected matroid M, and $1 \leqslant i < n$, then, $(P_{i+1}, P_{i+2}, \ldots, P_n)$ is a strict path of 2-separations in either $M\backslash(P_0 \cup P_1 \cup \cdots \cup P_i)$ or $M^*\backslash(P_0 \cup P_1 \cup \cdots \cup P_i)$.*

Lemma 10.5. *Let $\mathbf{P} = (P_0, P_1, \ldots, P_l)$ be a strict path of 2-separations of the connected matroid M, let J be a proper nonempty subset of $\{1, \ldots, l\}$ and let $Z = \bigcup_{j \in J} P_j$. Then the following hold.*

(i) *If there exists $i \in \{0, 1, \ldots, l-1\}$ such that $J = \{0, 1, \ldots, i\}$ or $J = \{i+1, i+2, \ldots, l\}$, then $\lambda(Z) = 1$.*
(ii) *If there exist $i, j \in \{1, 2, \ldots, l-1\}$ with $i \leqslant j$ and $J = \{i, i+1, \ldots, j\}$ or $J = \{0, 1, \ldots, i-1, j+1, j+2, \ldots, l\}$, then $\lambda(Z) = 2$.*
(iii) *If neither (i) nor (ii) holds, then $\lambda(Z) > 2$.*

Proof. Part (i) follows from the definition. Part (ii) is just Lemma 10.2(ii). Consider part (iii). Assume that Z satisfies neither (i) nor (ii). The result is vacuous if $\mathbf{P}$ has length one. Assume for induction that (iii) holds if $\mathbf{P}$ has length at most $l-1$. Up to complementation we may assume that $0 \notin J$. Let $i+1$ be the least integer in J. Let $Y = E(M) - Z$ and let $Y' = P_0 \cup P_1 \cup \cdots \cup P_i$. By Lemma 10.2(i) and Lemma 10.4, we may assume that $M\backslash Y'$ is connected and that $(P_{i+1} \cup P_{i+2} \cup \cdots \cup P_n)$ is a strict path of 2-separations in this matroid. By the choice of J and either (ii) or the induction assumption we have $\lambda_{M\backslash Y'}(Z, Y - Y') \geqslant 2$. Note that $\sqcap(Y', Y - Y') = 0$, so that $r(Y) = r(Y - Y') + r(Y')$. But $r(M) = r(M\backslash Y') + r(Y') - 1$. The lemma now follows from an easy calculation. □

Let $\mathbf{P} = (P_0, P_1, \ldots, P_l)$ be a path of 2-separations in the matroid M. A subset X of $E(M)$ is

(i) an *initial* set of $\mathbf{P}$ if $X = P_0 \cup P_1 \cup \cdots \cup P_i$ for some $i \in \{0, 1, \ldots, l\}$;
(ii) a *terminal* set of $\mathbf{P}$ if $E(M) - X$ is an initial set;
(iii) a *consecutive* set of $\mathbf{P}$ if $X = P_i \cup P_{i+1} \cup \cdots \cup P_j$ where $0 \leqslant i \leqslant j \leqslant l$; and
(iv) a *coconsecutive* set of $\mathbf{P}$ if $E(M) - X$ is a consecutive set.

The next lemma is a version of our old friend Lemma 3.31 for paths rather than flowers.

Lemma 10.6. *Let e be an element of the matroid M such that $M\backslash e$ is connected with a strict path $\mathbf{P} = (P_0, P_1, \ldots, P_n)$ of 2-separations. Let X and Y be subsets of $E(M\backslash e)$. Assume that both X and Y are unions of steps and that $e \in \mathrm{cl}(X)$ and $e \in \mathrm{cl}(Y)$.*

(i) *If X is the union of an initial set of steps, Y is the union of a terminal set of steps, and $X \cap Y \neq \emptyset$, then $e \in \mathrm{cl}(X \cap Y)$.*
(ii) *If X and Y are the union of either a consecutive or coconsecutive set of petals, $X \cup Y \neq E(M) - \{e\}$, and $X \cap Y \neq \emptyset$, then $e \in \mathrm{cl}(X \cap Y)$.*

Proof. In either case it suffices to show that X and Y form a modular pair. In case (i) we have $\lambda(X) = \lambda(Y) = 1$, $\lambda(X \cap Y) = 2$ and $\lambda(X \cup Y) = 0$. Thus (i) holds.

Consider (ii). Since $X \cup Y \neq E(M) - \{e\}$ and $X \cap Y \neq \emptyset$, it is readily deduced that both $X \cup Y$ and $X \cap Y$ are unions of consecutive or coconsecutive sets of steps of $\mathbf{P}$. It now follows from Lemmas 10.2(iii) and 10.5(ii), that X and Y form a modular pair. □

The next lemma establishes a connection between strict paths of 2-separations and swirl-like flowers. We omit the easy proof.

Lemma 10.7. *Let e be an element of the matroid M such that $M\backslash e$ is connected with a strict path $\mathbf{P} = (P_0, P_1, \ldots, P_l)$ of 2-separations. If $e \in \mathrm{cl}(P_0 \cup P_l)$, but $e \notin \mathrm{cl}(P_0)$ and $e \notin \mathrm{cl}(P_l)$, then $(P_l \cup P_0 \cup \{e\}, P_1, \ldots, P_{l-1})$ is a swirl-like flower in M.*

If $\mathbf{P} = (P_1, P_2, \ldots, P_n)$ is a swirl-like flower in the connected matroid M, then it is easily seen that, if P_1 is coclosed, then $M\backslash P_1$ is connected. We also omit the easy proof of the next lemma.

Lemma 10.8. *Let $\mathbf{P} = (P_1, P_2, P_3, \ldots, P_n)$ be a swirl-like flower in the connected matroid M. If P_1 is coclosed, then $(P_2, P_3, \ldots, P_n)$ is a strict path of 2-separations in the connected matroid $M\backslash P_1$.*

Lemma 10.9. *Let $\mathbf{P} = (P_0, P_1, \ldots, P_l)$ be a strict path of 2-separations in the connected matroid M, and let $\{p_i, q_i\}$ be a $\mathbf{P}$-strong clonal pair. Then, $\kappa(\{p_i, q_i\}, P_0 \cup P_l) = 2$.*

Proof. If the lemma does not hold, then there is a 2-separation (X, Y) of M with $P_0 \cup P_l \subseteq X$ and $\{p_i, q_i\} \subseteq Y$. Such a separation is not $\mathbf{P}$-relevant so it must be the case that there is a petal P_i such that $Y \subseteq P_i$. But this contradicts the definition of $\mathbf{P}$-strong clonal pair. □

Lemma 10.10. *Let N be a connected minor of the connected matroid M and let $\mathbf{P} = (P_0, P_1, \ldots, P_l)$ be a strict path of 2-separations in N. Assume that every $\mathbf{P}$-relevant 2-separation of N is bridged in M, and that*

$\mathbf{Q} = (Q_0, Q_1, \ldots, Q_m)$ *is a partition of* $E(M)$ *such that,* $P_0 \subseteq Q_0$, $P_m \subseteq Q_m$ *and* $\lambda_M(Q_0 \cup Q_1 \cup \cdots \cup Q_i) = 2$ *for* $i \in \{0, 1, \ldots, m-1\}$. *Then the following hold.*

(i) $\mathbf{Q}$ *is a path of* 3*-separations in* M.
(ii) *If* $\{p, q\}$ *is an* M*-clonal pair of* N *that is* $\mathbf{P}$*-strong, then it is also* $\mathbf{Q}$*-strong.*

Proof. Assume that (Q'_0, Q'_m) is a 2-separation of M with $Q_0 \subseteq Q'_0$ and $Q_m \subseteq Q'_m$. Then (Q'_0, Q'_m) is induced by a 2-separation (P'_0, P'_l) with $P_0 \subseteq P'_0$ and $P_l \subseteq P'_l$. But by the definition of strict path, such 2-separations are $\mathbf{P}$-relevant and hence bridged in M. Part (i) follows from this contradiction. Part (ii) follows from Lemma 10.9. □

3. A first certificate

We now develop some certificates for showing that a matroid is not in $\mathcal{E}(q)$. Interpreted in this way the next lemma says that if we block all the 2-separations displayed by a strict path of 2-separations, and we do not keep a large displayed swirl-like flower, then we are not in $\mathcal{E}(q)$.

Lemma 10.11. *Let* M *be a matroid in* $\mathcal{E}(q)$ *with an element* b *such that* $M\backslash b$ *is connected with a strict path* $\mathbf{P} = (P_0, P_1, \ldots, P_n)$ *of* 2*-separations, each internal step of which contains a* $\mathbf{P}$*-strong* M*-clonal pair. Assume that* b *blocks both* P_0 *and* P_n*. Then there is a function* $f_{10.11}(m, q)$ *such that, if* $n \geqslant f_{10.11}(m, q)$*, then the following holds. There exists an* $s \in \{1, 2, \ldots, n-2\}$ *such that* $s + m < n$ *with the property that*

$$\mathbf{R} = \big(P_0 \cup P_1 \cup \cdots \cup P_s \cup P_{s+m} \cup P_{s+m+1} \cup \cdots \cup P_n \cup \{b\}, P_{s+1}, \ldots, P_{s+m-1}\big)$$

is a swirl-like flower in M *of order* m*. Moreover, each petal of* $\mathbf{R}$ *contains an* $\mathbf{R}$*-strong* M*-clonal pair.*

Proof. Let $f_{10.11}(m, q) = f_{7.36}(m+1, q) + 2$. Assume that $n \geqslant f_{10.11}(m, q)$. Note that, if (X, Y) is a 2-separation of $M\backslash b$ where $P_0 \subseteq X$ and $P_n \subseteq Y$, then b blocks (X, Y). Hence $\mathbf{P}' = (P_0 \cup \{b\}, P_1, \ldots, P_n)$ is a path of 3-separations in M as otherwise b fails to block both P_0 and P_n. Say $i \in \{1, 2, \ldots, n-1\}$. Then, by the definition of $\mathbf{P}$-strong for strict paths of 2-separations, the step P_i of $\mathbf{P}$ contains an M-clonal pair $\{p_i, p'_i\}$ such that $\kappa_{M\backslash b}(\{p_i, p'_i\}, P_0 \cup \cdots \cup P_{i-1} \cup P_{i+1} \cup \cdots \cup P_n) = 2$. Hence $\kappa_M(\{p_i, p'_i\}, \{b\} \cup P_0 \cup \cdots \cup P_{i-1} \cup P_{i+1} \cup \cdots \cup P_n) = 2$. It follows that $\{p_i, p'_i\}$ is $\mathbf{P}'$-strong in the sense of $\mathbf{P}'$-strong defined for paths of 3-separations.

Let $\mathbf{P}'' = (\{b\} \cup P_0 \cup P_1, P_2, \ldots, P_{n-2}, P_{n-1} \cup P_n)$. Then each step of $\mathbf{P}''$ contains a $\mathbf{P}''$ strong clonal pair and $\mathbf{P}''$ has length $n-2$. By Lemma 7.36, $\mathbf{P}''$ displays a swirl-like flower of order $m+1$. Let $\mathbf{Q}$ be a maximal such flower, say $\mathbf{Q}$ has order $\mu + 1 \geqslant m + 1$. Let $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_{\mu+1})$. Then, by Lemma 7.14, there is an integer $s \geqslant 2$, with $s + \mu \leqslant n - 2$ such that $\{Q_1, Q_2, \ldots, Q_{\mu+1}\} = \{\{b\} \cup P_0 \cup P_1 \cup \cdots \cup P_s, P_{s+1}, \ldots, P_{s+\mu-1}, P_{s+\mu} \cup P_{s+\mu+1} \cup \cdots \cup P_{n-1} \cup P_n\}$.

Recall that if R and R' are petals of a swirl-like flower, then $\sqcap(R, R') > 0$ if and only if R and R' are adjacent. Say $\{i, i+1\} \subseteq \{s+1, s+2, \ldots, s+\mu-1\}$. Then, by the definition of strict 2-path, $\sqcap_{M\backslash b}(P_i, P_{i+1}) = 1$, so that $\sqcap_M(P_i, P_{i+1}) = 1$. Hence P_i and P_{i+1} are adjacent in $\mathbf{Q}$. Moreover, $\sqcap_M(\{b\} \cup P_0 \cup P_1 \cup \cdots \cup P_s, P_{s+1}) \geqslant 1$, so these two petals of $\mathbf{P}$ are adjacent in $\mathbf{Q}$. Also $\sqcap(P_{s+\mu-1}, P_{s+\mu} \cup P_{s+\mu+1} \cup \cdots \cup P_{n-1} \cup P_n) \geqslant 1$, and these two petals are adjacent in $\mathbf{Q}$. By elimination $\{b\} \cup P_0 \cup P_1 \cup \cdots \cup P_s$ and $P_{s+\mu-1} \cup P_{s+\mu} \cup \cdots \cup P_{n-1} \cup P_n$ are adjacent. Hence $\mathbf{Q} = (\{b\} \cup P_0 \cup P_1 \cup \cdots \cup P_s, P_{s+1}, \ldots, P_{s+\mu-1}, P_{s+\mu} \cup \cdots \cup P_{n-1} \cup P_n)$. Let $\mathbf{Q}' = (\{b\} \cup P_0 \cup P_1 \cup \cdots \cup P_s \cup P_{s+\mu} \cup \cdots \cup P_{n-1} \cup P_n, P_{s+1}, P_{s+2}, \ldots, P_{s+\mu-1})$. Every petal of this flower contains a $\mathbf{Q}'$-strong clonal pair. As $\mu \geqslant m$, the lemma is satisfied by taking $\mathbf{R}$ to be an appropriate concatenation of $\mathbf{Q}'$. □

4. A second certificate

Our next certificate requires somewhat more work.

Lemma 10.12. *Let M be a matroid with a set $B = \{b_2, b_3, \dots, b_n\}$ of elements such that $M\backslash B$ is connected with a strict path $\mathbf{P} = (P_0, P_1, \dots, P_n)$ of 2-separations each internal step of which contains a $\mathbf{P}$-strong M-clonal pair. Assume that, for $i \in \{2, 3, \dots, n\}$, the element b_i is in $\mathrm{cl}(P_i \cup P_0)$, but not in $\mathrm{cl}(P_0)$ or $\mathrm{cl}(P_i)$. Then there is a function $f_{10.12}(q)$ such that, if $n \geqslant f_{10.12}(q)$, then $M \notin \mathcal{E}(q)$.*

We begin by proving Lemma 10.12 in two special cases. For the first case was assume that P_0 consists of a single element.

Lemma 10.13. *Let M be a matroid with a set $B = \{b_2, b_3, \dots, b_n\}$ of elements such that $M\backslash B$ is connected with a strict path $\mathbf{P} = (\{p_0\}, P_1, \dots, P_n)$ of 2-separations each internal step of which contains a $\mathbf{P}$-strong M-clonal pair. Assume that, for $i \in \{2, 3, \dots, n\}$, the element b_i is in $\mathrm{cl}(P_i \cup \{p_0\})$, but not in $\mathrm{cl}(\{p_0\})$ or $\mathrm{cl}(P_i)$. Then there is a function $f_{10.13}(q)$ such that, if $n \geqslant f_{10.13}(q)$, then $M \notin \mathcal{E}(q)$.*

Proof. Consider the partition

$$\mathbf{P}' = \big(\{p_0\} \cup P_1, P_2 \cup \{b_2\}, \dots, P_{n-1} \cup \{b_{n-1}\}, P_n \cup \{b_n\}\big)$$

of $E(M)$. We first prove that $\mathbf{P}'$ is a path of 3-separations in M.

Say that $j \in \{1, 2, \dots, n-1\}$. Note that $p_0 \notin \mathrm{cl}(P_{j+1} \cup P_{j+2} \cup \dots \cup P_n)$, as otherwise $\lambda_{M\backslash B}(\{p_0\} \cup P_{j+1} \cup \dots \cup P_n) = 1$, contradicting Lemma 10.5. Also $b_i \in \mathrm{cl}(P_i \cup \{p_0\})$ for all $i \in \{1, 2, \dots, n-1\}$. Thus

$$r\big(\{p_0\} \cup P_1 \cup P_2 \cup \{b_2\} \cup \dots \cup P_{j-1} \cup \{b_{j-1}\}\big) = r\big(\{p_0\} \cup P_1 \cup \dots \cup P_{j-1}\big)$$

and

$$r\big(P_j \cup \{b_j\} \cup \dots \cup P_n \cup \{b_n\}\big) = r\big(P_j \cup \dots \cup P_n \cup \{p_0\}\big) = r(P_j \cup \dots \cup P_n) + 1.$$

Hence

$$\lambda_M\big(\{p_0\} \cup P_1 \cup P_2 \cup \{b_2\} \cup \dots \cup P_{j-1} \cup \{b_{j-1}\}\big) = 2.$$

To prove that $\mathbf{P}'$ is a path of 3-separations in M. It remains to show that $\kappa_M(\{p_0\} \cup P_1, P_n \cup \{b_n\}) = 2$. Assume not. Then, for some $i \in \{1, 2, \dots, n\}$ and subset P'_i of $P_i \cup \{b_i\}$, we have $\lambda_M(\{p_0\} \cup P_1 \cup P_2 \cup \{b_2\} \cup \dots \cup P_{i-1} \cup \{b_{i-1}\} \cup P'_i) = 1$. But $\lambda_{M\backslash B}(\{p_0\} \cup P_1 \cup P_2 \cup \dots \cup P_{i-1} \cup P'_i) = 1$, so that b_n does not block the 2-separation corresponding to this 2-separating set. Hence either $b_n \in \mathrm{cl}(\{p_0\} \cup P_1 \cup P_2 \cup \dots \cup P_{i-1} \cup P'_i)$ or $b_n \in \mathrm{cl}((P_i - P'_i) \cup P_{i+1} \cup \dots \cup P_n)$. But $b_n \in \mathrm{cl}(\{p_0\} \cup P_n)$. By Lemma 10.6, we obtain the contradiction that either $b_n \in \mathrm{cl}(\{p_0\})$ or $b_n \in \mathrm{cl}(P_n)$.

If $i \in \{2, 3, \dots, n\}$, then $b \in \mathrm{cl}(P_i \cup \{b_i\})$. Therefore, if P'_i and P'_j are distinct steps of the path $\mathbf{P}'$, then $\sqcap(P'_i, P'_j) > 0$. This shows that $\mathbf{P}'$ does not display any 4-petal swirl-like flower. Moreover, it is evident that if $i \in \{2, 3, \dots, n-1\}$, then any $\mathbf{P}$-strong clonal pair in P_i is $\mathbf{P}'$-strong. The lemma now follows by letting $f_{10.13}(q) = f_{7.36}(4, q) + 2$. □

For the second special case we assume that P_0 consists of a series class that is blocked in a particular way.

Lemma 10.14. *Let M be a matroid with a set $B = \{b_2, b_3, \dots, b_n\}$ such that $M\backslash B$ is connected with a strict path $\mathbf{P} = (P_0, P_1, \dots, P_n)$ of 2-separations, where $P_0 = \{s_2, s_3, \dots, s_n\}$ is a series class and each internal step of $\mathbf{P}$ contains a $\mathbf{P}$-strong M-clonal pair. Assume that, for $i \in \{2, 3, \dots, n\}$, the element b_i is in $\mathrm{cl}(P_i \cup \{s_i\})$, but $b_i \notin \mathrm{cl}(P_i)$ and $b_i \notin \mathrm{cl}(\{s_i\})$. Then there is a function $f_{10.14}(q)$ such that, if $n \geqslant f_{10.14}(q)$, then $M \notin \mathcal{E}(q)$.*

Proof. Clearly we may assume that $n \geqslant 3$. We first note:

10.14.1.

(i) *If* $2 \leqslant i \leqslant n$, *then* $b_i \notin \mathrm{cl}(P_0 \cup P_1 \cup \cdots \cup P_{i-2})$. *In particular* $\mathrm{cl}(P_0) \cap B = \emptyset$.
(ii) *Say* $P'_0 \subseteq P_0$ *and* $|P'_0| \geqslant 2$. *If* $s_i \in P'_0$, *then* $b_i \notin \mathrm{cl}((P_0 - P'_0) \cup P_1 \cup P_2 \cup \cdots \cup P_n)$.

Subproof. Assume that $b_i \in \mathrm{cl}(P_0 \cup P_1 \cup \cdots \cup P_{i-2})$. Consider the pair of sets $P_0 \cup P_1 \cup \cdots \cup P_{i-2}$ and $P_i \cup \{s_i\}$. As s_i is in a non-trivial series class of $M \backslash B$ that does not meet P_i, we have $r(P_i \cup \{s_i\}) = r(P_i) + 1$. The union of the two sets under consideration is $P_0 \cup P_1 \cup \cdots \cup P_{i-2} \cup P_i$. By Lemma 10.1 the rank of this set is $r(P_0 \cup P_1 \cup \cdots \cup P_{i-2}) + r(P_i)$. Moreover, the intersection of the sets is $\{s_i\}$. We deduce that the two sets form a modular pair. As b_i is in the span of each we obtain the contradiction that $b_i \in \mathrm{cl}(\{s_i\})$. Hence (i) holds.

Consider (ii). Assume that $|P'_0| \geqslant 2$ and that $s_i \in P'_0$. As $b_i \in \mathrm{cl}(P_i \cup \{s_i\})$ but not in $\mathrm{cl}(P_i)$, we have $s_i \in \mathrm{cl}(P_i \cup \{b_i\})$. Thus, if $b_i \in \mathrm{cl}((P_0 - P'_0) \cup P_1 \cup \cdots \cup P_n)$, then we have $s_i \in \mathrm{cl}((P_0 - P'_0) \cup P_1 \cup \cdots \cup P_n)$, contradicting the fact that P'_0 is a series set in $M \backslash B$. □

10.14.2. *If* (X, Y) *is a* 2*-separation of* M, *then, for some* $Z \in \{X, Y\}$, *either* $Z \subseteq P_i$ *for an* $i \in \{1, 2, \ldots, n-2\}$ *or* $Z \subseteq P_{n-1} \cup P_n$.

Subproof. Consider the 2-separation (X, Y). Let $(X', Y') = (X - B, Y - B)$. Assume that $|X'| \leqslant 1$. Then, as $r(M \backslash B) = r(M)$, and $M \backslash B$ is connected, we see that $X \subseteq \mathrm{cl}(Y)$. Hence $r(X) = 1$. But it is evident that no member of B is in a non-trivial parallel class. Hence $|X - B| > 1$ and also $|Y - B| > 1$, so that (X', Y') is a 2-separation of $M \backslash B$.

Assume that (X', Y') is **P**-relevant, so that, up to labels, $(X', Y') = (P_0 \cup P_1 \cup \cdots \cup P_{i-1} \cup P'_i, P''_i \cup P_{i+1} \cup \cdots \cup P_n)$ for some $i \in \{1, 2, \ldots, n-1\}$. If $i \neq n-1$, then, by 10.14.1, (X', Y') is blocked by a member of B. Thus $i = n-1$, and $Y' \subseteq P_{n-1} \cup P_n$. In this case, it also follows from 10.14.1 that $\mathrm{cl}_M(Y') \cap B = \emptyset$, so that $Y' = Y$ and the claim holds in this case.

On the other hand, if (X', Y') is not **P**-relevant then, by the definition of strict path of 2-separation, we may assume that $X' \subseteq P_i$ for some $i \in \{0, 1, \ldots, n\}$. Again by 10.14.1, $\mathrm{cl}_M(X') \cap B = \emptyset$, so that $X' = X$. If $X' \subseteq P_0$, then it follows from 10.14.1 that X' is blocked by at least one member of B. Thus $X' \nsubseteq P_0$ and the sublemma follows. □

10.14.3. *For all* $i \in \{1, 2, \ldots, n-1\}$, *we have* $\lambda_M(P_1 \cup P_2 \cup \cdots \cup P_i \cup \{s_2, s_3, \ldots, s_i\} \cup \{b_2, b_3, \ldots, b_i\}) = 2$.

Subproof. By the definition of paths of 2-separations, $\lambda_{M \backslash B}(P_1 \cup P_2 \cup \cdots \cup P_i \cup \{s_2, s_3, \ldots, s_n\}) = 1$. Since $\{s_2, s_3, \ldots, s_n\}$ is a series class of $M \backslash B$, and $\sqcap(\{s_2, s_3, \ldots, s_n\}, P_1) = 1$, we have $r(P_1 \cup P_2 \cup \cdots \cup P_i \cup \{s_2, s_3, \ldots, s_n\}) = r(P_1 \cup P_2 \cup \cdots \cup P_i) + n - 1$. But $r(P_1 \cup P_2 \cup \cdots \cup P_i \cup \{s_2, s_3, \ldots, s_i\}) = r(P_1 \cup P_2 \cup \cdots \cup P_i) + i$, and $r(P_{i+1} \cup P_{i+2} \cup \cdots \cup P_n \cup \{s_{i+1}, s_{i+2}, \ldots, s_n\}) = r(P_{i+1} \cup P_{i+2} \cup \cdots \cup P_n) + n - i$. Hence $\lambda_{M \backslash B}(P_1 \cup P_2 \cup \cdots \cup P_i \cup \{s_2, s_3, \ldots, s_i\}, P_{i+1} \cup P_{i+2} \cup \cdots \cup P_n \cup \{s_{i+1}, s_{i+2}, \ldots, s_n\}) = 2$. The sublemma is a straightforward consequence of this observation and the fact that $b_j \in \mathrm{cl}(P_j \cup \{s_j\})$ for all $j \in \{2, 3, \ldots, n\}$. □

By 10.14.3 and 10.14.2, $\mathbf{P}' = (P_1 \cup P_2 \cup \{s_2, b_2\}, P_3 \cup \{s_3, b_3\}, \ldots, P_{n-2} \cup \{s_{n-2}, b_{n-2}\}, P_{n-1} \cup P_n \cup \{s_{n-1}, s_n, b_{n-1}, b_n\})$ is a path of 3-separations in M. If this path displays a swirl-like flower with at least four petals, then there exists an $i \in \{3, 4, \ldots, n-2\}$ such that $\lambda_M(P_i \cup \{s_i, b_i\}) = 2$, so that $\lambda_{M \backslash B}(P_i \cup \{s_i\}) = 2$. But, if $i \in \{3, 4, \ldots, n-2\}$, then $s_i \notin \mathrm{cl}(P_i)$ and $s_i \in \mathrm{cl}(P_1 \cup \{s_2, s_3, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n\})$ so that $\lambda_{M \backslash B}(P_i \cup \{s_i\}) = \lambda_{M \backslash B}(P_i) + 1$. By Lemma 10.2, $\lambda_{M \backslash B}(P_i) = 2$. Hence $\lambda_{M \backslash B}(P_i \cup \{s_i\}) = 3$ and we conclude that $\mathbf{P}'$ does not display any 4-petal swirl-like flowers.

To complete the proof we need to show that our M-clonal pairs are $\mathbf{P}'$-strong. Say $i \in \{2, 3, \ldots, n-2\}$, and $\{p_i, p'_i\} \subseteq P'_i$ is a **P**-strong M-clonal pair. If $\kappa_M(\{p_i, p'_i\}, E(M) - (P_i \cup \{b_i, s_i\})) < 2$, then there is a 2-separating set Z of M such that $\{p_i, p'_i\} \subseteq Z \subseteq P_i \cup \{b_i, s_i\}$. By 10.14.2, $Z \subseteq P_i$ and is

2-separating in $M\backslash B$ contradicting the definition of $\mathbf{P}$-strong. Thus $\{p_i, p'_i\}$ is a $\mathbf{P}'$-strong clonal pair as required.

The lemma now follows by letting $f_{10.14}(q) = f_{7.36}(4, q) + 3$. $\square$

Before proving Lemma 10.12 we note an elementary fact.

Lemma 10.15. *Let C be a circuit of the connected matroid M. If $x \in E(M) - C$ then there is an $N \in \{M\backslash x, M/x\}$ such that N is connected and C is a circuit of N.*

Proof. If $M\backslash x$ is connected, let $N = M\backslash x$. Otherwise say (X, Y) is a separation of $M\backslash x$, where $C \subseteq X$. If $x \in \mathrm{cl}(C)$, then $(X \cup \{x\}, Y)$ is a separation of M. Thus $x \notin \mathrm{cl}(C)$ and the lemma holds by setting $N = M/x$. $\square$

We also recall the theorem of Lemos and Oxley [17] that we have stated in this paper as Theorem 9.59. The next result is a straightforward consequence of this theorem. We omit the routine proof.

Corollary 10.16. *Let (A, B) be a 2-separation of the connected matroid $M \in \mathcal{E}(q)$. Then there is a function $f_{10.16}(m, q)$ such that, if $|A| \geqslant f_{10.16}(m, q)$, then A contains either a circuit, a cocircuit, a parallel set, or a series set with at least m elements.*

Proof of Lemma 10.12. Let $\rho = \max\{f_{10.13}(q), f_{10.14}(q)\}$ and let $f_{10.12}(q) = f_{10.13}(q) f_{10.16}((q+1)\rho)$. Assume that $n \geqslant f_{10.12}(q)$.

To facilitate the proof, we make some local definitions. Let M' be a minor of M obtained by removing a proper subset of elements of P_0, let $N' = M'\backslash B$, and let $P'_0 = P_0 \cap E(N')$. Then M' is an *allowable* minor of M if N' is connected and B is coindependent in M'.

We will always use the convention that, if M' is an allowable minor of M, then $P'_0 = P_0 \cap E(M')$ and $N' = M'\backslash B$. The next claim is evident.

10.12.1. *Let M' be an allowable minor of M. Say $e \in P'_0$ and $|P'_0| > 1$. If $N'\backslash e$ is connected, then $M'\backslash e$ is allowable and if N'/e is connected, then M'/e is allowable. Moreover, either $M'\backslash e$ or M'/e is allowable.*

Let M'' be an arbitrary minor of M obtained by removing elements of P_0 and let $P''_0 = E(M'') \cap P_0$. For an element x of P''_0, define

$$d_{M''}(x) = \bigl|\bigl\{P_i\colon\ i \in \{2, 3, \ldots, m\},\ x \in \mathrm{cl}_{M''}\bigl(P_i \cup \{b_i\}\bigr)\bigr\}\bigr|$$

and define

$$d(M'') = \bigl|\bigl\{P_i\colon\ i \in \{2, 3, \ldots, n\},\ b_i \in \mathrm{cl}_{M''}\bigl(P''_0 \cup P_i\bigr),\ b_i \notin \mathrm{cl}_{M''}(P_i)\bigr\}\bigr|.$$

Let x and z be distinct elements of P''_0. The next sublemma is elementary.

10.12.2.

(i) *$d_{M''\backslash z}(x) = d_{M''}(x)$ and, if z is not a coloop of M'', then $d(M''\backslash z) = d(M'')$.*
(ii) *$d_{M''/z}(x) \geqslant d_{M''}(x)$ and $d(M''/z) = d(M'') - d_{M''}(z)$.*

10.12.3. *Let M' be an allowable minor of M, and let p be an element of P'_0. If $d_{M'}(p) > f_{10.13}(q)$, then $M \notin \mathcal{E}(q)$.*

Subproof. Consider the partition

$$\mathbf{P}' = \big(\{p\}, \big(P'_0 - \{p\}\big) \cup P_1, P_2, \dots, P_n\big).$$

It is easily checked that this is a strict path of 2-separations each internal step of which contains a $\mathbf{P}'$-strong, M'-clonal pair. But now the hypotheses of Lemma 10.13 hold for M', N' and $\mathbf{P}'$, so by that lemma $M' \notin \mathcal{E}(q)$. Hence $M \notin \mathcal{E}(q)$. $\square$

10.12.4. *There is an allowable minor M' of M that has the following properties: P'_0 has no non-trivial parallel classes; $d(M') = d(M)$; and $d_{M'}(p) > 0$ for all $p \in P'_0$.*

Subproof. Let M' be an allowable minor of M that has the properties that $d(M') = d(M)$ and that $|E(M')|$ is minimal. It follows from 10.12.1 and 10.12.2 that M' satisfies the claim. $\square$

From now on we assume that M' is an allowable minor of M satisfying 10.12.4.

10.12.5. *If $r_{M'}(P'_0) \leqslant f_{10.16}((q+1)\rho)$, then $M \notin \mathcal{E}(q)$.*

Subproof. Recall that $n \geqslant f_{10.13}(q) f_{10.16}((q+1)\rho)$. If M' contains an allowable minor M'' with an element $z \in E(M'') \cap P_0$ such that $d_{M''(x)} \geqslant f_{10.13}(q)$, then the claim follows from 10.12.3. Thus we may assume that this never occurs. It follows from this that we may apply 10.12.1 and 10.12.2 to obtain an allowable minor M'', where $E(M'') \cap P_0 = \{p_0\}$. But, in this case $d_{M''}(p_0) = d(M'')$ and $d(M'') \geqslant f_{10.13(q)}$. It now follows from Lemma 10.13 that $M \notin \mathcal{E}(q)$. $\square$

10.12.6. *If P'_0 contains a series class of N' of size at least $f_{10.14}(q)$, then $M \notin \mathcal{E}(q)$.*

Proof. Let $P'_0 = S$, $P'_1 = (P_0 - P'_0) \cup P_1$, and $\mathbf{P}' = (P'_0, P'_1, P_2, \dots, P_n)$. Then $\mathbf{P}'$, N' and M' satisfy the hypotheses of Lemma 10.14, so that, by that lemma $M \notin \mathcal{E}(q)$. $\square$

10.12.7. *If P'_0 contains a circuit C of N' with at least $(q+1) f_{10.14}(q)$ elements, then $M \notin \mathcal{E}(q)$.*

Subproof. Say $P'_0 \neq C$. Then by Lemma 10.15 and 10.12.1, there is an element $z \in P'_0$ such that either $M' \backslash z$ or M'/z is allowable with C as a circuit. By this fact and 10.12.2, we lose no generality in assuming that $P'_0 = C$. As $\lambda_{N'}(C) = 1$ we have $r^*_{N'}(C) = 2$. If $M \in \mathcal{E}(q)$, then C contains at most $q+1$ series classes of N'. As $|C| \geqslant (q+1) f_{10.14}$, there is a series class S in C of size at least $f_{10.14}(q)$. By 10.12.6 $M \notin \mathcal{E}(q)$. $\square$

An easy argument that we omit shows that

10.12.8. *If M'' is an allowable minor of M' and the element $p \in P'_0 \cap E(M'')$ belongs to a parallel class of size l, then $d_{M''}(p) \geqslant l$.*

10.12.9. *If P'_0 has a cocircuit C of N' with at least $(q+1) f_{10.13}(q)$ elements, then $M \notin \mathcal{E}(q)$.*

Subproof. Arguing as in 10.12.7, but using the dual of Lemma 10.15, we obtain an allowable minor M'' with $P'_0 \cap E(M'') = C$ such that C is a cocircuit in M''. But then, $r_{N'}(C) \leqslant 2$. So that C has a parallel set of size at least $f_{10.13}(q)$. Say that p is an element of such a parallel set. Then, by 10.12.8, $d_{M''}(p) \geqslant f_{10.13}(q)$. Let $\mathbf{P}'' = (\{p\}, P_1 \cup P_0 - \{p\}, P_2, \dots, P_n)$. With this path of 2-separations in $M'' \backslash B$ the hypotheses of Lemma 10.13 are satisfied. Thus $M \notin \mathcal{E}(q)$. $\square$

If $r(P'_0) \leqslant f_{10.16}((q+1)\rho)$, then $M \notin \mathcal{E}(q)$ by 10.12.5. Thus we may assume that $r(P'_0) > f_{10.16}((q+1)\rho)$. By definition P'_0 contains no non-trivial parallel sets of N', so, by the definition of ρ and Corollary 10.16, P_0' contains one of the following: a series set of N' of size at least $f_{10.14}(q)$, a circuit of size at least $(q+1)f_{10.14}(q)$, or a cocircuit of N' of size at least $(q+1)f_{10.13}(q)$. By 10.12.6, 10.12.7 and 10.12.8 respectively, we deduce in each case that $M \notin \mathcal{E}(q)$. $\square$

5. A third certificate

Let $\mathbf{P} = (P_0, P_1, \ldots, P_n)$ be a strict path of 2-separations. Recall that a 2-separation (X, Y) is $\mathbf{P}$-*relevant* if either X or Y is of the form $P_0 \cup P_1 \cup \cdots \cup P_{i-1} \cup P_i'$ for some subset P_i' of P_i for some $i \in \{1, 2, \ldots, n-1\}$.

Lemma 10.17. *Let M be a matroid with a set coindependent set B such that $M \backslash B$ has a strict path $\mathbf{P} = (P_0, P_1, \ldots, P_n)$ of 2-separations, each internal step of which contains a $\mathbf{P}$-strong M-clonal pair. Assume that the following hold.*

(i) *For all $b \in B$, there exists $i \in \{1, 2, \ldots, n\}$ such that $b \in \mathrm{cl}(P_{i-1} \cup P_i)$.*
(ii) *Every $\mathbf{P}$-relevant 2-separation of $M \backslash B$ is bridged in M.*

Then there is a function $f_{10.17}(q)$ such that, if $n \geqslant f_{10.17}(q)$, then $M \notin \mathcal{E}(q)$.

Proof. We may assume that B is minimal in that, for all $b \in B$, there is a $\mathbf{P}$-relevant 2-separation that is induced in $M \backslash b$. For $i \in \{1, 2, \ldots, n\}$, let $Z_i = \{b \in B\colon\ b \in \mathrm{cl}(P_0 \cup P_1 \cup \cdots \cup P_i), b \notin \mathrm{cl}(P_0 \cup P_1 \cup \cdots \cup P_{i-1})\}$.

10.17.1. *$(Z_1, Z_2, \ldots, Z_n)$ partitions B into nonempty subsets.*

Subproof. It is immediate from the definition that the members of $(Z_1, Z_2, \ldots, Z_n)$ are pairwise disjoint. Say $i \in \{1, 2, \ldots, n\}$. We now prove that $Z_i \neq \emptyset$. Consider $(P_0 \cup P_1 \cup \cdots \cup P_{i-1}, P_i \cup P_{i+1} \cup \cdots \cup P_n)$. This 2-separation of $M \backslash B$ is not induced in M, so some member b of B blocks it. For such an element b we have $b \notin \mathrm{cl}(P_0 \cup P_1 \cup \cdots \cup P_{i-1})$ and $b \notin \mathrm{cl}(P_i \cup P_{i+1} \cup \cdots \cup P_n)$. But $b \in \mathrm{cl}(P_{j-1} \cup P_j)$ for some j, so we have $j = i$ and it follows that $b \in \mathrm{cl}(P_0 \cup P_1 \cup \cdots \cup P_i)$ so that $b \in Z_i$. $\square$

10.17.2. *If $b \in Z_k$, then $b \in \mathrm{cl}(P_{k-1} \cup P_k)$.*

Subproof. There is a $j \in \{1, 2, \ldots, n\}$ such that $b \in \mathrm{cl}(P_{j-1} \cup P_j)$. By the definition of Z_k we have $j > k-1$. If $j = k$ the claim holds. Say $j > k$. Then $b \in \mathrm{cl}(P_k \cup P_{k+1} \cup \cdots \cup P_n)$, so by Lemma 10.6, $b \in \mathrm{cl}(P_k)$. Hence $b \in \mathrm{cl}(P_{k-1} \cup P_k)$ in this case too. $\square$

10.17.3. *Say $j \in \{1, 2, \ldots, n\}$ and $b \in Z_j$. Let (X, Y) be a $\mathbf{P}$-relevant 2-separation of $M \backslash B$ that induces a 2-separation (X', Y') of $M \backslash b$. Then the following hold.*

(i) *$P_0 \cup P_1 \cup \cdots \cup P_{j-2} \subseteq X$ and $P_{j+1} \cup P_{j+2} \cup \cdots \cup P_n \subseteq Y$.*
(ii) *$Z_1 \cup Z_2 \cup \cdots \cup Z_{j-3} \subseteq X'$ and $Z_{j+2} \cup Z_{j+3} \cup \cdots \cup Z_n \subseteq Y'$.*

Subproof. By the definition of $\mathbf{P}$-relevant,

$$(X, Y) = \big(P_0 \cup P_1 \cup \cdots \cup P_{i-1} \cup P_i', P_i'' \cup P_{i+1} \cup \cdots \cup P_n\big)$$

for some $i \in \{1, 2, \ldots, n-1\}$. If $i < j-1$, then by 10.17.2, $b \in \mathrm{cl}(Y)$. If $i > j$, then $b \in \mathrm{cl}(X)$ by the definition of Z_j. In either case we contradict the fact that (X, Y) is not induced in M. Thus $i \in \{j-1, j\}$ and (i) holds.

Consider (ii). Say $z \in Z_1 \cup Z_2 \cup \cdots \cup Z_{j-3}$. Assume that $z \in Y'$. Then $z \in \mathrm{cl}(P_{j-1} \cup P_j \cup \cdots \cup P_n)$ as otherwise (X, Y) is not induced in $M \backslash b$. But, by definition, $z \in \mathrm{cl}(P_0 \cup P_1 \cup \cdots \cup P_{j-3})$. Now, by

Lemma 10.1, $\sqcap(P_0 \cup P_1 \cup \cdots \cup P_{j-3}, P_{j-1} \cup P_j \cup \cdots \cup P_n) = 0$, contradicting the fact that b is not a loop of M. Thus $z \in X'$. A similar argument proves that if $z \in Z_{j+2} \cup Z_{j+3} \cup \cdots \cup Z_n$, then $z \in Y'$. □

Let $t = \lfloor (n-1)/4 \rfloor$. For $i \in \{1, 2, \ldots, t\}$, let b_{4i} be an element of Z_{4i}. Let (X'_{4i}, Y'_{4i}) be a 2-separation of $M \backslash b_{4i}$ that is induced by a **P**-relevant 2-separation (X_{4i}, Y_{4i}) of $M \backslash B$. Note that $(X'_{4i}, \{b_{4i}\} \cup Y'_{4i})$ is a 3-separation of M.

10.17.4. *If $1 \leqslant i < j \leqslant t$, then $X'_{4i} \subseteq X'_{4j}$.*

Subproof. By Lemma 10.17(i), $P_{4i+1} \cup P_{4i+2} \cup \cdots \cup P_n \subseteq Y_{4i} \subseteq Y'_{4i}$. By Lemma 10.17(ii), $Z_{4i+2} \cup Z_{4i+3} \cup \cdots \cup Z_n \subseteq Y'_{4i}$. Similarly $P_1 \cup P_2 \cup \cdots \cup P_{4j-2} \subseteq X'_{4j}$ and $Z_1 \cup Z_2 \cup \cdots \cup Z_{4j-3} \subseteq X'_{4j}$. But $j > i$, so $4j - 3 \geqslant 4i + 1$. Thus $Y'_{4i} \cup X'_{4j} = E(M)$ and hence $X'_{4i} \subseteq X'_{4j}$. □

Let $R_1 = X'_4$, $R_t = \{b_t\} \cup Y'_t$, and for $i \in \{2, 3, \ldots, t-1\}$, let $R_i = X'_{4i} - X'_{4i-4}$. Note that $\lambda_M(R_1 \cup R_2 \cup \cdots \cup R_i) = 2$ for $i \in \{1, 2, \ldots, t-1\}$. By this fact and 10.17.4, we deduce that $\mathbf{R} = (R_1, R_2, \ldots, R_t)$ is a path of 3-separations in M. Each internal step of **R** contains an internal step of **P** and hence contains a **P**-strong, M-clonal pair. Such a clonal pair is clearly **R**-strong.

10.17.5. *If $i, j \in \{1, 2, \ldots, t\}$, and $i < j - 2$, then $R_1 \cup R_2 \cup \cdots \cup R_i$ and $R_j \cup R_{j+1} \cup \cdots \cup R_t$ are skew.*

Proof. Consider X_{4i} and Y_{4j-4}. Then $4i < 4j - 4$, so, by Lemma 10.1, X_{4i} and Y_{4j-4} are skew in $M \backslash B$. Now $X'_{4i} \subseteq \mathrm{cl}(X_{4i})$ and $Y'_{4j} \cup \{b_{4j}\} \subseteq \mathrm{cl}(Y_{4j-4})$. Moreover $X'_{4i} = R_1 \cup R_2 \cup \cdots \cup R_i$ and $Y'_{4j} \cup \{b_{4j}\} \subseteq \mathrm{cl}(Y_{4j-4})$. This establishes the claim. □

It is a straightforward consequence of 10.17.5 that **R** does not display any swirl-like flowers of order 4. The lemma now follows by letting $f_{10.17}(q) = 4(f_{7.36}(4, q) + 2)$. □

6. Simply-bridged paths of 2-separations

Let $\mathbf{P} = (P_0, P_1, \ldots, P_n)$ be a strict path of 2-separations of the connected matroid N and let M be a matroid with an N-minor. Then we say that M *simply bridges* **P** if there is a labelling $V = \{v_0, \ldots, v_n\}$ of $E(M) - E(N)$ and a sequence $N = M_0, M_1, \ldots, M_n = M$ of minors of M such that the following hold.

(i) For all $i \in \{0, 1, \ldots, n-1\}$, the matroid M_{i+1} is either a single-element extension or coextension of M_i by v_i.
(ii) For all $i \in \{0, 1, \ldots, n-1\}$, the partition $(P_0 \cup P_1 \cup \cdots \cup P_i \cup \{v_0, v_1, \ldots, v_{i-1}\}, P_{i+1} \cup P_{i+2} \cup \cdots \cup P_n)$ is a 2-separation of M_i that is bridged by v_i in M_{i+1}.
(iii) Every **P**-relevant 2-separation of N is bridged in M.

The goal of this section is to prove the following lemma.

Lemma 10.18. *Let $\mathbf{P} = (P_0, P_1, \ldots, P_n)$ be a strict path of 2-separations in the connected matroid N that is simply bridged by the matroid M. Assume that each internal step of* **P** *contains a* **P***-strong M-clonal pair. Then there is a function $f_{10.18}(q)$ such that, if $n \geqslant f_{10.18}(q)$, then $M \notin \mathcal{E}(q)$.*

For the remainder of this section we assume that we are under the hypotheses of Lemma 10.18 with labelling for the sequence of minors and set V of bridging elements as given in the definition of "simply bridges".

For $i \in \{0, 1, \ldots, n-1\}$, we say that v_i is a *delete* (respectively *contract*) element of V if M_{i+1} is an extension (respectively coextension) of M_i. We denote the set of delete and contract elements by D and C respectively.

Say that $S = \{s_1, s_2, \dots, s_\omega\}$ is a subset of V. We use the notation $N[S]$ or $N[s_1, s_2, \dots, s_\omega]$ for the matroid $M/(C - S)\backslash(D - S)$. Thus we have $M_i = N[v_1, v_2, \dots, v_{i-1}]$. For clarity we will use the latter notation from now on. The next lemma follows easily from the definition of simply bridges.

Lemma 10.19. *C is independent in M and D is coindependent in M. Hence, if S is a subset of V, then $C \cap S$ is independent in $N[S]$ and $D \cap S$ is coindependent in $N[S]$.*

Define the function α on $\{0, 1, \dots, n-1\}$ as follows: $\alpha(i) = 0$ if v_i bridges P_0 in $N[v_i]$. Otherwise $\alpha(i)$ is the least integer j such that the 2-separating set

$$P_0 \cup P_1 \cup \dots \cup P_{j+1} \cup \{v_0, v_1, \dots, v_j\}$$

of $N[v_0, v_1, \dots, v_j] = M_{j+1}$ is bridged in $N[v_0, v_1, \dots, v_j, v_i]$. It follows from the definition of simply bridged that α is well defined. Moreover, if $\alpha(i) > 0$, then v_i does not bridge the 2-separating set

$$P_0 \cup P_1 \cup \dots \cup P_{\alpha(i)} \cup \{v_0, v_1, \dots, v_{\alpha(i)-1}\}$$

of $N[v_0, \dots, v_{\alpha(i)-1}]$ in $N[v_0, \dots, v_{\alpha(i)-1}, v_i]$.

For an integer i we will denote the collection of delete elements whose indices are less than i by D_i^- and the set whose indices are greater than or equal to i by D_i^+. The sets C_i^- and C_i^+ are defined analogously.

Lemma 10.20. *Say that v_i is a delete element of V. Then*:

(i) $v_i \in \mathrm{cl}_M(P_{\alpha(i)+1} \cup P_{\alpha(i)+2} \cup \dots \cup P_n \cup C^+_{\alpha(i)})$.
(ii) $v_i \in \mathrm{cl}_M(P_0 \cup P_1 \cup \dots \cup P_{i+1} \cup C^-_{i+1})$.

Proof. As the 2-separation $(P_0 \cup P_1 \cup \dots \cup P_{\alpha(i)} \cup \{v_0, v_1, \dots, v_{\alpha(i)-1}\}, P_{\alpha(i)+1} \cup P_{\alpha(i)+2} \cup \dots \cup P_n)$ of $N[v_0, v_1, \dots, v_{\alpha(i)-1}]$ is induced in $N[v_0, v_1, \dots, v_{\alpha(i)-1}, v_i]$, we see that

$$v_i \in \mathrm{cl}_{N[v_0, v_1, \dots, v_{\alpha(i)-1}, v_i]}(P_{\alpha(i)+1} \cup P_{\alpha(i)+2} \cup \dots \cup P_n).$$

Part (i) follows from this fact. We omit the easy proof of (ii). □

Let $\mathbf{Q} = (Q_0, Q_1, \dots, Q_t)$ be a concatenation of $\mathbf{P}$. Then $\mathbf{Q}$ is *tidy* if for all $i \in \{0, 1, \dots, n-1\}$ the following holds: whenever $P_i \subseteq Q_j$, then $P_{\alpha(i)} \subseteq Q_{j-1} \cup Q_j$. We now consider the situation when we can find a tidy concatenation of sufficient length.

Lemma 10.21. *Let $\mathbf{Q} = (Q_0, Q_1, \dots, Q_t)$ be a tidy concatenation of $\mathbf{P}$. Then there is a function $f_{10.21}(q)$ such that if $t \geqslant f_{10.21}(q)$, then $M \notin \mathcal{E}(q)$.*

Proof. We first show that if t is sufficiently large, then we can get s large such that we have one of two cases in M/C.

10.21.1. *There is a function $f_{10.21.1}(s)$ such that if $t \geqslant f_{10.21.1}(s)$, then one of the following holds.*

(i) *There are indices β, γ with $\gamma \geqslant s$ and a concatenation*

$$\mathbf{Q}' = (Q_0 \cup \dots \cup Q_\beta, Q_{\beta+1}, Q_{\beta+2}, \dots, Q_{\beta+\gamma-1}, Q_{\beta+\gamma} \cup \dots \cup Q_t)$$

of $\mathbf{Q}$ such that every $\mathbf{Q}'$-relevant 2-separation of N is bridged in M/C.

(ii) *There is a path* $\mathbf{R}' = (R'_0, R'_1, \dots, R'_s)$ *of* $\mathbf{Q}$*-relevant* 2*-separations of* N *such that,*
  (a) *for all* $i \in \{0, 1, \dots, s\}$*, there is a* $j \in \{0, 1, \dots, t\}$ *such that* $Q_j \cup Q_{j+1} \subseteq R'_i$*, and*
  (b) $\mathbf{R}'$ *induces a strict path* $\mathbf{R} = (R_0, R_1, \dots, R_s)$ *of* 2*-separations in* M/C.

Subproof. Let $f_{10.21.1}(s) = s(4s+3)$ and assume that $t \geqslant f_{10.21.1}(s)$. Assume that (i) does not hold. Then, in any concatenation

$$\mathbf{Q}_k = (Q_0 \cup \dots \cup Q_k, Q_{k+1}, \dots, Q_{s+k-1}, Q_{s+k} \cup \dots \cup Q_t)$$

of $\mathbf{Q}$, there is a $\mathbf{Q}_k$-relevant 2-separation that is induced in M/C. Thus there is a path $\mathbf{R}'' = (R''_0, R''_1, \dots, R''_{4s+3})$ of $\mathbf{Q}$-relevant 2-separations such that

(1) for all $i \in \{0, 1, \dots, 4s+2\}$, if

$$R''_0 \cup R''_1 \cup \dots \cup R''_i = Q_0 \cup Q_1 \cup \dots \cup Q_{j-1} \cup Q'_j,$$

and

$$R''_0 \cup R''_1 \cup \dots \cup R''_{i+1} = Q_0 \cup Q_1 \cup \dots \cup Q_{k-1} \cup Q'_k,$$

then $j < k$; and
(2) every displayed $\mathbf{R}''$-relevant 2-separation is induced in M/C.

For $i \in \{0, 1, \dots, s\}$, let $R'_i = R''_{4i} \cup R''_{4i+1} \cup R''_{4i+2} \cup R''_{4i+3}$, and let $\mathbf{R}' = (R'_0, R'_1, \dots, R'_s)$. Evidently each step of $\mathbf{R}'$ contains two consecutive steps of $\mathbf{Q}$ so that (a) holds. Let $R_0 = \mathrm{cl}_{M/C}(R'_0)$ and, for $i \in \{0, 1, \dots, s\}$, let $R_i = \mathrm{cl}_{M/C}(R'_0 \cup R'_1 \cup \dots \cup R'_i) - \mathrm{cl}_{M/C}(R'_0 \cup R'_1 \cup \dots \cup R'_{i-1})$. It follows from the definition of $\mathbf{R}'$, Lemma 10.20, and Lemma 10.6, that $\mathbf{R}$ is a path of 2-separations in M/C.

There remains the irritating possibility that $\mathbf{R}$ is not a strict path of 2-separations. We now show that we may choose $\mathbf{R}'$ and $\mathbf{R}$ so that $\mathbf{R}$ is indeed strict. If $\mathbf{R}$ is not strict, then there exists a 2-separation (X, Y) of M/C that is not $\mathbf{R}$-relevant such that neither X nor Y is contained in a step of $\mathbf{R}$. By Lemma 10.19, D is a coindependent set of M/C. If $X \subseteq D$, then X is a parallel class and is contained in a step of $\mathbf{R}$ by the definition of $\mathbf{R}$. Thus we may assume that neither X nor Y is contained in D. Let $X' = X - D$ and $Y' = Y - D$. As D is coindependent in M/C and N is connected, we have $\mathrm{cl}_{M/C}(X') \supseteq X$, $\mathrm{cl}_{M/C}(Y') \supseteq Y$, and (X', Y') is a 2-separation in N. As $\mathbf{Q}$ is a strict path in N, we may assume, up to labels, that $X \subseteq Q_\omega$ for some $\omega \in \{1, 2, \dots, t-1\}$. Then, by the definition of $\mathbf{R}$, there is a $\delta \in \{1, 2, \dots, s\}$ such that $Q_\omega \subseteq R'_\delta \cup R'_{\delta+1}$. Moreover, by uncrossing, $\lambda_{M/C}(R'_0 \cup R'_1 \cup \dots \cup R'_\delta \cup X) = 1$. Consider the path

$$\big(R'_0, R'_1, \dots, R'_{\delta-1}, R'_\delta \cup X', R'_{\delta+1} - X', R'_{\delta+2}, \dots, R'_s\big).$$

Observe that the 2-separation $(R'_0 \cup R'_1 \cup \dots \cup R'_\delta \cup X', (R'_{\delta+1} - X') \cup R'_{\delta+2} \cup \dots \cup R'_s)$ is induced in M/C. Repeat the process. As we are always moving sets from a set of higher index to one of lower index, the process must terminate. When it does, and we perform an appropriate relabelling we have found the required paths $\mathbf{R}'$ and $\mathbf{R}$ satisfying the sublemma. $\square$

Assume that we are in case (ii) of 10.21.1 and consider the path $\mathbf{R} = (R_0, R_1, \dots, R_s)$. For $i \in \{1, 2, \dots, t\}$, let $\beta(i)$ denote the least positive integer k such that $Q_i \subseteq R_k \cup R_{k-1}$. For clarity we give an alternative description of β. Say $Q_i \cap R_l \neq \emptyset$. If $Q_i \subseteq R_l$, then $\beta(i) = l$. Otherwise, either $Q_i \subseteq R_l \cup R_{l+1}$, in which case $\beta(i) = l+1$, or $Q_i \subseteq R_l \cup R_{l-1}$, in which case $\beta(i) = l$.

10.21.2. *If* $v_i \in D$*, and* $P_{i+1} \subseteq Q_j$*, then* $v_i \in R_{\beta(j)} \cup R_{\beta(j)-1}$.

Subproof. From the definition of β and the fact that each member of $\mathbf{R}$ contains a member of $\mathbf{Q}$ we see that $Q_j \cup Q_{j-1} \subseteq R_{\beta(j)} \cup R_{\beta(j)-1}$. By Lemma 10.20 and the definition of $\mathbf{Q}$, we see that $v_i \in \mathrm{cl}_M(Q_0 \cup Q_1 \cup \cdots \cup Q_j \cup C)$ and $v_i \in \mathrm{cl}_M(Q_{j-1} \cup Q_j \cup \cdots \cup Q_s \cup C)$. Therefore $v_i \in \mathrm{cl}_{M/C}(Q_0 \cup Q_1 \cup \cdots \cup Q_j)$ and $v_i \in \mathrm{cl}_{M/C}(Q_{j-1} \cup Q_j \cup \cdots \cup Q_s)$. By Lemma 10.6 $v_i \in \mathrm{cl}_{M/C}(Q_{j-1} \cup Q_j)$. Hence $v_i \in \mathrm{cl}_{M/C}(R'_{\beta(j)} \cup R'_{\beta(j)-1})$.

We now prove that $v_i \in R_{\beta(j)} \cup R_{\beta(j)-1}$. If this is not the case, then $v_i \in \mathrm{cl}_{M/C}(R'_0 \cup R'_1 \cup \cdots \cup R'_{\beta(j)-2})$. Observe that, by construction, $Q_{j-2} \cup Q_{j-1} \subseteq R_{\beta(j)} \cup R_{\beta(j)-1}$. But, by the fact that $\mathbf{Q}$ is a tidy concatenation of $\mathbf{P}$, we have $v_i \in \mathrm{cl}_{M/C}(Q_{j-1} \cup \cdots \cup Q_j)$. This yields the contradiction that v_i is a loop of M/C and the sublemma follows. $\square$

10.21.3. *If $v_i \in C$ and $P_{i+1} \subseteq Q_j$, then $v_i \in \mathrm{cl}^*_M(R_{\beta(j)} \cup R_{\beta(j)-1})$.*

Subproof. If $P_{i+1} \subseteq Q_j$, then $P_{\alpha(i)} \subseteq Q_j \cup Q_{j-1}$. By the dual of Lemma 10.20 $v_i \in \mathrm{cl}^*_M(Q_j \cup Q_{j+1} \cup \cdots \cup Q_t \cup D^+_{\alpha(i)})$. By 10.21.2, $D^+_{\alpha(i)} \subseteq R_{\beta(j)-1} \cup R_{\beta(j)} \cup \cdots \cup R_s$. Hence $v_i \in \mathrm{cl}^*_M(R_{\beta(j)-1} \cup R_{\beta(j)} \cup \cdots \cup R_s)$.

Again by the dual of Lemma 10.20 we have $v_i \in \mathrm{cl}^*_M(Q_1 \cup Q_2 \cup \cdots \cup Q_j \cup D^-_i)$, so that $v_i \in \mathrm{cl}^*_M(R_1 \cup R_2 \cup \cdots \cup R_{\beta(j)})$.

This proves that v_i does not coblock either $R_{\beta(j)-1} \cup R_{\beta(j)} \cup \cdots \cup R_s$ or $R_1 \cup R_2 \cup \cdots \cup R_{\beta(j)}$. By the dual of Lemma 10.6 we conclude that $v_i \in \mathrm{cl}^*_M(R_{\beta(j)} \cup R_{\beta(j)-1})$ as required. $\square$

Assume that (i) holds in 10.21.1. By Lemma 10.17, if $s \geqslant f_{10.17}(q)$, then $M \notin \mathcal{E}(q)$. On the other hand if (ii) holds in 10.21.1 then we may apply the dual of Lemma 10.17 and conclude that if $s \geqslant f_{10.17}(q)$, then $M \notin \mathcal{E}(q)$. Thus the lemma holds by letting $f_{10.21}(q) = f_{10.21.1}(f_{10.17}(q))$. $\square$

Proof of Lemma 10.18. Consider the path $\mathbf{P} = (P_1, P_2, \ldots, P_n)$ satisfying the hypotheses of the lemma.

10.18.1. *Assume that $M \in \mathcal{E}(q)$. Then there is a function $f_{10.18.1}(m, l, q)$ such that, if $n \geqslant f_{10.18.1}(m, l, q)$, then there exists an integer $t(m) \in \{1, 2, \ldots, n\}$, and a concatenation $\mathbf{Q} = (Q_0, Q_1, \ldots, Q_{m-1}, Q_m)$ of $\mathbf{P}$, where $Q_m = P_{t(m)} \cup P_{t(m)+1} \cup \cdots \cup P_n$, such that the following hold.*

(i) *$(P_1, P_2, \ldots, P_{t(m)-1}, Q_m)$ is simply bridged by $\{v_1, v_2, \ldots, v_{t(m)}\}$ in $N[v_1, v_2, \ldots, v_{t(m)}]$.*
(ii) *For all i, if $P_i \subseteq Q_j$, then $P_{\alpha(i)} \subseteq Q_{j-1} \cup Q_j$.*
(iii) *Q_{m-1} contains at least l steps of $\mathbf{P}$.*

Subproof. Let $N = 2f_{10.12}(q)$.

We define $f_{10.18.1}(m, l, q)$ inductively. To begin let $f_{10.18.1}(3, l, q) = (l + 1)N$. Assume that $n \geqslant (l + 1)N$. Let S be the subset of V with indices i for which $\alpha(i) = 0$. If $|S| \geqslant N$, then there are either at least $f_{10.12}(q)$ members of D in S or at least $f_{10.12}(q)$ members of C in S. In either case by Lemma 10.12, or its dual, we contradict the assumption that $M \in \mathcal{E}(q)$. Thus there are at most N indices i such that $\alpha_i = 0$. Thus, if $n \geqslant N(l+1)$, there are indices s and t with $t - s \geqslant m$ such that, for all $j \in \{s, s+1, \ldots, t\}$ we have $\alpha_j \geqslant 1$. Let $Q_1 = P_1$, $Q_2 = P_2 \cup P_3 \cup \cdots \cup P_{s-1}$, $Q_3 = P_s \cup P_{s+1} \cup \cdots \cup P_{t-1}$ and $Q_4 = P_t \cup P_{t+1} \cup \cdots \cup P_n$. Clearly (Q_0, Q_1, Q_2, Q_3) satisfies the claim with $m = 3$.

For $m \geqslant 4$, let $f_{10.18.1}(m, l, q) = f_{10.18.1}(m - 1, N(l + 1), q)$. A repeat of essentially the same argument as in the base case establishes the claim. $\square$

Let $m = f_{10.21}(q)$ and let $f_{10.18}(q) = f_{10.18.1}(m, 1, q)$. If $M \in \mathcal{E}(q)$, then 10.18.1 implies that $\mathbf{P}$ has a tidy concatenation $\mathbf{Q}$ of length $f_{10.21}(q)$. By Lemma 10.21, $M \notin \mathcal{E}(q)$. $\square$

7. Sequentially bridged paths of 2-separations

Let $\mathbf{P} = (P_0, P_1, \ldots, P_n)$ be a strict path of 2-separations in the connected matroid N and let M be a matroid having N as a minor. Then $\mathbf{P}$ is *sequentially bridged* in M if M has a sequence $M_0 = N, M_1, \ldots, M_n$ of minors such that the following hold.

(i) For all $i \in \{0, 1, \ldots, n-1\}$, the matroid M_i is a minor of M_{i+1} and M_n is a minor of M. Denote $E(M_{i+1}) - E(M_i)$ by V_i.
(ii) For all $i \in \{0, 1, \ldots, n-1\}$, the 2-separation $(P_0 \cup P_1 \cup \cdots \cup P_{i-1}, P_i \cup \cdots \cup P_n)$ of N is bridged in M_i, and $\lambda_{M_i}(P_1 \cup \cdots \cup P_i \cup V_0 \cup \cdots \cup V_{i-1}) = 1$.
(iii) Every $\mathbf{P}$-relevant 2-separation of N is bridged in M.

Lemma 10.22. *Let N be a matroid with a strict path $\mathbf{P} = (P_0, P_1, \ldots, P_n)$ of 2-separations that is sequentially bridged in the matroid M. Assume that each step of N is an M-clonal pair. Then there is a function $f_{10.22}(q)$ such that, if $n \geqslant f_{10.22}(q)$, then $M \notin \mathcal{E}(q)$.*

Proof. We use the notation of the definition of sequentially bridges. Thus we have matroids $N = M_0, M_1, \ldots, M_n$ such that, for all $i \in \{0, 1, \ldots, n-1\}$ the matroid M_i is a minor of M_{i+1} and M_n is a minor of M. We also have $V_i = E(M_{i+1}) - E(M_i)$. It is straightforwardly seen that we lose no generality in assuming that for all $i \in \{0, 1, \ldots, n-1\}$, the set V_i is a bridging sequence for the 2-separation $(P_0 \cup \cdots \cup P_i \cup V_0 \cup \cdots \cup V_{i-1}, P_{i+1} \cup \cdots \cup P_n)$ of M_i. For $i \in \{0, 1, \ldots, n\}$, let v_i denote the first element of V_i. Let $V = \{v_i\colon\ i \in \{0, 1, \ldots, n-1\}\}$ and let C and D denote the elements of V that are respectively contract and delete elements of their bridging sequences. Let $N' = M_n/C\backslash D$. Observe that

$$\mathbf{P}' = \big(P_0, P_1 \cup \big(V_0 - \{v_0\}\big), \ldots, P_n \cup \big(V_{n-1} - \{v_{n-1}\}\big)\big)$$

is a strict path of 2-separations in N'. Moreover, this path is simply bridged in M_n. Evidently, if $i \in \{1, 2, \ldots, n-1\}$, then P_i is a $\mathbf{P}'$-strong M-clonal pair. The lemma now follows by letting $f_{10.22}(q) = f_{10.18}(q)$. $\square$

Chapter 11. Last rites

It will take only a few more lemmas and we will finally be able to prove the main theorems of the paper.

1. Sequentially bridged paths of 3-separations

We have defined "sequentially bridges" for paths of 2-separations. The extension to paths of 3-separations has no surprises.

Let $\mathbf{P} = (P_0, P_1, \ldots, P_n)$ be a path of 3-separations in the matroid M. A 3-separation (X, Y) of M is $\mathbf{P}$-*relevant* if there is a $j \in \{1, 2, \ldots, n-1\}$ such that either X or Y has the form $P_0 \cup P_1 \cup \cdots \cup P_{j-1} \cup P'_j$ for some subset P'_j of P_j.

Let $\mathbf{P} = (P_0, P_1, \ldots, P_n)$ be a path of 3-separations in the connected matroid N and let M be a matroid having N as a minor. Then $\mathbf{P}$ is *sequentially bridged* in M if M has a sequence $M_0 = N$, $M_1, \ldots, M_n$ of minors such that the following hold.

(i) For all $i \in \{0, 1, \ldots, n-1\}$, the matroid M_i is a minor of M_{i+1} and M_n is a minor of M. Denote $E(M_{i+1}) - E(M_i)$ by V_i.
(ii) For all $i \in \{0, 1, \ldots, n-1\}$, the 3-separation $(P_0 \cup P_1 \cup \cdots \cup P_{i-1}, P_i \cup \cdots \cup P_n)$ of N is bridged in M_i, and $\lambda_{M_i}(P_1 \cup \cdots \cup P_i \cup V_0 \cup \cdots \cup V_{i-1}) \leqslant 2$.
(iii) Every $\mathbf{P}$-relevant 3-separation of N is bridged in M.

Lemma 11.1. *Let M be a matroid with a connected minor N that has a swirl-like flower $\mathbf{P} = (P_0, P_1, \ldots, P_n)$ such that each petal of $\mathbf{P}$, apart from P_0 and P_n are M-clonal pairs, and such that the path $\mathbf{P}$ is sequentially bridged in M. Then there is a function $f_{11.1}(q)$ such that, if $n \geqslant f_{11.1}(q)$, then $M \notin \mathcal{E}(q)$.*

Proof. As $\mathbf{P}$ is sequentially bridged, there is a sequence of minors $N = M_0, M_1, \ldots, M_n$ of M satisfying the definition of sequentially bridged. Let $V_i = E(M_i) - E(M_{i-1})$. For $i \in \{0, 1, \ldots, n-1\}$ let $A_i =$

$V_0 \cup V_2 \cup \cdots \cup V_i$. Observe that, as both P_{n-1} and P_{n-3} are clonal pairs, the set P_{n-2} is coclosed in N, so that $N\backslash P_{n-2}$ is connected. Let $P'_0 = P_0 \cup P_n \cup P_{n-1}$, and let $\mathbf{P}' = (P'_0, P_1, \ldots, P_{n-3})$. By Lemma 10.8 $\mathbf{P}'$ is a strict path of 2-separations in $N\backslash P_{n-2}$.

11.1.1. $\mathbf{P}'$ *is sequentially bridged in* M.

Subproof. Say $i \in \{1, 2, \ldots, n-4\}$. Then $\lambda_{M_i}(P_0 \cup P_1 \cup \cdots \cup P_i \cup A_{i-1}) = 2$ and it follows that $(P_0 \cup P_1 \cup \cdots \cup P_i \cup A_{i-1}, P_{i+1}, \ldots, P_n)$ is a swirl-like flower in M_i. Hence $(P'_0 \cup P_1 \cup \cdots \cup P_i \cup A_{i-1}, P_{i+1}, \ldots, P_{n-3}, P_{n-2})$ is a swirl-like flower in M_i so that $(P'_0 \cup P_1 \cup \cdots \cup P_i \cup A_{i-1}, P_{i+1}, \ldots, P_{n-3})$ is a path of 2-separations in $M_i\backslash P_{n-2}$. Thus $\lambda_{M_i\backslash P_{n-2}}(P'_0 \cup P_1 \cup \cdots \cup P_i \cup A_{i-1}) = 1$.

We now prove that the 2-separation $(P'_0 \cup P_1 \cup \cdots \cup P_{i-1}, P_i \cup \cdots \cup P_{n-3})$ of $\mathbf{P}'$ is bridged in $M_i\backslash P_{n-2}$. Assume otherwise. Then there is a partition (A', A'') of A_{i-1} such that

$$\lambda_{M_i\backslash P_{n-2}}\big(P'_0 \cup P_1 \cup \cdots \cup P_{i-1} \cup A', P_i \cup \cdots \cup P_{n-3} \cup A''\big) = 1.$$

But it is easily seen that

$$r_{M_i}\big(P_i \cup \cdots \cup P_{n-3} \cup A'' \cup P_{n-2}\big) \leqslant r_{M_i}\big(P_i \cup \cdots \cup P_{n-3} \cup A''\big) + 1.$$

Thus $\lambda_{M_i}(P'_0 \cup P_1 \cup \cdots \cup P_{i-1} \cup A') = 2$. By definition $\lambda_{M_i}(P_0 \cup P_1 \cup \cdots \cup P_i \cup A_{i-1}) = 2$. Uncrossing these two 3-separating sets proves that $\lambda_{M_i}(P_0 \cup P_1 \cup \cdots \cup P_{i-1} \cup A') = 2$. Hence $P_0 \cup P_1 \cup \cdots \cup P_{i-1}$ is not bridged in M_i, contradicting the definition of sequentially bridges.

The claim follows from the above facts and the fact that every $\mathbf{P}'$-relevant 2-separation is displayed by $\mathbf{P}'$. □

Let $f_{11.1}(q) = f_{10.22}(q) + 3$. The lemma now follows from Lemma 10.22. □

2. Bridged swirls

The goal of this section is to prove the following theorem.

Theorem 11.2. *Let M be a 4-connected matroid in $E(q)$ and let N be a free-swirl minor of M all of whose clonal pairs are clonal in M. Then there is a function $f_{11.2}(q)$ such that $|E(N)| \leqslant f_{11.2}(q)$.*

Given Lemma 11.1, it is clear that the task is to prove the following lemma.

Lemma 11.3. *Let M be a 4-connected matroid in $\mathcal{E}(q)$ with a Δ_n-minor N whose ground set consists of M-clonal pairs. Then there is a function $f_{11.3}(m, q)$ such that, if $n \geqslant f_{11.3}(m, q)$, then M has a minor with a swirl-like flower $\mathbf{Q} = (Q_0, Q_1, \ldots, Q_m)$, each petal of which apart from Q_0 is an M-clonal pair, having the property that the path $\mathbf{Q}$ is sequentially bridged in M.*

In fact this task is quite straightforward, although notationally unwieldy. In the light of Lemma 10.11, there are no surprises in the next lemma.

Lemma 11.4. *Let M be a matroid in $\mathcal{E}(q)$ with an element b such that $M\backslash b$ is connected with a swirl-like flower $\mathbf{P} = (P_0, P_1, \ldots, P_n)$. Assume that all petals of $\mathbf{P}$, apart from P_0, are M-clonal pairs and that P_0 is blocked by b. Then there is a function $f_{11.4}(m, q)$ such that, if $n \geqslant f_{11.4}(m, q)$, then the following holds. There exist i and j in $\{1, 2, \ldots, n\}$ with $j - i \geqslant m$ such that*

$$\big(P_0 \cup P_1 \cup \cdots \cup P_{i-1} \cup P_{j+1} \cup \cdots \cup P_n \cup \{b\}, P_i, P_{i+1}, \ldots, P_j\big)$$

is a swirl-like flower in M.

Proof. Let $f_{11.4}(m,q) = f_{10.11}(m,q) + m + 2$. Observe that P_{n-1} is coclosed in $M\backslash b$, so by Lemma 10.8, $(P_0 \cup P_n, P_1, \dots, P_{n-2})$ is a strict path of 2-separations in $M\backslash b\backslash P_{n-1}$. Assume that $P_{n-m-2} \cup P_{n-m-1} \cup \cdots \cup P_{n-2}$ is not blocked by b in $M\backslash P_{n-1}$. Then the partition $(\{b\} \cup P_0 \cup P_1 \cup \cdots \cup P_{n-m-3} \cup P_{n-1} \cup P_n, P_{n-m-2}, P_{n-m-1}, \dots, P_{n-2})$ of $E(M)$ clearly satisfies the lemma. Assume then, that $P_{n-m-2} \cup P_{n-m-1} \cup \cdots \cup P_{n-2}$ is blocked by b in $M\backslash P_{n-1}$. Consider the path $(P_n \cup P_0, P_1, \dots, P_{n-m-2} \cup P_{n-m-1} \cup \cdots \cup P_{n-2})$ of 2-separations in $M\backslash b\backslash P_2$. This path satisfies the hypotheses of Lemma 10.11. Applying that lemma gives a swirl-like flower in $M\backslash P_{n-1}$ of the form

$$\big(P_0 \cup P_1 \cup \cdots \cup P_{i-1} \cup P_{j+1} \cup \cdots \cup P_{n-2} \cup P_n \cup \{b\}, P_i, P_{i+1}, \dots, P_j\big),$$

where $j - i \geqslant m$. To complete the proof of the lemma it suffices to observe that $P_{n-1} \subseteq \mathrm{cl}(P_{n-2} \cup P_n)$. $\square$

Extending from a single blocking element to a bridging sequence we obtain:

Lemma 11.5. *Let M be a matroid in $\mathcal{E}(q)$ with a minor N that is connected and has a swirl-like flower $\mathbf{P} = (P_0, P_1, \dots, P_n)$. Assume that all petals of $\mathbf{P}$ except P_0 are M-clonal pairs. Let V_0 be an M-bridging sequence for P_0. Then there is a function $f_{11.5}(m,q)$ such that if $n \geqslant f_{11.5}(m,q)$, then the following holds. There exist i and j in $\{1, 2, \dots, n\}$ with $j - i \geqslant m$ such that*

$$(P_0 \cup P_1 \cup \cdots \cup P_{i-1} \cup P_{j+1} \cup \cdots \cup P_n \cup V_0, P_i, P_{i+1}, \dots, P_j)$$

is a swirl-like flower in $N[V_0]$.

Proof. Say that bridging sequence is $(v_0, v_1, \dots, v_t)$. Up to duality we may assume that v_t is a deletion element. Observe that $(P_0 \cup \{v_0, v_1, \dots, v_{t-1}\}, P_1, P_2, \dots, P_n)$ is a swirl-like flower in $N[V_0]\backslash v_t$ all of whose petals are $N[V_0]$-clonal pairs apart from the initial petal. Moreover, the initial petal is blocked by v_t. The lemma follows from Lemma 11.4 by letting $f_{11.5}(m,q) = f_{11.4}(m,q)$. $\square$

To facilitate the proof of Lemma 11.3 we introduce some terminology. Let $\mathbf{P} = (P_0, P_1, \dots, P_n)$ be a swirl-like flower in a matroid N all of whose petals except P_0 consist of clonal pairs. If $i \in \{1, 2, \dots, n\}$, and $P_i = \{p_i, q_i\}$, then $\mathbf{P} = (P_0, \dots, P_{i-1}, P_{i+1}, \dots, P_n)$ is a swirl-like flower in $N' = N\backslash p_i/q_i = N/p_i\backslash q_i$. In this case we say that N' and $\mathbf{P}'$ are obtained by *removal of* P_i. More generally, if N' and $\mathbf{P}'$ are obtained by a sequence of such operations, then we say that they are obtained from M and $\mathbf{P}$ by *petal removal*.

The next lemma, while lengthy to state, is an immediate corollary of Lemma 2.30.

Lemma 11.6. *Let $\mathbf{P} = (P_0, P_1, \dots, P_n)$ be a swirl-like flower in the matroid N. Assume that all petals of $\mathbf{P}$ except P_0 are clonal pairs. Let V_0 be an M-bridging sequence for P_0. Assume that $1 \leqslant i < j \leqslant n$ and that*

$$(P_0 \cup P_1 \cup \cdots \cup P_{i-1} \cup P_{j+1} \cup \cdots \cup P_n \cup V_0, P_i, P_{i+1}, \dots, P_j)$$

is a maximal swirl-like flower in $N[V_0]$. Then either $i > 1$ or $j < n$.

If $i > 1$ and $N/C\backslash D$ is the matroid obtained by removing the petals $P_1, P_2, \dots, P_{i-2}, P_{j+1}, P_{j+2}, \dots, P_n$ from $\mathbf{P}$ then the following hold.

(i) *The petal P_0 of the swirl-like flower $(P_0, P_{i-1}, P_i, \dots, P_{j-1}, P_j)$ of the matroid $N/C\backslash D$ is bridged in $N[V_0]/C\backslash D$.*
(ii) $\lambda_{N[V_0]/C\backslash D}(P_i \cup P_{i+1} \cup \cdots \cup P_j) = 2$.

On the other hand, if $j < n$ and N' is the matroid obtained by removing the petals $P_1, \dots, P_{i-1}, P_{j+2}, \dots, P_n$ from $\mathbf{P}$ (say $N' = N/C'\backslash D'$), then the following hold.

(iii) *The petal* P_0 *of the swirl-like flower* $(P_0, P_i, \dots, P_j, P_{j+1})$ *of the matroid* $N/C'\backslash D'$ *is bridged in* $N[V_0]/C'\backslash D'$.
(iv) $\lambda_{N[V_0]/C'\backslash D'}(P_i \cup P_{i+1} \cup \cdots \cup P_j) = 2$.

In the next lemma we use the notation $N \preccurlyeq M$ to indicate that N is a minor of M. The lemma is somewhat stronger than we need, but it is set up to facilitate an inductive proof. Note that $n+1 = 0 \bmod n+1$.

Lemma 11.7. *Let* M *be a* 4*-connected matroid in* $\mathcal{E}(q)$ *and let* N *be a minor of* M *with a swirl-like flower* $\mathbf{P} = (P_0, P_1, \dots, P_n)$ *all of whose petals apart from* P_0 *are* M*-clonal pairs. Then there is a function* $f_{11.7}(m,t,q)$ *such that, if* $n \geqslant f_{11.7}(m,t,q)$*, then* N *has a minor* N' *with a swirl-like flower* $(P_0, Q_1, Q_2, \dots, Q_s)$ *obtained by removing petals from* $\mathbf{P}$ *such that there are indices*

$$0 \leqslant l_1 \leqslant l_2 \leqslant \cdots \leqslant l_m < r_m \leqslant r_{m-1} \leqslant \cdots \leqslant r_1 \leqslant s+1$$

and minors $N' = N_0 \preccurlyeq N_1 \preccurlyeq \cdots \preccurlyeq N_m \preccurlyeq M$*, where* $E_i = E(N_i) - E(N_0)$*, such that the following hold.*

(i) $r_m - l_m \geqslant t$.
(ii) *If* $i \in \{2,3,\dots,m\}$*, then* $(l_i - l_{i-1}) + (r_{i-1} - r_i) = 1$*, and* $(l_1, r_1) \in \{(0,s),(1,0)\}$.
(iii) $\lambda_{N_i}(P_0 \cup P_{l_1} \cup \cdots \cup P_{l_i} \cup P_{r_1} \cup \cdots \cup P_{r_i} \cup E_i) = 2$ *for* $i \in \{1,2,\dots,m\}$.
(iv) $P_0 \cup P_{l_1} \cup \cdots \cup P_{l_{i-1}} \cup P_{r_1} \cup \cdots \cup P_{r_{i-1}} \cup E_{i-1}$ *is bridged in* N_i *for* $i \in \{1,2,\dots,m\}$.

Proof. Define $f_{11.7}(m,t,q)$, by $f_{11.7}(1,t,q) = f_{11.5}(t,q)$ and, for $m > 1$ by $f_{11.7}(m,t,q) = f_{11.7}(m-1, f_{11.5}(t,q), q)$.

It follows from Lemmas 11.5 and 11.6 that the lemma holds for $m = 1$ with $N_1 = N[V_0]$ as defined in Lemma 11.6. Say $k \geqslant 1$ and assume that the lemma holds for all t whenever $m \leqslant k$. Let $t' = f_{11.5}(t,q)$ and say that $n \geqslant f_{11.7}(k,t',q)$. Let N_0 be a minor of M with a swirl-like flower $(P_0, Q_1, \dots, Q_{s'})$ for which there exists a sequence of minors $N_0 \preccurlyeq N_1 \preccurlyeq \cdots \preccurlyeq N_k \preccurlyeq M$ with indices $l_1, l_2, \dots, l_k, r_1, r_2, \dots, r_k$ that satisfy the conclusion of the lemma where $r_k - l_k \geqslant t'$. Let

$$Q_0 = P_0 \cup Q_1 \cup \cdots \cup Q_{l_k} \cup Q_{r_k} \cup Q_{r_k+1} \cup \cdots \cup Q_{s'} \cup \big(E(N_k) - E(N_0)\big),$$

and let

$$\mathbf{Q} = (Q_0, Q_{l_k+1}, Q_{l_k+2}, \dots, Q_{r_k-1}).$$

Then $\mathbf{Q}$ is a swirl-like flower in N_k, each petal of which, apart from Q_0 is an M-clonal pair. Therefore, by Lemmas 11.5, 11.6, and the choice of t', there is a minor $N_k/C\backslash D$ of N_k, obtained by removing petals from $\mathbf{Q}$, with a swirl-like flower $\mathbf{Q}' = (Q_0, Q'_1, \dots, Q'_s)$, where $s \geqslant t+1$, for which the following hold.

(i) $N_k/C\backslash D \preccurlyeq N_{k+1} \preccurlyeq M$.
(ii) Q_0 is bridged in N_{k+1}.
(iii) Either $\lambda_{N_{k+1}}(Q'_1 \cup Q'_2 \cup \cdots \cup Q'_{s-1}) = 2$ or $\lambda_{N_{k+1}}(Q'_2 \cup Q'_3 \cup \cdots \cup Q'_s) = 2$.

Consider the sequence

$$N_0/C\backslash D, N_1/C\backslash D, \dots, N_k/C\backslash D, N_{k+1},$$

and the swirl-like flower in $N_0/C\backslash D$ obtained by removing the petals of $(P_0, Q_1, \dots, Q_s)$ that are contained in $C \cup D$. It is now easily verified that the conclusions of the lemma hold for this choice of flower and sequence of minors. $\square$

Proof of Lemma 11.3. Let $f_{11.3}(m,q) = f_{11.7}(2m,1,q)$. By Lemma 11.7 we have the following: a sequence of minors

$$N_0 \preccurlyeq N_1 \preccurlyeq \cdots \preccurlyeq N_{2m} \preccurlyeq M,$$

where $E_i = E(N_i) - E(N_0)$; a swirl-like flower $\mathbf{P} = (P_0, P_1, \ldots, P_n)$ in N_0, all of whose petals apart from P_0 are M-clonal pairs; and a sequence of indices

$$0 \leqslant l_1 \leqslant l_2 \leqslant \cdots \leqslant l_{2m} < r_{2m} \leqslant \cdots \leqslant r_1 \leqslant 2m+1$$

such that the following hold.

(i) If $i \in \{2,3,\ldots,2m\}$, then $(l_i - l_{i-1}) + (r_{i-1} - r_i) = 1$, and $(l_1, r_1) \in \{(0,n),(1,0)\}$.
(ii) $\lambda_{N_i}(P_0 \cup P_{l_1} \cup \cdots \cup P_{l_i} \cup P_{r_1} \cup \cdots \cup P_{r_i} \cup E_i) = 2$ for $i \in \{1,2,\ldots,2m\}$.
(iii) $P_0 \cup P_{l_1} \cup \cdots \cup P_{l_{i-1}} \cup P_{r_1} \cup \cdots \cup P_{r_{i-1}} \cup E_{i-1}$ is bridged in N_i for $i \in \{1,2,\ldots,2m\}$.

Let $\mathcal{N} = \{N_0, N_1, \ldots, N_{2m}\}$ For $i \in \{0,1,\ldots,2m-1\}$ say that $N_i \in \mathcal{N}$ is *bridged on the left* if $l_{i+1} = l_i + 1$; otherwise it is *bridged on the right*. Let l denoted the number of matroids in $\mathcal{N}$ that are bridged on the left. We lose no generality in assuming that $l \geqslant m$. Let $N'_0, N'_1, \ldots, N'_l$ denote the matroids that are bridged on the left, where $N'_0 \preccurlyeq N'_1 \preccurlyeq \cdots \preccurlyeq N'_l$. Let $E'_i = E(N'_i) - E(N'_{i-1})$. The next claim is an immediate consequence of the definitions.

11.3.1. *Say $i \in \{1,2,\ldots,l\}$. Then there is a sequence $n > j_1 \geqslant j_2 \geqslant \cdots \geqslant j_l \geqslant l+2$ such that*

(i) $\lambda_{N'_i}(P_0 \cup P_1 \cup \cdots \cup P_i \cup P_n \cup P_{n-1} \cup \cdots \cup P_{j_i} \cup E'_i) = 2$, *but*
(ii) $P_0 \cup P_1 \cup \cdots \cup P_i \cup P_n \cup P_{n-1} \cup \cdots \cup P_{j_{i-1}} \cup E'_{i-1}$ *is bridged in* N'_i.

Let $P'_0 = P_{l+2} \cup P_{l+3} \cup \cdots \cup P_n \cup P_0 \cup (E(N'_0) - E(N_0))$. An easy uncrossing argument proves

11.3.2. *Say $i \in \{0,1,\ldots,l\}$. Then*

(i) $\lambda_{N'_i}(P'_0 \cup P_1 \cup \cdots \cup P_i \cup E'_i) = 2$, *but*
(ii) $P'_0 \cup P_1 \cup \cdots \cup P_{i-1} \cup E'_{i-1}$ *is bridged in* N'_i.

Let $\mathbf{P}' = (P'_0, P_1, P_2, \ldots, P_{l+1})$. Then $\mathbf{P}'$ is a swirl-like flower in N'_0 all of whose petals, apart from P'_0, are M-clonal pairs. It follows from 11.3.2 that the path $\mathbf{P}'$ is sequentially bridged by the matroids $N'_0, N'_1, \ldots, N'_l$. The lemma now follows from the fact that $l \geqslant m$. $\square$

Proof of Theorem 11.2. Let $f_{11.2}(q) = f_{11.3}(f_{11.1}(q)+1, q)$. Assume that M has a Δ_n-minor all minor of M all of whose petals are M-clonal pairs, where $n \geqslant f_{11.2}(q)$. Then, by Lemma 11.3, for some $m \geqslant f_{11.1}(q)+1$, the matroid M has a minor with a swirl-like flower $\mathbf{Q} = (Q_0, Q_1, \ldots, Q_m)$ all of whose petals, apart from Q_0, are M-clonal pairs with the property that the path $\mathbf{Q}$ is sequentially bridged in M. By Lemma 11.1 $M \notin \mathcal{E}(q)$. $\square$

3. The interment

At long last we are able to bury our skeletons and prove that there are only a finite number of k-skeletons in $\mathcal{E}(q)$.

Theorem 11.8. *Let q and $k \geqslant 5$ be integers. Then there is a function $f_{11.8}(k,q)$ such that, if M is a k-skeleton in $\mathcal{E}(q)$, then M has at most $f_{11.8}(k,q)$ elements.*

Proof. Let $n_1 = f_{11.2}(q) + 1$, let $n_2 = f_{9.1}(n_1, q)$, and let $f_{11.8}(k, q) = f_{8.1}(n_2, k, q)$.

Let M be a k-skeleton and assume that $|E(M)| > f_{11.8}(k, q)$. By Theorem 8.1, M has a 4-connected minor N with a set of n_2 pairwise-disjoint clonal pairs. By Theorem 9.1, N has an N-clonal Δ_{n_1}-minor. By Theorem 11.2, $M \notin \mathcal{E}(q)$. □

Chapter 12. Applications to matroid representability

We are finally in a position to obtain consequences for inequivalent representations of matroids. We begin with results that bound the number of inequivalent representations. Two representations of a matroid over a field $\mathbb{F}$ are *equivalent* if one can be obtained from another by elementary row operations and column scalings. This differs from the definition given in [19] where field automorphisms are also allowed. For the results presented here the difference is not significant—for a finite field, a bound on the number of inequivalent representations with respect to one notion of equivalence implies a bound with respect to the other.

1. Bounding inequivalent representations

Our main theorems bounding inequivalent representations are corollaries of the next theorem.

Theorem 12.1. *Let $k \geqslant 5$ and $q \geqslant 3$ be integers and let $\mathbb{F}$ be a finite field. Then there is a function $f_{12.1}(k, q, \mathbb{F})$ such that a k-coherent member of $\mathcal{E}(q)$ has at most $f_{12.1}(k, q, \mathbb{F})$ inequivalent $\mathbb{F}$-representations.*

To prove Theorem 12.1, we need a few more easy facts. Let $\mathbb{F}$ be a field. Suppose that z is fixed in M, and consider two $\mathbb{F}$-representations of M of the form $[A, y]$ and $[A, y']$, where A represents $M \backslash z$. The matrix $[A, y, y']$ represents a single-element extension of M and it is easily checked that y and y' are clones in this matroid. Since z is fixed, $\{y, y'\}$ is a parallel pair. Thus $[A, y]$ and $[A, y']$ are equivalent. This shows that, up to equivalence, any representation of $M \backslash z$ extends to at most one representation of M. Part (i) of the next lemma follows from this argument. Part (ii) is the dual of part (i).

Lemma 12.2. *Let z be an element of the matroid M and $\mathbb{F}$ be a finite field.*

(i) *If z is fixed in M, then the number of inequivalent $\mathbb{F}$-representations of M is at most the number of inequivalent $\mathbb{F}$-representations of $M \backslash z$.*
(ii) *If z is cofixed in M, then the number of inequivalent $\mathbb{F}$-representations of M is at most the number of inequivalent $\mathbb{F}$-representations of M/z.*

Recall that wheels have one k-skeleton minor, namely $U_{2,3}$ and whirls have an additional one, namely $U_{2,4}$.

Lemma 12.3. *Let M be a k-coherent matroid and let $\mathbb{F}$ be a finite field. Then the number of inequivalent $\mathbb{F}$-representations of M is bounded above by the maximum of the number of inequivalent $\mathbb{F}$-representations of members of the set of k-skeleton minors of M.*

Proof. Assume that M is not a k-skeleton. If M is a wheel, then M is uniquely $\mathbb{F}$-representable, as is $U_{2,3}$. If M is a whirl, then it is well known and easily seen that the number of inequivalent $\mathbb{F}$-representations of M is equal to that of $U_{2,4}$. So the lemma holds in these trivial cases. Assume that M is not a wheel or a whirl. Then by Corollary 4.4, up to duality, there is an element $x \in E(M)$ such that x is fixed in M and $M \backslash x$ is k-coherent and the lemma holds by Lemma 12.2 and an obvious induction. □

Proof of Theorem 12.1. By Theorem 11.8 there is a finite number of k-skeletons in $\mathcal{E}(q)$. Let $f_{12.1}(k, q, \mathbb{F})$ denote the maximum of the number of inequivalent $\mathbb{F}$-representations of a k-skeleton in $\mathcal{E}(q)$. It follows from Lemma 12.3 that a k-coherent matroid in $\mathcal{E}(q)$ has at most $f_{12.1}(k,q,\mathbb{F})$ inequivalent $\mathbb{F}$-representations as required. $\square$

It follows from [12, Lemma 11.6] that, if p is a prime that exceeds 3, then Λ_p is not $GF(p)$-representable. Certainly neither $U_{2,p+2}$ nor $U_{p,p+2}$ is $GF(p)$-representable. We therefore have

Lemma 12.4. *Let $p \geqslant 3$ be a prime. Then the class of $GF(p)$-representable matroids is contained in $\mathcal{E}(p)$.*

The next corollary follows immediately from Theorem 12.1 and Lemma 12.4.

Corollary 12.5. *Let $k \geqslant 5$ be an integer, p be a prime number and $\mathbb{F}$ be a finite field. Then a k-coherent $GF(p)$-representable matroid has at most $f_{12.1}(k,p,\mathbb{F})$ inequivalent representations over $\mathbb{F}$.*

A special case of Corollary 12.5 is Theorem 1.3. We restate it here for convenience.

Theorem 12.6. *Let $k \geqslant 5$ be an integer and p be a prime number. Then there is a function $f_{12.6}(k,p)$ such that a k-coherent matroid has at most $f_{12.6}(k,p)$ inequivalent representations over $GF(p)$.*

Finally Theorem 1.1, stated in the introduction, follows from Theorem 1.3 as 4-connected matroids are k-coherent.

2. Excluding a free swirl

If, as well as excluding a free spike we exclude a free swirl, we bound the number of inequivalent representations of 3-connected matroids over a finite field.

A 3-connected matroid M is *totally free* if it is not a wheel or a whirl of rank at least three, and has the properties that, for all $x \in E(M)$, if $M\backslash x$ is 3-connected, then x is not fixed in M, and if M/x is 3-connected, then x is not cofixed in M. If, in addition to excluding $U_{2,q+2}$, $U_{q,q+2}$ and Λ_q, we also exclude Δ_q, then, as well as having only a finite number of k-skeletons, we have only a finite number of totally-free matroids. It is proved in [12], and easily seen, that the maximum number of inequivalent representations of a 3-connected matroid over a finite field is bounded above by that of its totally-free minors. Let $EX(U_{2,q+2}, U_{q,q+2}, \Lambda_q, \Delta_q)$, denote the class of matroids with no $U_{2,q+2}$-, $U_{q,q+2}$-, Λ_q- or Δ_q-minor.

Theorem 12.7. *Let q be a positive integer. Then there are a finite number of totally-free matroids in $EX(U_{2,q+2}, U_{q,q+2}, \Lambda_q, \Delta_q)$.*

Proof. Let M be a totally-free matroid. Observe that, if M is k-coherent, then M is also a k-skeleton. By Theorem 11.8, there are a finite number of k-skeletons in $\mathcal{E}(q)$ for any fixed k. Thus, if the theorem fails, there is a totally-free matroid $M \in EX(U_{2,q+2}, U_{q,q+2}, \Lambda_q, \Delta_q)$ that is not q-coherent. Let $l = |E(M)|$. Then any 3-connected minor of M is l-coherent. Hence M is an l-skeleton. As M is not q-coherent, M has a swirl-like flower of order q. By Corollary 6.12, M has a Δ_q-minor, contradicting the assumption that $M \in EX(U_{2,q+2}, U_{q,q+2}, \Lambda_q, \Delta_q)$. $\square$

The next corollary follows from Theorem 12.7 and the observation prior to it.

Corollary 12.8. *Let $q \geqslant 3$ be an integer and $\mathbb{F}$ be a finite field. Then there is a function $f_{12.8}(q,\mathbb{F})$ such that a 3-connected matroid in $E(U_{2,q+2}, U_{q,q+2}, \Lambda_q, \Delta_q)$ has at most $f_{12.8}(q,\mathbb{F})$ inequivalent $\mathbb{F}$-representations.*

Excluding both a free swirl and a free spike is a significant constraint, but it is not so severe that we lose all interesting classes. We give one illustration. Let p be a prime. Recall that p is a *Mersenne prime* if $p = 2^m - 1$ for some integer m. It is a well-known and widely believed conjecture that the number of Mersenne primes is infinite. We first note that if p is a Mersenne prime, then not all free swirls are representable over $GF(p+1)$. While this is widely known, there does not appear to be a proof in the literature so we give one here. Readers familiar with bias matroids of group-labelled graphs, see for example Zaslavsky [31], will find the proof particularly obvious. The bound that the proof provides is certainly not tight. Note that the converse of Lemma 12.9 holds in that, if $\mathbb{F}$ is a finite field and $|\mathbb{F}|-1$ is not prime, then all free swirls are $\mathbb{F}$-representable.

Lemma 12.9. *Let $\mathbb{F}$ be a finite field such that $|\mathbb{F}|-1$ is prime. Then there is an integer $f_{12.9}(|\mathbb{F}|)$, such that, if $n \geqslant f_{12.9}(|\mathbb{F}|)$, then Δ_n is not $\mathbb{F}$-representable.*

Proof. Let M_n denote a matroid whose ground set consists of a basis $B=\{b_1,b_2,\dots,b_n\}$ together with a set $\{e_1,f_1,e_2,f_2,\dots,e_n,f_n\}$ such that, for $i\in\{1,2,\dots,n-1\}$, the elements e_i and f_i are placed freely on the line spanned by $\{b_i,b_{i+1}\}$ and e_n, f_n are placed freely on the line spanned by $\{b_n,b_1\}$.

Recall that $\Delta_n \cong M_n\backslash B$. We will refer to the members of $\{\{e_1,f_1\},\{e_2,f_2\},\dots,\{e_n,f_n\}\}$ as the *legs* of M_n. Note that a representation of Δ_n induces a representation of M_n by adding the points of intersection of the legs of Δ_n. Therefore M_n is representable over a field $\mathbb{F}$ if and only if Δ_n is. Thus we lose no generality in focussing on representations of M_n.

Let $n=|\mathbb{F}|^2$. We show that M_n is not $\mathbb{F}$-representable. Note that $\Delta_3\cong U_{3,6}$ and that $U_{3,6}$ is not $GF(3)$-representable, so the lemma holds in this case. Thus we may assume that $\mathbb{F}$ has even order and hence that $-a=a$ for all $a\in\mathbb{F}$.

Assume that M_n is $\mathbb{F}$-representable. Consider a standard representation of M_n relative to the basis B. For convenience we identify elements of the legs of M_n with the column vectors that they label. Up to scaling we may assume, for $i\in\{1,2,\dots,n-1\}$, that $e_i=(0,\dots,0,1,1,0,\dots,0)^T$ and that $f_i=(0,\dots,0,1,\alpha_i,0,\dots,0)^T$, where $\alpha_i\neq 0$ and the nonzero entries are in the ith and $(i+1)$th coordinates. We may also assume that $f_n=(\alpha_n,0,\dots,0,1)^T$. Note that e_n plays no role in this argument.

Consider a transversal T of the legs of M_n that contains f_n. It is easily verified that the matrix labelled by T has determinant $\Pi_{f_i\in T}(\alpha_i)$. As $n=|\mathbb{F}|^2$, there is an element $\alpha\in\mathbb{F}^*$ such that $\alpha_i=\alpha$ for at least $|\mathbb{F}^*|$ members of $\{1,2,\dots,n-1\}$. As M_n is simple, $\alpha\neq 1$. As $\mathbb{F}^*$ has prime order, α generates $\mathbb{F}^*$. Thus, for some $l\in\{1,2,\dots,|\mathbb{F}|^*\}$, we have $\alpha^l=\alpha_n^{-1}$. It now follows that there is a transversal of the legs that labels a square matrix whose determinant is 0, contradicting the assumption that we have a representation of M_n. □

For a prime power q, let $\mathcal{R}(q)$ denote the class of matroids representable over all fields of size at least q. Thus, $\mathcal{R}(2)$ and $\mathcal{R}(3)$ are the classes of regular and near-regular matroids respectively.

Theorem 12.10. *Let $\mathbb{F}$ be a finite field and q be a prime power. If there is a Mersenne prime greater than or equal to $q-1$, then there is a function $f_{12.10}(\mathbb{F},q)$ such that a 3-connected matroid in $\mathcal{R}(q)$ has at most $f_{12.10}(\mathbb{F},q)$ inequivalent $\mathbb{F}$-representations.*

Proof. Assume that there is a Mersenne prime greater than or equal to $q-1$. Then, by Lemma 12.9, there is a free swirl that is not in $\mathcal{R}(q)$. Certainly there is a prime p greater than q, so there is a free spike that is not in $\mathcal{R}(q)$. It follows that there is an integer q' such that the members of $\mathcal{R}(q)$ are in $EX(U_{2,q'+2},U_{q',q'+2},\Lambda_{q'},\Delta_{q'})$. The theorem now follows from Corollary 12.8. □

Corollary 12.11. *There are infinitely many Mersenne primes if and only if, for each prime power q, there is a number $\rho(q)$ such that a 3-connected member of $\mathcal{R}(q)$ has at most $\rho(q)$ inequivalent $GF(7)$-representations.*

Proof. If there are infinitely many Mersenne primes, then the corollary follows from Theorem 12.10. Assume that there are a finite number of Mersenne primes. Let q be a prime power larger than the largest Mersenne prime. Then $\mathcal{R}(q)$ contains all free swirls. But it is shown in [25] that Δ_n has at least 2^n inequivalent $GF(7)$-representations, so that no bound can be placed on the number of inequivalent $GF(7)$-representations of members of $\mathcal{R}(q)$. □

3. Certifying non-representability

As noted in the introduction, one of the main applications of the material in this paper is in [14] where the following theorem is proved.

Theorem 12.12. *(See [14, Theorem 1.1].) For any prime p, proving that an n-element matroid is not representable over $GF(p)$ requires at most $O(n^2)$ rank evaluations.*

The purpose of this section is to adumbrate a straightforward consequence of earlier material from this paper that is optimised for the application in [14]. We consider a slight augmentation of the class of k-coherent matroids. A matroid M is *near* k-coherent if it is connected and either $\mathrm{si}(M)$ or $\mathrm{co}(M)$ is k-coherent.

Corollary 12.13. *Let p be a prime number and $k \geqslant 5$ be an integer. Then the following hold.*

(i) *If M is a nonempty near k-coherent matroid, then there is an element $e \in E(M)$ such that either $M\backslash e$ or M/e is near k-coherent.*
(ii) *Let M be a near k-coherent and $e \in E(M)$. Then the following hold.*
    (a) *If M/e is not near k-coherent, then there is a 4-separating partition (A, B) of $E(M)$ such that $e \in \mathrm{cl}_M(A - \{e\}) \cap \mathrm{cl}_M(B - \{e\})$.*
    (b) *If $M\backslash e$ is not near k-coherent, then there is a 4-separating partition (A, B) of $E(M)$ such that $e \in \mathrm{cl}^*_M(A - \{e\}) \cap \mathrm{cl}^*_M(B - \{e\})$.*
(iii) *There is an integer μ_p such that a near k-coherent matroid has at most μ_p inequivalent representations over $GF(p)$.*

Proof. Consider (i). If M is not k-coherent, then we may either delete an element in a non-trivial parallel class or contract an element in a non-trivial series class to preserve near k-coherence. If M is k-coherent then it follows that from Corollary 4.4 that M has an element e such that either $M\backslash e$ or M/e is k-coherent unless M is a wheel or a whirl. Assume that M is a wheel or a whirl. Let e be a rim element. Observe that M/e is near k-coherent.

Consider condition (ii). Let M be a near k-coherent matroid. Assume that $e \in E(M)$ and that M/e is not near k-coherent. Then e is not in a non-trivial series class. Assume that e is in a non-trivial parallel class. Then e is in the guts of a 2-separation in M and (ii) holds. If M is 3-connected, then it follows from Lemma 4.2 that e is in the guts of either a 3-separation or a 4-separation in M. The upgrade to the case when M has non-trivial series classes or parallel classes which do not contain e is elementary and is omitted. The case when $M\backslash e$ is not near k-coherent follows by duality.

Consider (iii). Note that we can either delete fixed elements or contract cofixed elements from a near k-coherent matroid to obtain a k-coherent matroid. Part (iii) follows from this observation, Lemma 12.2 and Theorem 12.6. □

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