Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Combinatorial Theory, Series B 96 (2006) 405–425
Journal of Combinatorial Theory Series B
www.elsevier.com/locate/jctb

# On Rota's conjecture and excluded minors containing large projective geometries

Jim Geelen[a], Bert Gerards[b, c], Geoff Whittle[d]

[a] *Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada*
[b] *CWI, Postbus 94079, 1090 GB Amsterdam, The Netherlands*
[c] *Department of Mathematics and Computer Science, Eindhoven University of Technology, Postbus 513, 5600 MB Eindhoven, The Netherlands*
[d] *School of Mathematical and Computing Sciences, Victoria University, Wellington, New Zealand*

Received 13 September 2004
Available online 9 November 2005

**Abstract**

We prove that an excluded minor for the class of $\mathrm{GF}(q)$-representable matroids cannot contain a large projective geometry over $\mathrm{GF}(q)$ as a minor.


*MSC:* 05B35

*Keywords:* Matroids; Rota's conjecture; Excluded minors

## 1. Introduction

We prove the following theorem.

**Theorem 1.1.** *For each prime power q, there exists an integer k such that no excluded minor for the class of* $\mathrm{GF}(q)$*-representable matroids contains a* $\mathrm{PG}(k, q)$*-minor.*

We recall that $\mathrm{PG}(k, q)$ is the rank-$(k+1)$ projective geometry over $\mathrm{GF}(q)$.

Rota's conjecture states that: *for any prime power q, there are only finitely many pairwise nonisomorphic excluded minors for the class of* $\mathrm{GF}(q)$*-representable matroids.* Theorem 1.1 shows that

0095-8956/$ - see front matter 
doi:10.1016/j.jctb.2005.09.005

excluded minors cannot contain large projective geometries. On the other hand, in [5] we prove that for any integer $k$ there are only finitely many excluded minors that do not contain the cycle matroid of a $k \times k$ grid. While there is still a big gap to bridge between grids and projective geometries, we are encouraged by these complementary results.

We conjecture the following strengthening of Theorem 1.1; however, it is not clear whether this stronger version would provide additional leverage toward resolving Rota's conjecture.

**Conjecture 1.2.** *For each prime power $q$, no excluded minor for the class of* GF$(q)$*-representable matroids contains a* PG$(2, q)$*-minor.*

Oxley, Vertigan, and Whittle [8] gave examples showing that, for each $q > 5$, there is no bound on the number of inequivalent representations for 3-connected matroids over GF$(q)$. This is in stark contrast with the following result, which plays a key role in the proof of Theorem 1.1.

**Theorem 1.3.** *If $M$ is a* 3*-connected* GF$(q)$*-representable matroid with a* PG$(q, q)$*-minor, then $M$ is uniquely* GF$(q)$*-representable.*

We conjecture that this result can be sharpened to:

**Conjecture 1.4.** *If $M$ is a* 3*-connected* GF$(q)$*-representable matroid with a* PG$(2, q)$*-minor, then $M$ is uniquely* GF$(q)$*-representable.*

We use the notation of Oxley [7], with the exception that the simplification of $M$ is denoted by si$(M)$ and the cosimplification of $M$ is denoted by co$(M)$.

## 2. Connectivity

Let $M$ be a matriod. For any subset $A$ of $E(M)$ we let $\lambda_M(A) = r_M(A) + r_M(E(M) - A) - r_M(E(M))$; $\lambda_M$ is the *connectivity function* of $M$. For sets $A, B \subseteq E(M)$, we have

(i) $\lambda_M(A) = \lambda_M(E(M) - A)$,
(ii) $\lambda_M(A) \leqslant \lambda_M(A \cup \{e\}) + 1$ for each $e \in E(M)$, and
(iii) $\lambda_M(A) + \lambda_M(B) \geqslant \lambda_M(A \cup B) + \lambda_M(A \cap B)$.

It can be easily verified that $\lambda_M(X) = r_M(X) + r_{M^*}(X) - |X|$ and, hence, that $\lambda_M(X) = \lambda_{M^*}(X)$.

We let $\kappa_M(X_1, X_2) = \min\,(\lambda_M(A) \,:\, X_1 \subseteq A \subseteq E(M) - X_2)$. Note that if $M'$ is a minor of $M$ and $X_1, X_2 \subseteq E(M')$, then $\kappa_{M'}(X_1, X_2) \leqslant \kappa_M(X_1, X_2)$. The following theorem provides a good characterization for $\kappa_M(X_1, X_2)$; this theorem is in fact a generalization of Menger's theorem.

**Theorem 2.1** (*Tutte's Linking Theorem [10]*). *Let $M$ be a matroid and let $X_1, X_2$ be disjoint subsets of $E(M)$. Then there exists a minor $M'$ of $M$, such that $E(M') = X_1 \cup X_2$ and $\lambda_{M'}(X_1) = \kappa_M(X_1, X_2)$.*

The following result shows that, if we apply Tutte's Linking Theorem when $\lambda_M(X_1) = \kappa_M(X_1, X_2)$, the resulting minor $M'$ satisfies $M|X_1 = M'|X_1$.

**Lemma 2.2.** *Let $M'$ be a minor of a matroid $M$ and let $X \subseteq E(M)$. If $\lambda_M(X) = \lambda_{M'}(X)$, then $M|X = M'|X$.*

**Proof.** Note that

$$\begin{aligned}\lambda_M(X) &= r_M(X) + r_{M^*}(X) - |X| \\ &\leqslant r_{M'}(X) + r_{M'^*}(X) - |X| \\ &= \lambda_{M'}(X).\end{aligned}$$

Therefore, if $\lambda_M(X) = \lambda_{M'}(X)$, then $r_M(X) = r_{M'}(X)$ and, hence, $M|X = M'|X$. □

3-*connectivity*: The rest of this section is devoted to the proof of a connectivity result, Lemma 2.8, that is needed in Section 6.

A matroid $M$ is *internally* 3-*connected* if $M$ is connected and for any 2-separation $(A, B)$ of $M$ either $|A| = 2$ or $|B| = 2$. We require the following well-known results on 3-connected matroids.

**Theorem 2.3** (*Bixby's Lemma [2]*). *If $e$ is an element of a* 3*-connected matroid, then either $M \setminus e$ or $M/e$ is internally* 3*-connected.*

**Theorem 2.4** (*Tutte's Triangle Lemma [11]*). *Let $T = \{a, b, c\}$ be a triangle in a* 3*-connected matroid $M$ with $|E(M)| \geqslant 4$. If neither $M \setminus a$ nor $M \setminus b$ is* 3*-connected, then there is a triad of $M$ that contains $a$ and exactly one of $b$ and $c$.*

**Theorem 2.5** (*Wheels and Whirls Theorem [11]*). *Let $M$ be a* 3*-connected matroid with $E(M) \neq \emptyset$. If $M$ is not a wheel or a whirl, then there exists $e \in E(M)$, such that $M \setminus e$ or $M/e$ is* 3*-connected.*

**Corollary 2.6.** *If $M$ is a* 3*-connected matroid with $E(M) \neq \emptyset$, then there exists $e \in E(M)$ such that* $\mathrm{si}(M/e)$ *is* 3*-connected.*

**Proof.** By the Wheels and Whirls Theorem, we can find a sequence of elements $e_1, \ldots, e_k$, such that

(i) $M \setminus e_1, \ldots, e_i$ is 3-connected for each $i \in \{1, \ldots, k\}$, and
(ii) either $M \setminus e_1, \ldots, e_k$ is a wheel or a whirl, or there exists an element $e$ of $M \setminus e_1, \ldots, e_k$ such that $(M \setminus e_1, \ldots, e_k)/e$ is 3-connected.

In both cases arising from (ii), there exists an element $e$ of $M \setminus e_1, \ldots, e_k$, such that $\mathrm{si}((M \setminus e_1, \ldots, e_k)/e)$ is 3-connected. But then $\mathrm{si}(M/e)$ is also 3-connected, as required. □

**Lemma 2.7.** *Let $T$ be a triangle in a* 3*-connected matroid $M$ with $|E(M)| \geqslant 4$. Then there exists $e \in T$ such that $M \setminus e$ is internally* 3*-connected.*

**Proof.** Suppose otherwise. The result can be readily checked on matroids with at most 6 elements, so we assume that $|E(M)| \geqslant 7$. By Tutte's Triangle Lemma, there exists a triad $T^*$ with $|T \cap T^*| = 2$; let $e \in T - T^*$. Note that, $(T^*, E(M) - T^*)$ is a 2-separation in $M/e$. Then $M/e$ is not internally 3-connected since $|E(M)| \geqslant 7$. So, by Bixby's Lemma, $M \setminus e$ is internally 3-connected. □

The following lemma is the main result of this section.

**Lemma 2.8.** *Let $M$ be a 3-connected matroid with $|E(M)|\geqslant 5$. Suppose that no element of $M$ is in both a triangle and a triad. Then there exist $u, v \in E(M)$ such that either*:

(1) *$M \setminus u$ and $M \setminus v$ are 3-connected, and $M \setminus u, v$ is internally 3-connected, or*
(2) *$M/u$ and $M/v$ are 3-connected, and $M/u, v$ is internally 3-connected.*

**Proof.** Suppose that $M$ is a counterexample. Let $\Lambda(M)$ denote the set of elements $e \in E(M)$ such that $M \setminus e$ is 3-connected, and let $\Lambda^*(M)$ denote $\Lambda(M^*)$. The first three claims are straightforward, we leave the details to the reader.

**2.8.1.** *$r(M)\geqslant 4$ and $r^*(M)\geqslant 4$.*

**2.8.2.** *If $e \in \Lambda(M)$, then $\Lambda(M \setminus e) = \emptyset$.*

**2.8.3.** *If $N$ is a 3-connected matroid, $e \in \Lambda(N)$, and $f \in \Lambda^*(N \setminus e)$, then either $f \in \Lambda^*(N)$ or there is a triangle of $N$ containing both $e$ and $f$.*

**2.8.4.** $\Lambda(M) \cup \Lambda^*(M) = E(M)$.

**Proof.** Suppose not; then there exists $e \in E(M)$ such that neither $M \setminus e$ nor $M/e$ is 3-connected. By Bixby's Lemma and duality, we may assume that $M/e$ is internally 3-connected. But then, since $M/e$ is not 3-connected, $e$ is in a triangle, say $T = \{e, a, b\}$. Now $M \setminus e$ is not 3-connected and neither $a$ nor $b$ is in a triad. Then, by Tutte's Triangle Lemma, both $M \setminus a$ and $M \setminus b$ are 3-connected. (We will obtain a contradiction by proving that $M \setminus a, b$ is internally 3-connected.) Let $(A, B)$ be a 2-separation in $M \setminus e$ with $a \in A$. Note that $b \in B$, since otherwise $(A \cup \{e\}, B)$ would be a 2-separation in $M$. Since neither $a$ nor $b$ is in a triad, $|A|, |B|\geqslant 3$. Moreover, since $|E(M)|\geqslant 8$, by possibly swapping $A$ and $B$ we may assume that $|A|\geqslant 4$. Note that, $(A, B \cup \{e\})$ is a 3-separation in $M$, and $a \in \mathrm{cl}_M(B \cup \{e\})$. Thus $(A - \{a\}, B \cup \{e\})$ is a 2-separation in $M/a$ and, hence $(A - \{a\}, (B \cup \{e\}) - \{b\})$ is a 2-separation in $M/a \setminus b$. Thus $(M \setminus b)/a$ is not internally 3-connected. However, $M \setminus b$ is 3-connected, so, by Bixby's Lemma, $M \setminus a, b$ is internally 3-connected. □

It follows from 2.8.4 that, if $e$ is in a triangle, then $M \setminus e$ is 3-connected, and if $e$ is in a triad, then $M/e$ is 3-connected.

**2.8.5.** *If $T$ is a triangle of $M$, then $\Lambda(M) \subseteq T$.*

**Proof.** Suppose, by way of contradiction, that there exists $e \in \Lambda(M) - T$. Thus $M \setminus e$ is 3-connected. Then, by Lemma 2.7, there exists $f \in T$ such that $M \setminus e, f$ is internally 3-connected. Moreover, by 2.8.4, $M \setminus f$ is 3-connected. □

**2.8.6.** *$M$ contains no triangles and no triads.*

**Proof.** Suppose otherwise; then, by duality, we may assume that $M$ has a triangle $T$. By 2.8.4 and 2.8.5, $\Lambda(M) = T$ and $\Lambda^*(M) = E(M) - T$. Thus $T$ is the only triangle of $M$, and, since $\Lambda^*(M) > 3$, $M$ contains no triads. Let $e \in E(M) - T$. By the duals of 2.8.2 and 2.8.3, $\Lambda^*(M/e) = \emptyset$ and $\Lambda(M/e) \subseteq T$.

Since $r(M) \geqslant 4$, there exists $f \in E(M/e) - \mathrm{cl}_{M/e}(T)$. As $f \notin T$ and $\Lambda(M/e) \subseteq T$, the minor $(M/e) \setminus f$ is not 3-connected. Moreover, since $M/e$ has no triads, $(M/e) \setminus f$ is not internally 3-connected. So, by Bixby's Lemma, $M/e, f$ is internally 3-connected. □

**2.8.7.** *If* $e \in \Lambda(M)$ *and* $f \in E(M \setminus e)$, *then* $M \setminus e, f$ *is not internally* 3-*connected.*

**Proof.** Suppose that $M \setminus e, f$ is internally 3-connected. Then $M \setminus f$ is not 3-connected. Let $(A, B)$ be a 2-separation in $M \setminus f$ with $e \in A$. Since $M$ has no triads, $|A|, |B| \geqslant 3$. However, $(A - \{e\}, B)$ is a 2-separation in $M \setminus e, f$ and $M \setminus e, f$ is internally 3-connected, so $|A| = 3$. But, $\lambda_M(A) = 2$ so $A$ is a triangle or a triad, contradicting 2.8.6. □

**2.8.8.** $\Lambda(M) = E(M)$ and $\Lambda^*(M) = E(M)$.

**Proof.** By symmetry we may assume that there exists $e \in \Lambda(M)$. By 2.8.7, for each $f \in E(M \setminus e)$, the minor $M \setminus e, f$ is not internally 3-connected. Then, by Bixby's Lemma, $M \setminus e/f$ is internally 3-connected. Moreover, since $M \setminus e$ has no triangles, $M \setminus e/f$ is 3-connected. Thus $\Lambda^*(M \setminus e) = E(M \setminus e)$. So, by 2.8.3 and 2.8.6, $E(M) - \{e\} \subseteq \Lambda^*(M)$. Now, since $|\Lambda^*(M)| \geqslant 2$, we can argue that $\Lambda(M) = E(M)$. Now $|\Lambda(M)| \geqslant 2$, so $\Lambda^*(M) = E(M)$. □

Let $e \in E(M)$. By Corollary 2.6, there exists $f \in E(M/e)$ such that $\mathrm{si}(M/e, f)$ is 3-connected. However, by the dual of 2.8.7, $M/e, f$ is not internally 3-connected. Thus, there is a 4-point line $L$ in $M/e$ that contains $f$. (That is, the restriction of $M/e$ to $L$ is isomorphic to $U_{2,4}$.) Note that $M/e$ has no triads. Then, by Tutte's Triangle Lemma, there exists $a \in L$ such that $M/e \setminus a$ is 3-connected. Now, by Lemma 2.7, there exists $b \in L - \{a\}$ such that $M/e \setminus a, b$ is internally 3-connected. If $M/e \setminus a, b$ were 3-connected, then $M \setminus a, b$ would be internally 3-connected, contradicting 2.8.7. Thus $M/e \setminus a, b$ has a series-pair $\{c, d\}$. Since $M/e$ has no triads, $\{a, b, c, d\}$ is a cocircuit of $M/e$. Since a circuit and a cocircuit cannot meet in exactly one element, $|L \cap \{a, b, c, d\}| \geqslant 3$. Moreover, since $M/e$ is 3-connected and has at least 7 elements, $L \neq \{a, b, c, d\}$. By symmetry, we may assume that $d \notin L$. Now $M/e \setminus d$ is not internally 3-connected. So, by Bixby's Lemma, $M/e, d$ is internally 3-connected, contradicting 2.8.7. □

## 3. Unique representation

In this section we prove Theorem 1.3.

Let $\mathbb{F}$ be a field and let $M$ be a matroid. Two $\mathbb{F}$-representations of $M$ are *algebraically equivalent* if one can be obtained from the other by elementary row operations, column scaling, and field automorphisms. A matroid $M$ is *uniquely* $\mathbb{F}$-representable if it is $\mathbb{F}$-representable and any two $\mathbb{F}$-representations of $M$ are algebraically equivalent. The following result is referred to as the Fundamental Theorem of Projective Geometry (see [1, p. 85]).

**Theorem 3.1.** *For each prime power* $q$ *and integer* $k \geqslant 2$, *the projective geometry* $\mathrm{PG}(k, q)$ *is uniquely* $\mathrm{GF}(q)$-*representable.*

Two $\mathbb{F}$-representations of $M$ are *projectively equivalent* if one can be obtained from the other by elementary row operations, and column scaling. Two representations that are not projectively equivalent are said to be *projectively inequivalent*. By Theorem 3.1, the number of projectively inequivalent representations of $\mathrm{PG}(k, q)$, for $k \geqslant 2$, is $|\mathrm{Aut}(\mathrm{GF}(q))|$ where $\mathrm{Aut}(\mathrm{GF}(q))$ is the automorphism group of GF($q$). Let $N$ be a minor of $M$. We say that $N$ *stabilizes* $M$ *over* $\mathbb{F}$ if no $\mathbb{F}$-representation of $N$ can be extended to two projectively inequivalent $\mathbb{F}$-representations of $M$.

*Clones*: Let $e$ and $f$ be distinct elements of $M$. We call $e$ and $f$ *clones* if there is an automorphism of $M$ that swaps $e$ and $f$ and that acts as the identity on all other elements of $M$; that is, $e$ and $f$ are clones if $r_M(X \cup \{e\}) = r_M(X \cup \{f\})$ for each set $X \subseteq E(M) - \{e, f\}$.

**Lemma 3.2.** *Let $e$ be an element of a matroid $M$ and let $\mathbb{F}$ be a field. If $M \setminus e$ does not stabilize $M$ over $\mathbb{F}$, then there exists an $\mathbb{F}$-representable matroid $M'$ with $E(M') = E(M) \cup \{f\}$ such that $M = M' \setminus f$, and $e$ and $f$ are independent clones in $M'$.*

**Proof.** If $M \setminus e$ does not stabilize $M$ over $\mathbb{F}$, then there is an $\mathbb{F}$-representation, say $A$, of $M \setminus e$ that extends to two projectively inequivalent $\mathbb{F}$-representations, say $[A, v_1]$ and $[A, v_2]$, of $M$. Let $M'$ be the $\mathbb{F}$-representable matroid represented by the matrix $[A, v_1, v_2]$ where the last two columns are indexed by $e$ and $f$, respectively. Clearly $e$ and $f$ are clones and, since the representations $[A, v_1]$ and $[A, v_2]$ are projectively inequivalent, $\{e, f\}$ is independent in $M'$. □

**Lemma 3.3.** *Let $M$ be a 3-connected GF($q$)-representable matroid and let $L \subseteq E(M)$ be a line of $M$. If $|L| \geqslant q$ and $e, f \in E(M) - L$, then $e$ and $f$ are not clones.*

**Proof.** Since $M$ is 3-connected, $\kappa_M(L, \{e, f\}) = 2$. Then, by Tutte's Linking Theorem, there exists a minor $N$ of $M$ with $E(N) = L \cup \{e, f\}$ and $\lambda_N(L) = 2$. Since $\lambda_N(L) = 2$, it follows that $r_N(\{e, f\}) = r_N(L) = 2$ and that $e, f \in \mathrm{cl}_N(L)$. Thus $r(N) = 2$. However, $N$ is GF($q$)-representable and $|E(N)| \geqslant q + 2$. Thus $N$ contains a parallel pair $\{x, y\}$. Now $\{e, f\}$ is not a parallel pair in $N$ and $N|L = M|L$, so $L$ does not contain a parallel pair. Thus $\{x, y\}$ contains one element of $\{e, f\}$ and one element of $L$. It follows that $e$ and $f$ are not clones in $N$, and, hence, they are not clones in $M$. □

**Lemma 3.4.** *Let $e$ and $f$ be clones in a matroid $M$. If $M \setminus e$ is 3-connected and $M$ is not 3-connected, then $e$ and $f$ are parallel.*

**Proof.** If $e$ and $f$ are clones and $M \setminus e$ is 3-connected, then $M \setminus f$ is also 3-connected and si($M$) is 3-connected. Thus, if $M$ is not 3-connected, then $e$ and $f$ are in parallel. □

The following lemma is a key step in the proof of Theorem 1.3.

**Lemma 3.5.** *Let $e$ and $f$ be elements of a 3-connected GF($q$)-representable matroid $M$. If $M/e, f$ is isomorphic to PG($q, q$), then $e$ and $f$ are not clones in $M$.*

**Proof.** Let $N = M/e, f$ and suppose that $e$ and $f$ are clones. By Lemma 3.5, $M$ has no $q$-point lines. So, if $L$ is a $(q + 1)$-point line of $N$, then $r_M(L) \in \{3, 4\}$. Moreover, since $M$ has 2-point lines, $q > 2$.

**3.5.1.** *There exists a rank-3 flat $P$ of $N$ such that $e, f \in \mathrm{cl}_M(P)$.*

**Subproof.** Suppose not. Then, for each line $L$ of $N$, we have $r_M(L) = 3$. Consider $M$ as a restriction of PG($q+2, q$), and let $Z$ be the line in PG($q+2, q$) spanned by $e$ and $f$. Each $(q+1)$-

point line $L$ of $N$ spans a plane in $\mathrm{PG}(q+2, q)$, and this plane intersects $Z$ in a point, say $z_L$. Suppose that there are two lines $L_1$ and $L_2$ of $N$ such that $z_{L_1} \neq z_{L_2}$. If $L_1$ and $L_2$ do not meet at a point, then consider a third line $L_3$ of $N$ that meets both $L_1$ and $L_2$. Note that either $z_{L_3} \neq z_{L_1}$ or $z_{L_3} \neq z_{L_2}$. Therefore, by possibly replacing one of $L_1$ and $L_2$ with $L_3$, we may assume that $L_1$ and $L_2$ meet at a point. Let $P = \mathrm{cl}_N(L_1 \cup L_2)$. Now $e$ and $f$ are spanned by $\{z_{L_1}, z_{L_2}\}$ and $z_{L_1}$ and $z_{L_2}$ are spanned by $L_1 \cup L_2$ in $\mathrm{PG}(q+2, q)$, so $e, f \in \mathrm{cl}_M(L_1 \cup L_2) \subseteq \mathrm{cl}_M(P)$. Now $P$ is a rank-3 flat of $N$ and $e, f \in \mathrm{cl}_M(P)$, as required.

Thus we may assume that there exists $z \in Z$, such that $z = z_L$ for each $(q+1)$-point line $L$ of $N$. Let $M'$ be the restriction of $\mathrm{PG}(q+2, q)$ obtained by adding $z$ to $M$. Now, since $\{e, f, z\}$ is a line, $M'/e, z \setminus f = M'/e, f \setminus z = N$. Since $M$ is 3-connected, $M'/z \setminus f$ is connected. Thus $e$ is in the closure of $E(N)$ in $M'/z \setminus f$. So there is a circuit $C$ of $N$ such that $C$ is independent in $M'/z$; among all such circuits we choose $C$ as small as possible. Note that, each line of $N$ is also a line of $M'/z$; thus $|C| > 3$. Let $(I_1, I_2)$ be a partition of $C$ into two sets with $|I_1|, |I_2| \geqslant 2$. Since $C$ is a circuit of $N$ and since $N$ is a projective geometry, there exists a unique element $a$ in $\mathrm{cl}_N(I_1) \cap \mathrm{cl}_N(I_2)$. Now $I_1 \cup \{a\}$ and $I_2 \cup \{a\}$ are both circuits of $N$ and are both smaller than $C$. Thus, by our choice of $C$, $I_1 \cup \{a\}$ and $I_2 \cup \{a\}$ are both circuits in $M'/z$. However, this implies that $C = I_1 \cup I_2$ is dependent in $M'/z$. This contradiction completes the proof. □

**3.5.2.** *If $P$ is a rank-3 flat of $N$, then there exists a restriction $K$ of $N$ such that $E(K) = P \cup L'$ where $L'$ is a $q$-point line in $K^*$.*

**Subproof.** Let $H$ be a matroid with $E(H) = L \cup \{a, b, c\}$, where $L$ is a $q$-point line of $H$ and $a$, $b$, and $c$ are placed in parallel with distinct elements of $L$ (recall that $q > 2$). Note that, $H$ is $\mathrm{GF}(q)$-representable, $H$ is cosimple, and $r^*(H) = q + 1$. Thus there is a spanning restriction $H'$ of $N$ that is isomorphic to $H^*$. Now let $E(H') = L' \cup \{a', b', c'\}$ where $a', b', c'$ are the elements corresponding to $a, b, c$. By the symmetry of $N$, we may assume that $a', b', c' \in P$. Finally, let $K = N|(L' \cup P)$; it is straightforward to check that $K$ has the desired properties. □

Let $P$ be the rank-3 flat of $N$ given by 3.5.1, let $K$ be the restriction of $N$ given by 3.5.2, and let $K'$ be the restriction of $M$ to $E(K) \cup \{e, f\}$. Thus $K'/e, f = K$. Since $e, f \in \mathrm{cl}_{K'}(P)$, the elements $e$ and $f$ are not in series. Then, by the dual of Lemma 3.4, $K'$ is 3-connected. Moreover, since $L'$ is a $q$-point coline of $K$, it is also a coline in $K'$. Thus, by applying the dual of Lemma 3.3 to $K'$ we obtain a final contradiction. □

*Stabilizers for a class of matroids*: We say that $N$ *stabilizes* a class $\mathcal{M}$ of matroids *over* $\mathbb{F}$ if $N$ stabilizes each 3-connected matroid in $\mathcal{M}$ that contains $N$ as a minor. For brevity, when $N$ stabilizes the class of $\mathbb{F}$-representable matroids over $\mathbb{F}$, we simply say that $N$ is a *stabilizer* for $\mathbb{F}$.

**Lemma 3.6.** *Let $q$ be a prime power and let $N$ be a uniquely* $\mathrm{GF}(q)$*-representable stabilizer for* $\mathrm{GF}(q)$. Then $N$ has $|\mathrm{Aut}(\mathrm{GF}(q))|$ *projectively inequivalent representations.*

**Proof.** This follows easily from Theorem 3.1 and the fact that $N$ is a stabilizer for all projective geometries of sufficiently large rank. □

The following result shows that to test whether $N$ stabilizes $\mathcal{M}$ we need only check matroids $M \in \mathcal{M}$ with $r(M) \leqslant r(N) + 1$ and $r^*(M) \leqslant r^*(N) + 1$.

**Theorem 3.7** (*Whittle [12]*). *Let $\mathcal{M}$ be a class of matroids that is closed with respect to taking minors, duality, and isomorphism. A* 3*-connected matroid $N \in \mathcal{M}$ stabilizes $\mathcal{M}$ with respect to a field $\mathbb{F}$ if and only if N stabilizes each* 3*-connected matroid $M \in \mathcal{M}$ satisfying one of the following conditions*:

(i) $N = M \setminus e$ *for some* $e \in E(M)$,
(ii) $N = M/e$ *for some* $e \in E(M)$, or
(iii) $N = M \setminus e/f$ *for some* $e, f \in E(M)$ *where* $M \setminus e$ *and* $M/f$ *are both* 3*-connected.*

We can now prove one of the main results of the paper.

**Theorem 3.8.** *For each prime power q*, $\mathrm{PG}(q, q)$ *is a stabilizer for* $\mathrm{GF}(q)$.

**Proof.** Let $M$ be a 3-connected $\mathrm{GF}(q)$-representable matroid with a minor $N$ isomorphic to $\mathrm{PG}(q, q)$. Since there are no 3-connected $\mathrm{GF}(q)$-representable extensions of $\mathrm{PG}(q, q)$, then, by Theorem 3.7, it suffices to consider the case that $N = M/e$ for some $e \in E(M)$.

Suppose that $M$ is not stabilized by $N$. Then, by applying the dual of Lemma 3.2, we see that there exists a matroid $M'$ with $E(M') = E(M) \cup \{f\}$ such that $M'/f = M$, the elements $e$ and $f$ are clones in $M'$, and $\{e, f\}$ is coindependent in $M'$. Since $\{e, f\}$ is coindependent in $M'$, $e$ and $f$ are not in series in $M'$. Then, by the dual of Lemma 3.4, $M'$ is 3-connected. This contradicts Lemma 3.5. □

Theorem 1.3 is an immediate consequence of Theorems 3.8 and 3.1.

## 4. Path-width

Let $M$ be a matroid on $E$. The *path-width* of $M$ is the least integer $k$, such that there exists an ordering $(e_1, \ldots, e_n)$ of $E$, such that $\lambda_M(\{e_1, \ldots, e_i\}) \leqslant k$ for all $i \in \{1, \ldots, n\}$. In the remainder of the paper we shift our attention from Theorem 1.1 to the following result.

**Theorem 4.1.** *For any prime power q, there exists an integer k such that, each excluded minor for the class of* $\mathrm{GF}(q)$*-representable matroids that contains a* $\mathrm{PG}(q+6, q)$*-minor has path-width at most k.*

Theorem 4.1 implies Theorem 1.1. Indeed, it is straightforward to show that $\mathrm{PG}(k+1, q)$ has path-width $k+2$, and that path-width is non-increasing with respect to taking minors. Then, by Theorem 4.1, there is no excluded minor for the class of $\mathrm{GF}(q)$-representable matroids that contains a $\mathrm{PG}(k+1, q)$-minor, proving Theorem 1.1.

Let $\mathcal{A} = (A_1, \ldots, A_l)$ be an ordered partition of $E$. We let $\rho_M(\mathcal{A}) = \max(\lambda_M(A_1 \cup \cdots \cup A_i) : i \in \{1, \ldots, l\})$. We use the following two lemmas to obtain bounds on the path-width.

**Lemma 4.2.** *Let M be a matroid,* $\mathcal{A} = (A_1, \ldots, A_l)$ *and* $\mathcal{B} = (B_1, \ldots, B_m)$ *be two ordered partitions of* $E(M)$*, and let* $\mathcal{C} = (A_1 \cap B_1, A_1 \cap B_2, \ldots, A_1 \cap B_m, \ldots, A_l \cap B_1, A_l \cap B_2, \ldots, A_l \cap B_m)$*. Then* $\rho_M(\mathcal{C}) \leqslant 2\rho_M(\mathcal{A}) + \rho_M(\mathcal{B})$.

**Proof.** For each $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, m\}$, we let

$$\widehat{A}_i = A_1 \cup \cdots \cup A_i,$$

$$\begin{aligned}\widehat{B}_j &= B_1 \cup \cdots \cup B_j, \text{ and}\\ S_{ij} &= ((A_1 \cap B_1) \cup \cdots \cup (A_1 \cap B_m)) \cup \cdots\\ &\quad \cup ((A_{i-1} \cap B_1) \cup \cdots \cup (A_{i-1} \cap B_m)) \cup \cdots\\ &\quad \cup ((A_i \cap B_1) \cup \cdots \cup (A_i \cap B_j))\\ &= \widehat{A}_{i-1} \cup (\widehat{A}_i \cap \widehat{B}_j).\end{aligned}$$

Now there exists $i \in \{1, \dots, l\}$ and $j \in \{1, \dots, m\}$, such that $\rho_M(\mathcal{C}) = \lambda_M(S_{ij})$. By submodularity,

$$\begin{aligned}\lambda_M(\widehat{A}_{i-1} \cup (\widehat{A}_i \cap \widehat{B}_j)) &\leqslant \lambda_M(\widehat{A}_{i-1}) + \lambda_M(\widehat{A}_i) + \lambda(\widehat{B}_j)\\ &\leqslant 2\rho_M(\mathcal{A}) + \rho_M(\mathcal{B}).\end{aligned}$$

Therefore $\rho_M(\mathcal{C}) = \lambda_M(S_{ij}) = \lambda_M(\widehat{A}_{i-1} \cup (\widehat{A}_i \cap \widehat{B}_j)) \leqslant 2\rho_M(\mathcal{A}) + \rho_M(\mathcal{B})$, as required. □

**Lemma 4.3.** *Let $A$, $B$, and $X$ be disjoint sets of elements in a matroid $M$ such that, for each $e \in X$, either $\kappa_{M\backslash e}(A, B) < \kappa_M(A, B)$ or $\kappa_{M/e}(A, B) < \kappa_M(A, B)$. Then there exists an ordering $(e_1, \dots, e_m)$ of $X$ and a partition $(Y_0, \dots, Y_m)$ of $E(M) - X$ such that $A \subseteq Y_0$, $B \subseteq Y_m$, and $\rho_M(Y_0, \{e_1\}, Y_1, \dots, \{e_m\}, Y_m) = \kappa_M(A, B)$.*

**Proof.** Let $k = \kappa_M(A, B)$. The result is vacuous when $X = \emptyset$. Suppose then that $X$ is non-empty and let $e \in X$. Now, inductively, we can find an ordering $(e_1, \dots, e_m)$ of $X - \{e\}$ and a partition $(Y_0, \dots, Y_m)$ of $E(M) - (X - \{e\})$ such that $A \subseteq Y_0$, $B \subseteq Y_m$, and $\rho_M(Y_0, \{e_1\}, Y_1, \dots, \{e_m\}, Y_m) = \kappa_M(A, B)$. Now $e \in Y_i$ for some $i \in \{0, \dots, m\}$. Define

$$A' = \begin{cases} A & \text{if } i = 0,\\ (Y_0 \cup \cdots \cup Y_{i-1}) \cup \{e_1, \dots, e_i\} & \text{if } i > 1\end{cases}$$

and

$$B' = \begin{cases} B & \text{if } i = m,\\ (Y_{i+1} \cup \cdots \cup Y_m) \cup \{e_{i+1}, \dots, e_m\} & \text{if } i < m.\end{cases}$$

By duality we may assume that $\kappa_{M/e}(A, B) < k$. Thus there exists a partition $(X_1, X_2)$ of $E(M/e)$ with $A \subseteq X_1$, $B \subseteq X_2$, and $\lambda_{M/e}(X_1) = k - 1$. It follows that $\lambda_M(X_1) = \lambda_M(X_1 \cup \{e\}) = k$ and that $e \in \mathrm{cl}_M(X_1) \cap \mathrm{cl}_M(X_2)$. If $A' = A$, then $A' \subseteq X_1$. On the other hand, if $A' \neq A$, then $\lambda_M(A') = k$. Then, by submodularity, $\lambda_M(A' \cap X_1) = k$ and $\lambda_M(A' \cup X_1) = k$. So, by replacing $X_1$ by $A' \cup X_1$, we get $A' \subseteq X_1$. Thus, in either case, we may assume that $A' \subseteq X_1$. Similarly, we may assume that $B' \subseteq X_2$. Finally, we get $\rho_M(Y_0, \{e_1\}, \dots, Y_{i-1}, \{e_{i-1}\}, Y_i \cap X_1, \{e\}, Y_i \cap X_2, \{e_{i+1}\}, Y_{i+1}, \dots, \{e_m\}, Y_m) = k$, as required. □

## 5. Final preparations

The following lemma is well-known; we prove it here for the sake of completeness.

**Lemma 5.1.** *Let $\mathbb{F}$ be a field and let $M$ be an excluded minor for the class of $\mathbb{F}$-representable matroids. If $|E(M)| \geqslant 5$ then no element of $M$ is in both a triangle and a triad.*

**Proof.** Suppose, by way of contradiction that $e \in E(M)$ is in both a triangle $T$ and a triad $T^*$. Note that $|T \cap T^*| \geqslant 2$. Since $M$ is 3-connected and $|E(M)| \geqslant 5$, we cannot have $T = T^*$. Thus $|T \cap T^*| = 2$; suppose that $T = \{e_1, e_2, e_3\}$ and $T^* = \{e_2, e_3, e_4\}$. Let $N$ be a matroid isomorphic to $M(K_4)$, where one of the triangles in $N$ is labelled by $\{e_1, e_2, e_3\}$. Now let $M'$ be obtained by taking the generalized parallel connection of $M/e_4$ and $N$ across the triangle $\{e_1, e_2, e_3\}$. Since $M/e_4$ is $\mathbb{F}$-representable, so is $M'$. However, $M' \setminus e_2, e_3$ is isomorphic to $M$. This contradiction completes the proof. □

**Lemma 5.2.** *Let $M$ be a* GF$(q)$*-representable matroid and let $N$ be a minor of $M$ isomorphic to* PG$(k+2, q)$*. Then for each $e \in E(M)$ there exists a restriction $N'$ of $N$ isomorphic to* PG$(k, q)$ *such that $N'$ is a minor of both $M \setminus e$ and $M/e$.*

**Proof.** By deleting or contracting the other elements in a way that keeps $N$ as a minor, we may assume that $E(M) = E(N) \cup \{e\}$. The result is straightforward if $e \in E(N)$; so assume that $e \notin E(N)$. We may also assume that $e$ is neither a loop nor a coloop.

First consider the case that $N = M \setminus e$. Since $M$ is GF$(q)$-representable, $e$ is in parallel with some element $e' \in E(N)$. Since $e' \in E(N)$, there is a restriction $N'$ of $N$ isomorphic to PG$(k, q)$ such that $N'$ is a minor of both $M \setminus e'$ and $M/e'$. Thus, since $e$ and $e'$ are in parallel, $N'$ is a minor of both $M \setminus e$ and $M/e$.

Now consider the case that $N = M/e$. Since $e$ is not a coloop of $M$, there exists some triangle $T$ of $N$ such that $e \in \mathrm{cl}_M(T)$. Choose a restriction $N'$ of $N$ isomorphic to PG$(k, q)$ such that $r_N(T \cup E(N')) = r(N') + 2$. Thus $N'$ is a minor of $N/T$ and hence also of $M/T$. However, $e$ is a loop in $M/T$. So $N'$ is a minor of both $M/e$ and $M \setminus e$. □

A matroid $M$ is called *stable* if it is connected and it cannot be written as the 2-sum of two non-binary matroids. This differs from the original definition in [4] since we require that $M$ is connected. Suppose that $\eta_q(M)$ denotes the number of GF$(q)$-representations of $M$ up to projective equivalence. It is easy to see that if $M$ is the 2-sum of $M_1$ and $M_2$, then $\eta_q(M) = \eta_q(M_1)\eta_q(M_2)$. Moreover, if $M$ is a binary matroid, then $\eta_q(M) = 1$. It follows that if $M$ is a stable GF$(q)$-representable matroid, then by repeatedly decomposing across 2-separations we will obtain a 3-connected matroid $M'$ such that $\eta_q(M) = \eta_q(M')$. It follows that if $N$ is a stabilizer for GF$(q)$, and if $M$ is a stable matroid that contains $N$ as a minor, then $N$ stabilizes $M$ over GF$(q)$.

The following two lemmas can be derived from results in [12]; we include direct proofs for completeness.

**Lemma 5.3.** *Let $M$ be a* 3*-connected matroid, let $u, v \in E(M)$ be such that $M \setminus u, v$ is stable, and suppose that $M \setminus u, v$ has a minor $N$ that is uniquely* GF$(q)$*-representable and is a stabilizer for* GF$(q)$*. If $M \setminus u$ and $M \setminus v$ are both* GF$(q)$*-representable, then there exists a* GF$(q)$*-representable matroid $M'$, such that $M' \setminus u = M \setminus u$ and $M' \setminus v = M \setminus v$.*

**Proof.** Let $B$ be a basis of $M$ containing neither $u$ nor $v$. Consider GF$(q)$-representations $A_1$ and $A_2$ of $M \setminus u$ and $M \setminus v$, respectively. By applying row operations we may assume that:

$$A_1 = \begin{pmatrix} \overset{B}{I} & C_1 & \overset{v}{y} \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} \overset{B}{I} & C_2 & \overset{u}{x} \end{pmatrix}.$$

Thus $(I, C_1)$ and $(I, C_2)$ are both GF$(q)$-representations of $M \setminus u, v$. However, $M \setminus u, v$ is uniquely GF$(q)$-representable since $N$ is a minor of $M \setminus u, v$. Therefore, by possibly applying a field automorphism and rescaling, we may assume that $C_1 = C_2$. Now let $M'$ be the matroid represented over GF$(q)$ by

$$\begin{matrix} B & & u & v \\ \big(I & C_1 & x & y\big). \end{matrix}$$

Clearly $M' \setminus u = M \setminus u$ and $M' \setminus v = M \setminus v$, as required. □

**Lemma 5.4.** *Let $M_1$ and $M_2$ be* GF$(q)$*-representable matroids on the same ground set and let $u, v \in E(M_1)$ be such that $M_1 \setminus u = M_2 \setminus u$* and *$M_1 \setminus v = M_2 \setminus v$. If $M_1 \setminus u$ and $M_2 \setminus v$ are both stable, $M_1 \setminus u, v$ is connected, and $M_1 \setminus u, v$ has a minor N that is uniquely* GF$(q)$*-representable and is a stabilizer for the class of* GF$(q)$*-representable matroids, then $M_1 = M_2$.*

**Proof.** Since $M_1 \setminus u$ and $M_1 \setminus v$ are connected, $\{u, v\}$ is co-independent. Thus there exists a basis $B$ of $M_1$ disjoint from $u$ and $v$. For each $i \in \{1, 2\}$, consider a GF$(q)$-representation $A_i$ of $M_i$ where:

$$\begin{matrix} & B & & u & v \\ A_i = & \big(I & C_i & x_i & y_i\big). \end{matrix}$$

Now $(I, C_1, x_1)$ and $(I, C_2, x_2)$ are both representations of $M_1 \setminus v$. However, $M_1 \setminus v$ is uniquely GF$(q)$-representable since it is stable and contains $N$ as a minor. Therefore, by possibly applying a field automorphism and rescaling, we may assume that $C_2 = C_1$ and $x_2 = x_1$. So we may assume that $A_2 = (I, C_1, x_1, y_2)$. Now we have two representations, $(I, C_1, y_1)$ and $(I, C_1, y_2)$, of $M_1 \setminus u$ and, since $M_1 \setminus u$ is stable and contains $N$ as a minor, these representations are algebraically equivalent. Consider the operations required to transform $(I, C_1, y_1)$ into $(I, C_1, y_2)$; we have at our disposal row operations, column scaling, and field automorphisms. The common identity matrix limits the row operations to row scaling. Since $M_1 \setminus u, v$ contains $N$ as a minor and since, by Theorem 3.1, $N$ has $|\mathrm{Aut}(\mathrm{GF}(q))|$ weakly inequivalent representations, we cannot apply field automorphisms (while keeping $(I, C_1)$ and $(I, C_2)$ projectively equivalent). Moreover, since $M_1 \setminus u, v$ is connected, the only scalings that we may apply to $(I, C_1)$ without changing it are trivial (that is, multiply every row by a constant $\alpha$ and divide all columns by $\alpha$). Therefore $y_2$ is obtained from $y_1$ by scaling, and, hence, $M_2 = M_1$. □

The next result is considerably harder to prove; we defer the proof to Sections 8–10. Before stating the result we need some definitions. If $M_1$ and $M_2$ are two matroids on a common ground set, then a set $B$ is said to *distinguish* $M_1$ from $M_2$ if $B$ is a basis of exactly one of $M_1$ and $M_2$. Let $X$ be a set of elements in a matroid $M$. We say that $X$ is *connected* in $M$ if $X$ is contained in a single component of $M$. We say that $X$ is *3-connected* in $M$ if $X$ is connected and for any partition $(X_1, X_2)$ of $X$ with $|X_1|, |X_2| \geqslant 2$ we have $\kappa_M(X_1, X_2) \geqslant 2$.

**Lemma 5.5.** *Let $M$, $M'$, and $N$ be matroids, let $B$ be a basis of $M$, let $u, v \in E(M) - B$, and let $a, b \in B$ be such that*

(1) *$M'$ is a* GF$(q)$*-representable matroid on the same ground set as $M$, $M' \setminus u = M \setminus u$, $M' \setminus v = M \setminus v$, and $(B - \{a, b\}) \cup \{u, v\}$ distinguishes $M$ from $M'$;*
(2) *$N$ is a uniquely* GF$(q)$*-representable stabilizer for* GF$(q)$ *and $N$ is a minor of $M \setminus u, v$; and*
(3) *$E(N) \cup \{a, b, u\}$ is* 3*-connected in $M \setminus v$ and $E(N) \cup \{a, b, v\}$ is* 3*-connected in $M \setminus u$.*

*Then $M$ is not* GF$(q)$*-representable.*

## 6. Proof of Theorem 4.1

Let $s$ denote the number of elements of $\mathrm{PG}(q,q)$, and let $t$ be the number of $\mathrm{PG}(q,q)$ restrictions of $\mathrm{PG}(q+2,q)$. In this section we prove Theorem 4.1 with $k = 24t2^{s+3} + 4$.

Let $M$ be an excluded minor for the class of $\mathrm{GF}(q)$-representable matroids. Suppose by way of contradiction that $M$ contains a $\mathrm{PG}(q+6,q)$- or a $\mathrm{PG}(q+6,q)^*$-minor and that the path-width of $M$ is greater than $k$. By Lemma 5.1, no element of $M$ is in both a triangle and a triad. Therefore, by Lemma 2.8 and by possibly replacing $M$ with $M^*$, we may assume that there exist elements $u, v \in E(M)$ such that $M \setminus u$ and $M \setminus v$ are 3-connected and $M \setminus u, v$ is internally 3-connected. By Lemma 5.2, $M \setminus u, v$ has a $\mathrm{PG}(q+2,q)$- or a $\mathrm{PG}(q+2,q)^*$-minor $N$. Therefore, by Lemma 5.3 and Theorem 3.8, there exists a $\mathrm{GF}(q)$-representable matroid $M'$ on the same ground set as $M$ such that $M' \setminus u = M \setminus u$ and $M' \setminus v = M \setminus v$. Moreover, by Lemma 5.4, $M'$ is unique.

**6.1.** *There exists a basis $B$ of $M$ and elements $a, b \in B$ such that $u, v \notin B$ and $(B - \{a,b\}) \cup \{u,v\}$ distinguishes $M$ from $M'$.*

**Proof.** Suppose that $B'$ distinguishes $M$ from $M'$. Since $M$ is 3-connected, there exists a basis $B$ of $M$ that is disjoint from $\{u,v\}$; we choose such $B$ minimizing $|B' - B|$. Note that $|B| = |B'|$ and that $u, v \in B' - B$; thus, if $|B' - B| = 2$, then 6.1 holds (take $a$ and $b$ to be the two elements in $B - B'$). Hence, we may assume that $|B' - B| > 2$; let $x \in (B' - B) - \{u,v\}$. By one of the standard basis exchange axioms, there exists $y \in B - B'$ such that $(B \cup \{x\}) - \{y\}$ is a basis of at least one of $M$ and $M'$; let $B'' = (B \cup \{x\}) - \{y\}$. Since $u, v \notin B''$, $B''$ does not distinguish $M$ from $M'$. Thus $B''$ is a basis of $M$ that contains neither $u$ nor $v$. However, $|B' - B''| < |B' - B|$, contradicting our choice of $B$. □

Let $N' \in \{N, N^*\}$ be isomorphic to $\mathrm{PG}(q+2,q)$, and let $N'_1, \ldots, N'_t$ be the $\mathrm{PG}(q,q)$-restrictions of $N'$. Now, for each $i \in \{1,\ldots,t\}$, let $N'_i = N_i$ if $N' = N$ and let $N'_i = N^*_i$ if $N' = N^*$. Let $Z = E(M) - \{a,b,u,v\}$. Now, for each $i \in \{1,\ldots,t\}$, let $Z_i$ denote the set of all elements $e \in Z$ such that $(M \setminus u, v) \setminus e$ and $(M \setminus u, v)/e$ both contain $N_i$ as a minor. By Lemma 5.2, each element in $Z$ is contained in at least one of $Z_1, \ldots, Z_t$.

For each $i \in \{1,\ldots,t\}$, let $\Pi_i(u)$ denote the set of all partitions $(A_1, A_2)$ of $E(N_i) \cup \{a,b,v\}$ such that $\kappa_{M\setminus u}(A_1,A_2) = 2$, and let $\Pi_i(v)$ denote the set of all partitions $(A_1,A_2)$ of $E(N_i) \cup \{a,b,u\}$ such that $\kappa_{M\setminus v}(A_1,A_2) = 2$. Recall that $|E(N_i)| = s$, so we trivially get $|\Pi_i(u)|, |\Pi_i(v)| \leqslant 2^{s+3}$.

**6.2.** *For each $e \in Z_i$ either*

(a) *there exists $(A_1,A_2) \in \Pi_i(u)$ such that either $\kappa_{(M\setminus u)\setminus e}(A_1,A_2) < 2$ or $\kappa_{(M\setminus u)/e}(A_1,A_2) < 2$; or*
(b) *there exists $(A_1,A_2) \in \Pi_i(v)$ such that either $\kappa_{(M\setminus v)\setminus e}(A_1,A_2) < 2$ or $\kappa_{(M\setminus v)/e}(A_1,A_2) < 2$.*

**Proof.** If $e \notin B$, then let

$$M_1 = M \setminus e,\ M'_1 = M' \setminus e,\ \text{and } B_1 = B.$$

If $e \in B$, then let

$$M_1 = M/e,\ M'_1 = M'/e,\ \text{and } B_1 = B - \{e\}.$$

Note that, $B_1$ is a basis of $M_1$. Moreover

(1) $M_1$ and $M_1'$ are GF($q$)-representable matroids on the same ground set, $M_1' \setminus u = M_1 \setminus u$, $M_1' \setminus v = M_1 \setminus v$, and $(B_1 - \{a, b\}) \cup \{u, v\}$ distinguishes $M_1$ from $M_1'$; and
(2) $N_i$ is a uniquely GF($q$)-representable stabilizer for GF($q$) and $N_i$ is a minor of $M_1 \setminus u, v$.

Then, by Lemma 5.5, either

(i) $E(N_i) \cup \{a, b, u\}$ is not 3-connected in $M_1 \setminus v$, or
(ii) $E(N_i) \cup \{a, b, v\}$ is not 3-connected in $M_1 \setminus u$.

However, $E(N_i) \cup \{a, b, u\}$ is 3-connected in $M \setminus v$ and $E(N_i) \cup \{a, b, v\}$ is 3-connected in $M \setminus u$. It follows that one of (a) and (b) hold. □

The result is now relatively straightforward, we just apply Lemmas 4.3 and 4.2 to bound the path-width of $M$.

For each $i \in \{1, \dots, t\}$, $w \in \{u, v\}$, and $\pi = (A_1, A_2) \in \Pi_i(w)$, let $Z_i(w, \pi)$ denote the set of all elements $e \in Z_i$ for which either $\kappa_{(M \setminus w) \setminus e}(A_1, A_2) < 2$ or $\kappa_{(M \setminus w)/e}(A_1, A_2) < 2$.

**6.3.** *For each* $i \in \{1, \dots, t\}$, $w \in \{u, v\}$, *and* $\pi = (A_1, A_2) \in \Pi_i(w)$ *there exists an ordering* $(e_1, \dots, e_m)$ *of* $Z_i(w, \pi)$ *and a partition* $(Y_0, \dots, Y_m)$ *of* $E(M) - Z_i(w, \pi)$, *such that* $\rho_M(Y_0, \{e_1\}, Y_1, \dots, \{e_m\}, Y_m) \leqslant 3$.

**Proof.** By Lemma 4.3, there exists an ordering $(e_1, \dots, e_m)$ of $Z_i(w, \pi)$ and a partition $(Y_0, \dots, Y_m)$ of $(E(M) - Z_i(w, \pi)) - \{w\}$ such that $\rho_{M \setminus w}(Y_0, \{e_1\}, Y_1, \dots, \{e_m\}, Y_m) \leqslant 2$. Adding $w$ to $Y_0$ gives the result. □

Now let $Z_i(w)$ denote the union of the sets $Z_i(w, \pi)$ over all $\pi \in \Pi_i(w)$. By 6.3 and Lemma 4.2, we get

**6.4.** *For each* $i \in \{1, \dots, t\}$ *and* $w \in \{u, v\}$, *there exists an ordering* $(e_1, \dots, e_m)$ *of* $Z_i(w)$ *and a partition* $(Y_0, \dots, Y_m)$ *of* $E(M) - Z_i(w)$, *such that* $\rho_M(Y_0, \{e_1\}, Y_1, \dots, \{e_m\}, Y_m) \leqslant 6|\Pi_i(w)| \leqslant 6\left(2^{s+3}\right)$.

Now, for each $e \in Z$, there exists $i \in \{1, \dots, t\}$ such that $e \in Z_i(u)$ or $e \in Z_i(v)$. Then, by 6.4 and Lemma 4.2, we get

**6.5.** *There exists an ordering* $(e_1, \dots, e_m)$ *of* $Z$ *and a partition* $(Y_0, \dots, Y_m)$ *of* $E(M) - Z$ *such that* $\rho_M(Y_0, \{e_1\}, Y_1, \dots, \{e_m\}, Y_m) \leqslant 24t2^{s+3}$.

Now $E(M) - Z = \{a, b, u, v\}$ so, by 6.5, $M \setminus \{u, v, a, b\}$ has path-width at most $24t2^{s+3}$. Hence, $M$ has path-width at most $24t2^{s+3} + 4 = k$. This contradiction completes the proof. □

## 7. Fixing a basis

In the proof of Lemma 5.5, we work with a pair $(M, B)$ where $B$ is a fixed basis of the matroid $M$. In this section we formalize the notion of a matroid viewed with respect to a fixed basis. The results given here were introduced in [4]; we use different notation in the hope of keeping a closer connection to more familiar matroid notions.

We denote the symmetric difference of sets $X$ and $Y$ by $X\Delta Y$; that is, $X\Delta Y = (X-Y)\cup(Y-X)$.

Let $B$ be a basis of a matroid $M$. A set $X \subseteq E(M)$ is a *feasible set* of $(M, B)$ if $X\Delta B$ is a basis of $M$. Duality is quite transparent in this setting, since $(M, B)$ and $(M^*, E(M) - B)$ have the same feasible sets.

*Representations*: An $\mathbb{F}$-*representation* of $(M, B)$ is a $B \times (E(M) - B)$ matrix $A$ over $\mathbb{F}$, such that

$$\begin{matrix} B & \\ \begin{pmatrix} I & A \end{pmatrix} \end{matrix}$$

is an $\mathbb{F}$-representation of $M$. (Elsewhere, $A$ is often called a *standard representation*.) Note that, $X \subseteq E(M)$ is a feasible set of $(M, B)$ if and only if $|X \cap B| = |X - B|$ and the submatrix $A[X \cap B, X - B]$ is non-singular. (Many of the results given below are straightforward for representable matroids.)

*Fundamental graphs*: The *fundamental graph* of $(M, B)$, denoted by $G_{(M,B)}$ or by $G_B$, is the graph whose vertex set is $E(M)$ and whose edge set is given by the 2-element feasible sets of $(M, B)$. Note that $G_B$ is bipartite with bipartition $(B, E(M) - B)$. For $X \subseteq E(M)$, we denote by $G_B[X]$ the subgraph of $G_B$ induced by the vertex set $X$. The following results relate feasible sets to the fundamental graph.

**Lemma 7.1** (*Brualdi [3]*). *If $X$ is a feasible set of $(M, B)$, then $G_B[X]$ has a perfect matching.*

**Lemma 7.2** (*Krogdahl [6]*). *If $G_B[X]$ has a unique perfect matching, then $X$ is a feasible set of $(M, B)$.*

*Minors*: For any $X \subseteq E(M)$, we let

$$M[X, B] = M \setminus (E(M) - (X \cup B))/(B - X);$$

such minors are said to be *visible* with respect to $B$. It is straightforward to show that, for any minor $N$ of $M$, there exists a basis $B'$ of $M$ such that $N = M[E(N), B']$. Note that $B \cap X$ is a basis of $M[X, B]$ and the fundamental graph of $(M[X, B], B \cap X)$ is $G_B[X]$. Moreover, if $A$ is a representation of $(M, B)$ then $A[B \cap X, X - B]$ is a representation of $(M[X, B], B \cap X)$.

*Pivoting*: We will need to change bases; for example, to make some minor visible. Suppose that $X$ is a feasible set of $(M, B)$. Then $B\Delta X$ is a basis of $M$. Now $Y$ is a feasible set of $(M, B\Delta X)$ if and only if $X\Delta Y$ is a feasible set of $(M, B)$. Typically we will shift from $(M, B)$ to $(M, B\Delta\{x, y\})$ for some edge $\{x, y\}$ of $G_B$; such a change is referred to as a *pivot on $xy$*. Let $B' = B\Delta\{x, y\}$. We can determine much of the structure of $G_{B'}$ from $G_B$. Note that $uv$ is an edge of $G_{B'}$ if and only if $\{u, v\}\Delta\{x, y\}$ is feasible in $(M, B)$. Thus

(i) $\{x, y\}$ is an edge of $G_{B'}$.
(ii) If $v \in E(M) - \{x, y\}$, then $xv$ is an edge of $G_{B'}$ if and only if $yv$ is an edge of $G_B$. Similarly, $yv$ is an edge of $G_{B'}$ if and only if $xv$ is an edge of $G_B$.
(iii) If $u, v \in E(M) - \{x, y\}$ and $v$ is adjacent to neither $x$ nor $y$ in $G_B$, then $uv$ is an edge of $G_{B'}$ if and only if $uv$ is an edge of $G_B$.
(iv) If $u, v \in E(M) - \{x, y\}$ where $ux$ and $vy$ are edges of $G_B$ but $uv$ is not, then $uv$ is an edge of $G_{B'}$.

This leaves only one problematic case: if $G_B[\{x, y, u, v\}]$ is a circuit, then we cannot determine whether $uv$ is an edge of $G_{B'}$ using only information from $G_B$. All we can say in this case is that, $uv$ is an edge of $G_{B'}$ if and only if $\{x, y, u, v\}$ is a feasible set of $(M, B)$.

A set $X \subseteq E(M)$ is a *twirl* of $(M, B)$ if $G_B[X]$ is an induced circuit and $X$ is feasible; it is easy to check that if $X$ is a twirl, then $M[X, B]$ is a whirl. We are only interested in 4-element twirls; these are precisely visible $U_{2,4}$-minors.

*Connectivity and fundamental graphs*: The following results help us identify 1- and 2-separations using fundamental graphs. In each of the these results, $B$ is a basis of a matroid $M$.

**Lemma 7.3.** *Let $Y \subseteq E(M)$. Then, $\lambda_M(Y) > 0$ if and only if there exists an edge $uv$ of $G_B$ with $u \in Y$ and $v \in V - Y$.*

**Corollary 7.4.** *$M$ is connected if and only if $G_B$ is connected.*

A partition $(X_1, X_2)$ of $E(M)$ is called a *split* of $G_B$ if $|X_1|, |X_2| \geqslant 2$ and the edges of $G_B$ connecting $X_1$ to $X_2$ induce a complete bipartite graph; that is, there exist $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$ such that each vertex in $Y_1$ is adjacent to each vertex in $Y_2$, and these are the only edges between $X_1$ and $X_2$.

**Lemma 7.5.** *If $(X_1, X_2)$ is a 2-separation in $M$, then $(X_1, X_2)$ is a split of $G_B$.*

A partial converse is given by the following result.

**Lemma 7.6** (*See [4, Proposition 4.12]*). *Let $(X_1, X_2)$ be a split in $G_B$ and let $x_1 \in X_1$ and $x_2 \in X_2$ where $x_1$ and $x_2$ are adjacent in $G_B$. Then, $(X_1, X_2)$ is a 2-separation in $M$ if and only if there is no twirl $\{x_1, x_2, y_1, y_2\}$ in $(M, B)$ with $y_1 \in X_1$ and $y_2 \in X_2$.*

*Series and parallel elements*: Suppose that $x$ and $y$ are parallel in $M$. We may assume that $y \notin B$. If $x \in B$, then $y$ is *pendant* to $x$ in $G_B$; that is, $x$ is the only neighbour of $y$. On the other hand, if $x \notin B$, then $x$ and $y$ are *twins* in $G_B$; that is, $x$ and $y$ have the same neighbours. Similarly, if $x$ and $y$ are in series in $M$ and $y \in B$, then either $x$ is pendant to $y$ in $G_B$ or $x$ and $y$ are twins. The converse need not be true. If $x$ and $y$ are twins in $G_B$, then $x$ and $y$ need not be in series or in parallel. However, by 7.6, if $x$ is pendant to $y$ in $G_B$, then either $x$ and $y$ are in series (when $x \in B$) or $x$ and $y$ are in parallel (when $x \notin B$).

## 8. 3-Connected sets and fundamental graphs

In this section we prove various connectivity results, most of which concern 3-connected sets in a matroid with a fixed basis. Let $X$ be a 3-connected set in a connected matroid $M$. Now let $\mathcal{F}_M(X) = \{Z \subseteq E(M) : \lambda_M(Z) \leqslant 1 \text{ and } |X \cap Z| \leqslant 1\}$ and let $\Pi_M(X)$ be the collection of maximal sets in $\mathcal{F}_M(X)$.

**Lemma 8.1.** *If $X$ is a 3-connected set in a connected matroid $M$ and $|X| \geqslant 4$, then $\Pi_M(X)$ is a partition of $E(M)$.*

**Proof.** Note that, for each $v \in E(M)$, we have $\{v\} \in \mathcal{F}_M(X)$. Thus it suffices to prove that, if $Z_1, Z_2 \in \mathcal{F}_M(X)$ and $Z_1 \cap Z_2 \neq \emptyset$, then $Z_1 \cup Z_2 \in \mathcal{F}_M(X)$. By submodularity, $\lambda_M(Z_1) + \lambda_M(Z_2) \geqslant \lambda_M(Z_1 \cap Z_2) + \lambda_M(Z_1 \cup Z_2)$. Since $Z_1, Z_2 \in \mathcal{F}_M(X)$, we have $\lambda_M(Z_1), \lambda_M(Z_2) \leqslant 1$. Moreover, since $Z_1 \cap Z_2 \neq \emptyset$ and since $M$ is connected, we have $\lambda_M(Z_1 \cap Z_2) \geqslant 1$. Therefore $\lambda_M(Z_1 \cup Z_2) \leqslant 1$. Now $|(Z_1 \cup Z_2) \cap X| \leqslant 2$ so $|X - (Z_1 \cup Z_2)| \geqslant 2$. Hence, since $X$ is a 3-connected set, we must have $|(Z_1 \cup Z_2) \cap X| \leqslant 1$ and, so, $Z_1 \cup Z_2 \in \mathcal{F}_M(X)$, as required. □

For any $\pi \subseteq E(M)$, we let $\partial_{(M,B)}(\pi)$ be the elements of $\pi$ that have a neighbour in $E(M) - \pi$ in $G_B$. For a partition $\Pi$ of $E(M)$, we let $\partial_{(M,B)}(\Pi)$ denote $(\partial_{(M,B)}(\pi) : \pi \in \Pi)$. Where there is no fear of ambiguity we denote $\partial_{(M,B)}$ by $\partial_B$. Now suppose that $B$ is a basis of $M$ and that $(X_1, X_2)$ is a 2-separation of $M$. Then, as noted in the previous section, $(X_1, X_2)$ is a split of $G_B$. Now let $x_1 \in \partial_B(X_1)$ and $x_2 \in \partial_B(X_2)$. It is straightforward to prove that $M$ is the 2-sum of $M[X_1 \cup \{x_2\}, B]$ and $M[\{x_1\} \cup X_2, B]$ (identifying $x_1$ with $x_2$) and that, up to isomorphism, these matroids do not depend on the particular choice of $x_1$ and $x_2$. Decomposing across each of the 2-separations given by the parts of $\Pi_M(X)$, we obtain the following lemma.

**Lemma 8.2.** *Let $B$ be a basis of a connected matroid $M$ and let $X$ be a* 3*-connected set of $M$ with $|X| \geqslant 4$. If $T$ is a transversal of $\partial_B(\Pi_M(X))$, then $M[T, B]$ is* 3*-connected. Moreover, if $N$ is a* 3*-connected minor of $M$ with $X \subseteq E(N)$, then $M[T, B]$ has a minor isomorphic to $N$.*

Lemma 8.2 provides a way of recognizing that certain minors are 3-connected; we also need to recognize that certain minors are stable.

**Lemma 8.3.** *Let $B$ be a basis in a connected matroid $M$ and let $X \subseteq E(M)$ be a* 3*-connected set in $M$ with $|X| \geqslant 4$. If $S \subseteq E(M)$ where $S \cap \pi \neq \emptyset$ for each $\pi \in \Pi_M(X)$ and each component of $G_B[S \cap \pi]$ is a tree containing exactly one element of $\partial_B(\pi)$, then $M[S, B]$ is stable.*

**Proof.** Note that, there is a transversal $T \subseteq S$ of $\partial_B(\Pi_M(X))$. By Lemma 8.2, $M[T, B]$ is 3-connected. Moreover, we can obtain $M[T, B]$ from $M[S, B]$ by repeated simplification and cosimplification. Thus $M[S, B]$ is stable. □

We need the following elementary fact about bipartite graphs; the easy proof is left to the reader.

**Lemma 8.4.** *If $G = (V, E)$ is a connected bipartite graph and $u, v, w \in V$, then there exists $A \subseteq V$, such that $u, v, w \in A$ and $G[A]$ is a tree.*

**Lemma 8.5.** *Let $B$ be a basis in a connected matroid $M$ and let $X \subseteq E(M)$ be a* 3*-connected set in $M$ with $|X| \geqslant 4$. If $\pi \in \Pi_M(X)$ and $Z \subseteq \pi$ with $|Z| \leqslant 2$, then there exists $S \subseteq \pi$, such that $Z \subseteq S$ and each component of $G_B[S]$ is a tree with exactly one vertex in $\partial_B(\pi)$.*

**Proof.** Let $v \in E(M) - \pi$ be a vertex of $G_B$ that has a neighbour in $\pi$. By Lemma 8.4, there exists $S \subseteq \pi$, such that $Z \subseteq S$ and $G_B[S \cup \{v\}]$ is a tree. Since $v$ is adjacent to every vertex in $\partial_B(\pi)$, each component of $G_B[S]$ is a tree with exactly one vertex in $\partial_B(\pi)$. □

**Lemma 8.6.** *Let $e$ be an element of a connected matroid $M$ and let $N$ be a* 3*-connected non-binary minor of $M \setminus e$. If $M \setminus e$ is stable but $M$ is not stable, then there exists $\pi \in \Pi_{M\setminus e}(E(N))$ such that $\lambda_M(\pi \cup \{e\}) = 1$.*

**Proof.** If $M$ is not stable, then $M$ can be expressed as the 2-sum of two non-binary matroids $M_1$ and $M_2$ on ground sets $X_1 \cup \{z\}$ and $X_2 \cup \{z\}$ respectively. By symmetry, we may assume that $e \in X_1$. Moreover, since $M \setminus e$ is stable, $M_1 \setminus e$ is binary. It follows that $|X_1 \cap E(N)| \leqslant 1$. Thus there exists $\pi \in \Pi_{M\setminus e}(E(N))$ such that $X_1 - \{e\} \subseteq \pi$. Now, since $\lambda_M(X_1) = \lambda_{M\setminus e}(X_1 - \{e\})$, we have $e \in \mathrm{cl}_M(X_1 - \{e\})$. Then $e \in \mathrm{cl}_M(\pi)$ and, hence, $\lambda_M(\pi \cup \{e\}) = 1$. □

We conclude this section with two easy connectivity results.

**Lemma 8.7.** *Let $(X, D, Y)$ be a partition of the ground set of a matroid $M$ where $D$ is co-independent in $M$. Then, $\lambda_M(X) = \lambda_{M\setminus D}(X)$ if and only if $D \subseteq \mathrm{cl}_M(Y)$.*

**Proof.** Note that,

$$\begin{aligned}\lambda_M(X) - \lambda_{M\setminus D}(X) &= (r_M(X) + r_M(D \cup Y) - r(M)) \\ &\quad -(r_M(X) + r_M(Y) - r_M(X \cup Y)) \\ &= (r_M(X) + r_M(D \cup Y) - r(M)) \\ &\quad -(r_M(X) + r_M(Y) - r(M)) \\ &= r_M(D \cup Y) - r_M(Y).\end{aligned}$$

Thus, $\lambda_M(X) = \lambda_{M\setminus D}(X)$ if and only if $D \subseteq \mathrm{cl}_M(Y)$. □

**Lemma 8.8.** *Let $X$ and $Y$ be disjoint sets of elements of a matroid $M$ and let $B$ be a basis of $M$. If $\lambda_M(X) > \lambda_{M[X\cup Y,B]}(X)$, then there exists $e \in E(M) - (X \cup Y)$, such that $\lambda_{M[X\cup Y\cup\{e\},B]}(X) > \lambda_{M[X\cup Y,B]}(X)$.*

**Proof.** Let $C = (E(M) - (X \cup Y)) \cap B$ and let $D = E(M) - (X \cup Y \cup C)$. By using duality, we may assume that $D$ is not empty. Now let $N = M/C$; thus $N \setminus D = M[X \cup Y, B]$. Suppose that $\lambda_N(X) > \lambda_{N\setminus D}(X)$. Then, by Lemma 8.7, there exists $e \in D$ such that $e \notin \mathrm{cl}_N(Y)$. Then, again by Lemma 8.7, $\lambda_{M[X\cup Y\cup\{e\},B]}(X) = \lambda_{N\setminus(D-\{e\})}(X) > \lambda_{N\setminus D}(X) = \lambda_{M[X\cup Y,B]}(X)$, as required. Therefore we may assume that $\lambda_N(X) = \lambda_{N\setminus D}(X)$. Then, by Lemma 8.7, $D \subseteq \mathrm{cl}_N(Y)$. However, since $N = M/C$, we have $D \subseteq \mathrm{cl}_M(Y \cup C)$. So, by Lemma 8.7, $\lambda_{M\setminus D}(X) = \lambda_M(X) > \lambda_{(M\setminus D)/C}(X)$. But $D \neq \emptyset$, so by replacing $M$ with $M \setminus D$ the result follows inductively. □

## 9. Proof of Lemma 5.5

Recall that $M$, $M'$, and $N$ are matroids, $B$ is a basis of $M$, $u, v \in E(M) - B$, and $a, b \in B$ sayisfying

(1) $M'$ is a GF($q$)-representable matroid on the same ground set as $M$, $M' \setminus u = M \setminus u$, $M' \setminus v = M \setminus v$, and $(B - \{a, b\}) \cup \{u, v\}$ distinguishes $M$ from $M'$;
(2) $N$ is a uniquely GF($q$)-representable stabilizer for GF($q$) and $N$ is a minor of $M \setminus u, v$; and
(3) $E(N) \cup \{a, b, u\}$ is 3-connected in $M \setminus v$ and $E(N) \cup \{a, b, v\}$ is 3-connected in $M \setminus u$.

We will need that $N$ is non-binary. It is straightforward to show that a binary matroid can only be a stabilizer over GF(2) or GF(3). On the other hand, Lemma 5.5 is straightforward when $q \in \{2, 3\}$. Therefore we may assume that $N$ is non-binary.

Note that $G_{(M,B)}$ and $G_{(M',B)}$ are the same; we denote this graph by $G_B$. Since $E(N)\cup\{u, a, b\}$ is 3-connected in $M \setminus v$, the set $E(N) \cup \{a, b\}$ is connected in $M \setminus u, v$. Thus $E(N) \cup \{a, b\}$ is contained in a component, say $H$, of $G_B - u - v$. Now it is easy to check that the hypotheses of Lemma 5.5 are satisfied when we replace $M$ and $M'$ by $M[V(H) \cup \{u, v\}, B]$ and $M'[V(H) \cup \{u, v\}, B]$, respectively. Thus we may assume that $M \setminus u, v$ is connected.

A set $F \subseteq E(M)$ *distinguishes* $(M, B)$ from $(M, B')$ if $F$ is a feasible set of exactly one of $(M, B)$ and $(M, B')$. Thus $\{a, b, u, v\}$ distinguishes $(M, B)$ from $(M, B')$. Since $M \setminus u = M' \setminus u$ and $M \setminus v = M' \setminus v$, both $u$ and $v$ are contained in any set that distinguishes $(M, B)$ from $(M, B')$. During the proof we change our choice of $a$, $b$, and $B$; however, we are careful that $a$, $b$, and $B$ are chosen such that they satisfy the following four conditions:

**9.1.** *$B$ is a basis of $M$ with $u, v \notin B$ and $a, b \in B$;*

**9.2.** *$\{a, b, u, v\}$ distinguishes $(M, B)$ from $(M', B)$;*

**9.3.** *no two of $a$, $b$, and $u$ are in the same part of $\Pi_{M\setminus v}(E(N))$; and*

**9.4.** *no two of $a$, $b$, and $v$ are in the same part of $\Pi_{M\setminus u}(E(N))$.*

Conditions 9.1 and 9.2 are trivially satisfied by our initial $a$, $b$, and $B$. Moreover, since $E(N) \cup \{a, b, u\}$ is 3-connected in $M \setminus v$ and, $E(N) \cup \{a, b, v\}$ is 3-connected in $M \setminus u$, conditions 9.3 and 9.4 are also satisfied.

Let $\Pi = \Pi_{M\setminus u,v}(E(N))$. For each $e \in E(M) - \{u, v\}$, we let $\pi_e$ denote the set in $\Pi$ that contains $e$. In this section we abbreviate $\partial_{(M\setminus u,v,B)}$ to $\partial$.

**9.5.** *If $X$ is a transversal of $\partial(\Pi)$, then $M[X, B]$ is 3-connected, uniquely* GF$(q)$*-representable, and is a stabilizer for* GF$(q)$.

**Proof.** By Lemma 8.2, $M[X, B]$ is 3-connected and contains an $N$-minor. Then, since $N$ is uniquely GF$(q)$-representable and is a stabilizer for GF$(q)$, $M[X, B]$ is uniquely GF$(q)$-representable and is a stabilizer for GF$(q)$. □

Since $\{a, b, u, v\}$ distinguishes $(M, B)$ from $(M, B')$, we see, by Lemmas 7.1 and 7.2, that:

**9.6.** *$G_B[\{u, v, a, b\}]$ is a circuit.*

**9.7.** *If $x$ is adjacent to both $a$ and $b$ in $G_B$, then $\{x, a, b, u\}$ and $\{x, a, b, v\}$ are both twirls of $(M, B)$.*

**Proof.** Suppose that $\{x, a, b, v\}$ is not a twirl of $(M, B)$. Then $x$ and $v$ are in parallel in $M[\{x, a, b, u, v\}, B]$ and, hence, also in $M'[\{x, a, b, u, v\}, B]$. Thus, $\{a, b, u, v\}$ is feasible in $(M, B)$ if and only if $\{x, a, b, u\}$ is feasible in $(M, B)$. Similarly, $\{a, b, u, v\}$ is feasible in $(M', B)$ if and only if $\{x, a, b, u\}$ is feasible in $(M', B)$. Then, since $\{a, b, u, v\}$ distinguishes $(M, B)$ and $(M', B)$, the set $\{a, b, u, v'\}$ also distinguishes $(M, B)$ and $(M', B)$. This contradicts the fact that $M \setminus v = M' \setminus v$. □

We rely on the following result to prove that $M$ is not GF$(q)$-representable.

**9.8.** *Let $X$ be a transversal of $\partial(\Pi)$ and let $S \subseteq E(M) - \{u, v\}$ with $X \cup \{a, b\} \subseteq S$. If $M[S \cup \{u\}, B]$ and $M[S \cup \{v\}, B]$ are stable and $M[S, B]$ is connected, then $M$ is not* GF$(q)$-*representable.*

**Proof.** Let $M_1 = M[S \cup \{u, v\}, B]$ and $M_2 = M'[S \cup \{u, v\}, B]$. Note that $M_1 \setminus u = M_2 \setminus u$ and $M_1 \setminus v = M_2 \setminus v$. However, $M_1 \neq M_2$ since $\{a, b, u, v\}$ distinguishes $(M, B)$ from $(M, B')$. Moreover, $M_1 \setminus u$ and $M_1 \setminus v$ are stable and $M_1 \setminus u, v$ is connected. Then, by Lemma 5.4, $M_1$ is not GF$(q)$-representable. $\square$

Henceforth, we assume that $M$ is GF$(q)$-representable, and, hence, there does not exist a set $S$ satisfying the hypotheses of 9.8. By 9.8 we can exclude an easy case.

**9.9.** *No transversal of $\partial(\Pi)$ contains both $a$ and $b$.*

**Proof.** Suppose that there is a transversal $X$ of $\partial(\Pi)$ with $a, b \in X$ and let $S = X \cup \{u, v\}$. By 9.5, $M[S - \{u, v\}, B]$ is 3-connected. Thus $M[S - \{u\}, B]$ and $M[S - \{v\}, B]$ are both internally 3-connected, and, hence, stable. Thus we have a contradiction to 9.8. $\square$

Currently $a$ and $b$ play interchangeable roles in the proof. By possibly swapping $a$ and $b$ we may assume that:

**9.10.** *If $b \in \partial(\pi_b)$, then $a \in \partial(\pi_b)$.*

**Proof.** Suppose that $b \in \partial(\pi_b)$. By the symmetry between $a$ and $b$ we may also suppose that $a \in \partial(\pi_a)$. If $\pi_a = \pi_b$, then the assumption holds. On the other hand, if $\pi_a \neq \pi_b$, then there is a transversal $X$ of $\partial(\Pi)$ that contains both $a$ and $b$, contradicting 9.9. $\square$

**9.11.** *Suppose that $b' \in \partial(\pi_b)$ such that if $a \in \partial(\pi_b)$ then $a = b'$. Now let $v' \in E(M) - (\{u, v\} \cup \pi_b)$ be a neighbour of $b'$. Then $\lambda_{M[\{b,b',v,v'\},B]}(\{b, b'\}) > 1$.*

**Proof.** By 9.10, $b' \neq b$. Suppose to the contrary that $\lambda_{M[\{b,b',v,v'\},B]}(\{b, b'\}) = 1$. Thus $(\{b, b'\}, \{v, v'\})$ is a split in $G_B[\{b, b', v, v'\}]$. However, note that $b$ is adjacent to $v$ and $b'$ is adjacent to $v'$. It follows that $b$ and $b'$ are both adjacent to $v$ and $v'$. Moreover, $\{b, b', v, v'\}$ is not a twirl in $(M, B)$. Since $b$ is adjacent to $v'$, we have $b \in \partial(\pi_b)$. Then, by 9.10, $a \in \partial(\pi_b)$. Hence, by our definition of $b'$, we have $b' = a$. Now $v'$ is adjacent to both $a$ and $b$ but $\{v', a, b, v\}$ is not a twirl in $(M, B)$, contradicting 9.7. $\square$

**9.12.** *Let $S \subseteq E(M) - \{u, v\}$ where $a, b \in S$, $M[S, B]$ is stable, and $S \cap \pi \neq \emptyset$ for each $\pi \in \Pi$. If $M[S \cup \{v\}, B]$ is not stable, then $\lambda_{M[\pi_b \cup \{v\} \cup S,B]}(\pi_b \cup \{v\}) = 1$.*

**Proof.** Let $\widehat{M} = M[S \cup \{v\}, B]$ and let $X \subseteq S$ be a transversal of $\partial(\Pi)$. By 9.5, $X$ is a 3-connected set in $\widehat{M} \setminus v$, so $\Pi_{\widehat{M} \setminus v}(X) = (S \cap \pi : \pi \in \Pi)$. If $M[S \cup \{v\}, B]$ is not stable, then, by Lemma 8.6, there exists $\pi \in \Pi_{\widehat{M} \setminus v}(X)$ such that $\lambda_{\widehat{M}}(\pi \cup \{v\}) = 1$. It follows that $v \in \text{cl}_{\widehat{M}}(\pi)$. Therefore, for any $\pi' \in \Pi_{\widehat{M} \setminus v}(X)$ where $\pi \neq \pi'$, we have $\lambda_{\widehat{M}}(\pi') = 1$. However, by 9.11, $\lambda_{\widehat{M}}(\pi_b \cap S) > 1$. Thus $\pi = S \cap \pi_b$. Suppose that $\lambda_{M[\pi_b \cup \{v\} \cup S,B]}(\pi_b \cup \{v\}) > 1$. We know that $\lambda_{M[S,B]}(\pi) = 1$. So, by Lemma 8.8, there exists $e \in (\pi_b \cup \{v\}) - \pi$ such that $\lambda_{M[S \cup \{e\},B]}(\pi \cup \{e\}) > 1$. Since $\lambda_{M[\pi_b \cup S,B]}(\pi_b) = 1$, it follows that $e = v$. But this contradicts the fact that $\lambda_{\widehat{M}}(\pi \cup \{v\}) = 1$. $\square$

Note that there is still symmetry between $u$ and $v$. Thus, an analogous result holds with the roles of $u$ and $v$ swapped in 9.12.

*Case* 1: $\pi_a = \pi_b$.

By Lemma 8.5, there exists $S_b \subseteq \pi_b$ such that $a, b \in S_b$ and each component of $G_B[S_b]$ is a tree containing exactly one element of $\partial(\pi_b)$. Now let $b' \in \partial(\pi_b) \cap S_b$ and let $X$ be a transversal of $\partial(\Pi)$ that contains $b'$. Finally, let $x$ be a neighbour of $b'$ in $G_B[X]$. By 9.4, $\lambda_{M\backslash u}(\pi_b \cup \{v\}) > 1 = \lambda_{M[\pi_b \cup \{v,x\},B]}(\pi_b \cup \{v\})$. Then, by Lemma 8.8, there exists $e_v \in E(M) - (\pi_b \cup \{u, v, x\})$ such that $\lambda_{M[\pi_b \cup \{e_v,v,x\},B]}(\pi_b \cup \{v\}) > 1$. Similarly, there exists $e_u \in E(M) - (\pi_b \cup \{u, v, x\})$ such that $\lambda_{M[\pi_b \cup \{e_u,u,x\},B]}(\pi_b \cup \{u\}) > 1$.

*Case* 1.1: *$e_u$ and $e_v$ are not both contained in $\pi_x$.*

By Lemmas 8.3 and 8.5, there exists $S \subseteq E(M) - \{u, v\}$ such that $M[S, B]$ is stable, $e_u, e_v, x \in S$, $S \cap \pi_b = S_b$, and $S \cap \pi \neq \emptyset$ for each $\pi \in \Pi$. Since $b', x, e_u, e_v \in S$, we have $\lambda_{M[\pi_b \cup \{u\} \cup S,B]}(\pi_b \cup \{u\}) > 1$ and $\lambda_{M[\pi_b \cup \{v\} \cup S,B]}(\pi_b \cup \{v\}) > 1$. Therefore, by 9.12, $M[S \cup \{u\}, B]$ and $M[S \cup \{v\}, B]$ are both stable, contradicting 9.8.

*Case* 1.2: $e_u, e_v \in \pi_x$.

Since $X$ is a transversal of $\partial(\Pi)$, the minor $M[X, B]$ is 3-connected. Hence, $G_B[X]$ has no vertices of degree one. Therefore $b'$ has a neighbour $x'$ in $G_B[X - \{x\}]$. Note that, $\lambda_{M[\pi_b \cup \{x,x',u,e_u\},B]}(\pi_b \cup \{u\}) > 1 = \lambda_{M[\pi_b \cup \{u,x'\},B]}(\pi_b \cup \{u\})$. Then, by Lemma 8.8, there exists $e'_u \in \{x, e_u\}$ such that $\lambda_{M[\pi_b \cup \{u,x',e'_u\},B]}(\pi_b \cup \{u\}) > 1$. Similarly, there exists $e'_v \in \{x, e_v\}$ such that $\lambda_{M[\pi_b \cup \{v,x',e'_v\},B]}(\pi_b \cup \{v\}) > 1$. Note that, $e'_u, e'_v \in \pi_x$ and that $\pi_x \neq \pi_{x'}$. Therefore replacing $x$, $e_u$, and $e_v$ with $x'$, $e'_u$, and $e'_v$ returns us to Case 1.1.

*Case* 2: $\pi_a \neq \pi_b$.

We choose $S_a \subseteq \pi_a$ such that $G_B[S_a]$ is a path connecting $a$ to some element $a' \in \partial(S_a)$. Now we choose $S_b \subseteq \pi_b$ such that $G_B[S_b]$ is a path connecting $b$ to some element $b' \in \partial(S_b)$. Now let $X$ be a transversal of $\partial(\Pi)$ containing both $a'$ and $b'$, and let $S = S_a \cup S_b \cup X$. By Lemma 8.3, $M[S, B]$ is stable. By 9.8 and by possibly swapping $u$ and $v$, we may assume that $M[S \cup \{u\}, B]$ is not stable. Then, by 9.12, $\lambda_{M[\pi_b \cup \{u\} \cup S,B]}(\pi_b \cup \{u\}) = 1$. Thus $(\pi_b \cup \{u\}, S - \pi_b)$ is a split in $G_B[\pi_b \cup \{u\} \cup S]$. Recall that $u$ is adjacent to $a$ in $G_B$. It follows that $a \in \partial(\pi_a)$ and that $a$ is adjacent to $b'$ in $G_B$.

Now let $\widehat{a} = b'$, $\widehat{b} = b$, and $\widehat{B} = B \Delta \{a, b'\}$. Observe that $\widehat{a}$ and $\widehat{b}$ are in the same part of $\Pi$. We will show that $\widehat{a}$, $\widehat{b}$, and $\widehat{B}$ satisfy 9.1, 9.2, 9.3, 9.4, and 9.9; thus reducing Case 2 to Case 1. Note that, $\widehat{a}$, $\widehat{b}$, and $\widehat{B}$ trivially satisfy 9.1. Moreover, as $\{\widehat{a}, \widehat{b}, u, v\} = \{a, b, u, v\} \Delta \{a, b'\}$ and $\{a, b, u, v\}$ distinguishes $(M, B)$ from $(M', B)$, the set $\{\widehat{a}, \widehat{b}, u, v\}$ distinguishes $(M, \widehat{B})$ from $(M', \widehat{B})$. Thus $\widehat{a}$, $\widehat{b}$, and $\widehat{B}$ also satisfy 9.2. Note that, $a$ and $b'$ remain adjacent in $G_{\widehat{B}}$, so $\widehat{a} \in \partial_{(M\backslash u,v,\widehat{B})}(\pi_b)$. Hence, $\widehat{a}$, $\widehat{b}$, and $\widehat{B}$ satisfy 9.9.

It remains to prove that $\widehat{a}$, $\widehat{b}$, and $\widehat{B}$ satisfy 9.3 and 9.4; suppose otherwise. By the symmetry between $u$ and $v$, we may assume that there exists $\pi \in \Pi_{M\backslash v}(E(N))$, such that $|\pi \cap \{\widehat{a}, \widehat{b}, u\}| \geqslant 2$. However, by 9.3, $\pi$ cannot contain both of $\widehat{b} = b$ and $u$. Thus $\widehat{a} = b' \in \pi$. Again using 9.3, since $\pi$ contains one of $u$ and $b$, we have $a \notin \pi$. Now $(\pi, E(M) - (\{v\} \cup \pi))$ is a split in $G_B - v$ and both of the edges $ub$ and $ab'$ cross this split. It follows that $u, b' \in \pi$, $a, b \notin \pi$, and that $u$ and $b'$ are both adjacent to $a$ and $b$. By 9.7, $\{b', a, b, u\}$ is a twirl of $(M, B)$; this contradicts the fact that $\lambda_{M\backslash v}(\pi) = 1$. This final contradiction completes the proof of Lemma 5.5. □

## Acknowledgements

This research was partially supported by grants from the Natural Sciences and Engineering Research Council of Canada and the Marsden Fund of New Zealand.

## References

[1] E. Artin, Geometric Algebra, Interscience, New York, 1957.
[2] R.E. Bixby, A simple theorem on 3-connectivity, Linear Algebra Appl. 45 (1982) 123–126.
[3] R.A. Brualdi, Comments on bases in dependence structures, Bull. Austral. Math. Soc. 1 (1989) 161–167.
[4] J.F. Geelen, A.M.H. Gerards, A. Kapoor, The excluded minors for GF(4)-representable matroids, J. Combin. Theory Ser. B 79 (2000) 247–299.
[5] J. Geelen, B. Gerards, G. Whittle, Excluding a planar graph from GF($q$)-representable matroids, Research Report 03-4, School of Mathematical and Computing Sciences, Victoria University of Wellington, 2003.
[6] S. Krogdahl, The dependence graph for bases in matroids, Discrete Math. 19 (1977) 47–59.
[7] J.G. Oxley, Matroid Theory, Oxford University Press, New York, 1992.
[8] J.G. Oxley, D.L. Vertigan, G. Whittle, On inequivalent representations of matroids over finite fields, J. Combin. Theory Ser. B 67 (1996) 325–343.
[10] W.T. Tutte, Menger's theorem for matroids, J. Res. Nat. Bur. Standards Section B 69 (1965) 49–53.
[11] W.T. Tutte, Connectivity in matroids, Canad. J. Math. 18 (1966) 1301–1324.
[12] G. Whittle, Stabilizers of classes of representable matroids, J. Combin. Theory Ser. B 77 (1999) 39–72.