Straight Line Embeddings of Cubic Planar Graphs With Integer Edge Lengths

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Received December 5, 2006; Revised January 16, 2008

Published online 10 April 2008 in Wiley InterScience(www.interscience.wiley.com).
DOI 10.1002/jgt.20304

Abstract: We prove that every simple cubic planar graph admits a planar embedding such that each edge is embedded as a straight line segment of integer length. © 2008 Wiley Periodicals, Inc. J Graph Theory 58: 270–274, 2008

Keywords: *planar graphs; straight line embedding*

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Contract grant sponsor: Natural Sciences and Engineering Research Council of Canada.

1. INTRODUCTION

We consider only simple finite graphs. A *straight line embedding* of a graph $G = (V, E)$ is an injective function $\phi : V \to \mathbb{R}^2$ such that for any two distinct edges $ab, cd \in E$ the straight line segments $(\phi(a), \phi(b))$ and $(\phi(c), \phi(d))$ are internally disjoint (that is, they may only meet at their ends). It is a well-known classical result that every planar graph admits a straight line embedding; see, for example, Wagner [7] or Fáry [3]. Given a straight line embedding of G, the *length* of an edge $uv \in E$ is the Euclidean distance between $\phi(u)$ and $\phi(v)$, which we denote by $\text{dist}(\phi(u), \phi(v))$.

In this article, we address a special case of the following conjecture of Kennitz and Harborth [5,4]; see also the book by Brass et al. [1].

Conjecture 1.1. *Every planar graph admits a straight line embedding with integer edge lengths.*

Note that, up to scaling, it suffices to find a straight line embedding with rational edge lengths. We prove Conjecture 1.1 for the class of cubic planar graphs. (A graph is *cubic* if each of its vertices has degree 3.) The result for cubic planar graphs relies on the following result for general cubic graphs.

Theorem 1.2. *Let $G = (V, E)$ be a cubic graph, let $\phi : V \to \mathbb{R}^2$, and let $\epsilon > 0$. Then there exists a function $\psi : V \to \mathbb{Q}^2$ such that*

1. *$\text{dist}(\psi(u), \psi(v)) \in \mathbb{Q}$ for each $uv \in E$, and*
2. *$\text{dist}(\psi(v), \phi(v)) \leq \epsilon$ for each $v \in V$.*

2. PRELIMINARIES

We require the following two theorems that are both of interest in their own right.

Theorem 2.1 (Berry [2]). *If $A, B, C \in \mathbb{R}^2$ are non-collinear points such that $\text{dist}(A, B)$, $\text{dist}(A, C)^2$, and $\text{dist}(B, C)^2$ are rational, then the set of points that are a rational distance from each of A, B, and C forms a dense subset of $\mathbb{R}^2$.*

Berry also notes that there are no points in the plane at rational distance from the three vertices of a triangle with sides $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$, so the condition that one side is rational is unavoidable.

Theorem 2.2. *If $A, B, C \in \mathbb{Q}^2$ are not collinear and $x \in \mathbb{R}^2$ such that $\text{dist}(x, y)^2$ is rational for each $y \in \{A, B, C\}$, then $x \in \mathbb{Q}^2$.*

Proof. By possibly translating the points, we may assume that $C = (0, 0)$. Let $\alpha = (A_1 - x_1)^2 + (A_2 - x_2)^2$, $\beta = (B_1 - x_1)^2 + (B_2 - x_2)^2$, and $\gamma = x_1^2 + x_2^2$. Thus, $\alpha = \text{dist}(x, A)^2$, $\beta = \text{dist}(x, B)^2$, and $\gamma = \text{dist}(x, C)^2$ are all rational. Note that

$$A_1x_1 + A_2x_2 = \frac{1}{2}\left(\gamma + A_1^2 + A_2^2 - \alpha\right).$$

$$B_1x_1 + B_2x_2 = \frac{1}{2}\left(\gamma + B_1^2 + B_2^2 - \beta\right).$$

Since A, B, and C are not collinear, A is not a scalar multiple of B. Considering x_1 and x_2 as variables, we have two rational linear equations with a unique rational solution. Therefore, x is rational. ■

3. THE MAIN RESULTS

We are interested in graphs $G = (V, E)$ satisfying:

Property 3.1. *For any function $\phi : V \to \mathbb{R}^2$ and any real number $\epsilon > 0$, there exists a function $\psi : V \to \mathbb{Q}^2$ such that*

1. *$dist(\psi(u), \psi(v)) \in \mathbb{Q}$ for each $uv \in E$, and*
2. *$dist(\psi(v), \phi(v)) \leq \epsilon$ for each $v \in V$.*

Lemma 3.2. *Let z be a vertex of degree 3 in a simple graph $G = (V, E)$ and let a, b, and c be the three neighbors of z. If $ab \in E$ and $G - z$ satisfies Property 3.1, then G satisfies Property 3.1.*

Proof. Let $\phi : V \to \mathbb{R}^2$ and $\epsilon > 0$. By possibly perturbing ϕ and adjusting ϵ accordingly, we may assume that ϕ is injective and that the image of ϕ does not contain three collinear points. Moreover, by possibly further decreasing ϵ, we may assume that there do not exist three collinear points $x_1, x_2, x_3 \in \mathbb{R}^2$ with $\text{dist}(x_i, \phi(u_i)) \leq \epsilon$ for $i \in \{1, 2, 3\}$.

Since $G - z$ satisfies Property 3.1, there is a function $\psi : V - \{z\} \to \mathbb{Q}^2$ such that

1. $\text{dist}(\psi(u), \psi(v)) \in \mathbb{Q}$ for each $uv \in E(G - z)$, and
2. $\text{dist}(\psi(v), \phi(v)) \leq \epsilon$ for each $v \in V - \{z\}$.

Note that $\psi(a), \psi(b), \psi(c) \in \mathbb{Q}^2$ and, since $ab \in E$, $\text{dist}(\psi(a), \psi(b))$ is rational. Hence, $\text{dist}(\psi(a), \psi(b))$, $\text{dist}(\psi(b), \psi(c))^2$, and $\text{dist}(\psi(a), \psi(c))^2$ are all rational. Therefore, by Theorem 2.1, there is a point $x \in \mathbb{R}^2$ with $\text{dist}(x, \psi(z)) \leq \epsilon$, that is, at a rational distance from each of $\phi(a)$, $\phi(b)$, and $\phi(c)$. By Lemma 2.2, the point x is rational. Now extend ϕ to a function $\phi : V \to \mathbb{Q}^2$ by defining $\phi(z) = x$. This shows that G satisfies Property 3.1, as required. ■

We are ready to prove Theorem 1.2. Our original proof was somewhat more convoluted, the simpler version presented here was suggested by a referee. We restate a mild strengthening of the result to facilitate induction.

Theorem 3.3. *Every simple graph with maximum degree ≤ 3 satisfies Property 3.1.*

Proof. Suppose that the result is false and let $G = (V, E)$ be a counterexample with $|V|$ minimum. Let $z \in V$. The case that z has degree ≤ 2 is straightforward, so we may assume that z has degree 3. Let a, b, and c be the three neighbors of z. Let $G' = (V, E')$ be the simple graph obtained from G by adding the edge ab (if ab was already an edge of G, then $G' = G$). Observe that $G' - z$ has maximum degree ≤ 3. Then, since G is a minimum counterexample, $G' - z$ satisfies Property 3.1. By Lemma 3.2, G' satisfies Property 3.1. Since G is a subgraph of G', G also satisfies Property 3.1. This contradicts that fact that G is a counterexample and, hence, the result holds. ■

Corollary 3.4. *Every cubic planar graph admits a straight line embedding with integer edge lengths.*

Proof. Let $\phi : V \to \mathbb{R}^2$ be a straight line embedding of a cubic planar graph $G = (V, E)$. Note that ϕ remains a straight line embedding under arbitrarily small perturbations. That is, there exists $\epsilon > 0$ such that, if $\psi : V \to \mathbb{R}^2$ is a function satisfying dist$(\phi(v), \psi(v))$ for each $v \in V$, then ψ is a straight line embedding of G. By Theorem 1.2, there is a straight line embedding ψ of G with rational edge lengths. A suitable scaling of ψ gives integer edge lengths. ■

4. CONCLUDING REMARKS

We do not know of a graph that does not satisfy Property 3.1, but it seems likely that such graphs exist. It seems reasonable to conjecture that all planar graphs satisfy the property. Using Lemma 3.2, it is easy to show that graphs of tree-width 3 satisfy the property. Thus, planar graphs of tree-width 3 satisfy Conjecture 1.1; this result is already implicit in Kemnitz and Harborth [5].

Property 3.1 is also of interest for small complete graphs, particularly K_8. A famous problem of Erdős asks: *How many points we can find in the plane with pairwise rational distances such that no three are on a line and no four are on a circle*? A collection of 7 such points has recently been discovered by Kreisel and Kurz [6], but the problem remains open for 8.

Theorem 2.1 plays a crucial role in our proof of Theorem 1.2. This suggests the following question.

Problem 4.1. *Let $A, B_1, \ldots, B_k \in \mathbb{Q}^2$ such that no three of these points are collinear and dist(A, B_i) is rational for each $i \in \{1, \ldots, k\}$. Does the set of points that are a rational distance from each of $A, B_1, \ldots, B_k$ form a dense subset of $\mathbb{R}^2$?*

An affirmative answer to Problem 4.1 for $k = 4$ would prove that planar graphs satisfy Property 3.1 and, hence, would verify Conjecture 1.1.

ACKNOWLEDGMENTS

The proof of Theorem 1.2 is much clearer due to a suggestion of a referee. This research was partially supported by grants from the Natural Sciences and Engineering Research Council of Canada.

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