Advances in Applied Mathematics 41 (2008) 628–637

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Some open problems on excluding a uniform matroid

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ARTICLE INFO

Article history:
Received 5 March 2008
Accepted 20 May 2008
Available online 8 August 2008

MSC:
05B35

Keywords:
Matroids
Uniform matroids
Spikes
Matroid minors

ABSTRACT

It would appear that minor-closed classes of matroids that are representable over any given finite field are very well behaved. This paper explores what happens when we go a little further to minor-closed classes of matroids that exclude a uniform minor. Numerous open problems of varying difficulty are posed.

1. Introduction

Over the past few years I have worked with Bert Gerards and Geoff Whittle on extending the Graph Minors Project of Neil Robertson and Paul Seymour to the class of matroids representable over any fixed finite field. That project is progressing well, and it is natural to consider where the techniques may lead in the future.

The Graph Minors Structure Theorem [12] lies at the heart of the Graph Minors Project; this theorem provides a constructive description for graphs contained in any minor-closed class. There is significant evidence that, for any finite field $\mathbb{F}$, similar results hold for minor closed classes of $\mathbb{F}$-representable matroids. I hope that the structural results will extend considerably further.

Problem 1.1. Let a and b be positive integers. Find a qualitative structure theorem for the class of matroids with no $U_{a,b}$-minor.

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0196-8858/$ – see front matter
doi:10.1016/j.aam.2008.05.002

What I have in mind here is an analogue of the Graph Minors Structure Theorem. The hope is to find a constructive structural description for all matroids with no $U_{a,b}$-minor; we allow the construction to generate some matroids with a $U_{a,b}$-minor, but the construction should not give matroids that contain arbitrary uniform matroids as minors. The construction should show that every matroid with no $U_{a,b}$-minor is *pieced together in a tree-like way* from pieces that are *essentially* contained in certain *basic classes*. The basic classes are the class of Dowling matroids (defined in Section 6), their duals, and the matroids representable over finite fields over which $U_{a,b}$ is not representable. I will not discuss the tree-like construction further here, but it is made precise in [6]. I will also remain vague about the meaning of *essentially*, but it includes *projection* and *lifting* described in Section 7.

Problem 1.1 is likely to be *extremely hard*, but it should be achievable by following a similar line of proof used in the Graph Minors Structure Theorem. Considering the proposed structure for matroids with no $U_{a,b}$-minor naturally gives rise to numerous problems of varying difficulty.

From the point of view of algorithms and well-quasi-ordering, there is little benefit to excluding a uniform minor; these classes are as wild as the class of all matroids. This is made precise in Section 3 by a construction due to Dirk Vertigan. However, there is still the potential for interesting applications. I propose two applications; the first concerns representability over the reals (see Section 4) and the second is a conjecture on intertwining (see Section 5).

I use the terminology of Oxley [10].

2. Representation over finite fields

There are three major conjectures, stated below, regarding matroids representable over finite fields. The first two, due to Robertson and Seymour, are natural extensions of the two main results in the Graph Minors Project [11,13], and the third is an older conjecture posed by Rota.

Conjecture 2.1 *(Well-Quasi-Ordering Conjecture). For any finite field* $\mathbb{F}$*, any infinite set of* $\mathbb{F}$*-representable matroids contains two matroids, one isomorphic to a minor of the other.*

Conjecture 2.2 *(Minor-Testing Conjecture). For any finite field* $\mathbb{F}$ *and* $\mathbb{F}$*-representable matroid* N*, there is a polynomial-time algorithm that tests for an* N*-minor in any* $\mathbb{F}$*-represented matroid.*

Conjecture 2.3 *(Rota's Conjecture, 1971). For any finite field* $\mathbb{F}$*, there are, up to isomorphism, only finitely many excluded minors for the class of* $\mathbb{F}$*-representable matroids.*

A qualitative structure theorem (like the Graph Minors Structure Theorem) for minor-closed classes of $\mathbb{F}$-representable matroids would provide considerable traction in solving each of these conjectures. The main thrust of my recent work with Gerards and Whittle is towards finding such a structure theorem. The structural results will hopefully have other applications; for example, they may help resolve the following conjecture.

Conjecture 2.4 *(Equivalence-Testing Conjecture). For any finite field* $\mathbb{F}$*, there is a polynomial-time algorithm that, given two matrices* A_1 *and* A_2 *over* $\mathbb{F}$ *with the same set of column indices, tests whether* $M(A_1) = M(A_2)$*.*

3. It all goes wrong for spikes

There is something utopian about matroids representable over finite fields. One does not need to go far outside the class before matroid theory reveals its true nature. In fact, all of the horrors are inherent to spikes. From a structural point of view, it is hard to imagine a more benign looking class: spikes have branch-width 3 (see [4] for a definition of branch-width) and spikes do not have $U_{2,6}$- or $U_{4,6}$-minors.

Let $S = (E, \mathcal{F})$ be a set-system on a finite set E such that $|E| \geqslant 3$ and no two sets in $\mathcal{F}$ differ in exactly one element. It is easy to verify that there is a unique matroid, denoted $\Lambda(S)$, with ground set $\{a_i\colon\ i \in E\} \cup \{b_i\colon\ i \in E\}$ such that:

- for each distinct $i, j \in E$, the set $\{a_i, b_i, a_j, b_j\}$ is both a circuit and a cocircuit, and
- for each $A \subseteq E$, the set $\{a_i\colon\ i \in A\} \cup \{b_j\colon\ j \in E - A\}$ is dependent if and only if $A \in \mathcal{F}$.

Such matroids are called *spikes*; the pair $\{a_i, b_i\}$ is a *leg* of the spike. For each $A \in \mathcal{F}$, the set $\{a_i :\ i \in A\} \cup \{b_j\colon\ j \in E - A\}$ is a *dependent transversal* of $\Lambda(S)$.

Our definition of spikes is a little more convoluted than necessary, but it is convenient for our purposes. One can show that a matroid is isomorphic to a spike if and only if its ground set can be partitioned into pairs such that the union of any two pairs is both a circuit and a cocircuit. From this it is clear that the dual of a spike is a spike. In fact, $\Lambda(S)$ is isomorphic to its dual; the isomorphism is the function that swaps the elements on each leg.

So here is the bad news. The partial order given by the minor relation on spikes contains an isomorphic copy of the partial order given by the minor relation on all matroids.

Theorem 3.1. *(Vertigan [19].) There is a function ψ from the set of all matroids with at least 3 elements to the set of all spikes such that, if N and M are matroids with at least 3 elements, then N is a minor of M if and only if $\psi(N)$ is a minor of $\psi(M)$.*

The function ψ is easy to define. For a matroid M, let $S(M) = (E(M), \mathcal{B}(M))$ where $\mathcal{B}(M)$ is the set of bases of M. Then $\psi(M)$ is defined as $\Lambda(S(M))$. We briefly sketch the proof below.

For $e \in E$, let $S \setminus e := (E - \{e\}, \mathcal{F} \setminus e)$ and $S/e := (E - \{e\}, \mathcal{F}/e)$ where

$$\mathcal{F} \setminus e := \{F \in \mathcal{F}\colon\ e \notin F\} \quad \text{and}$$
$$\mathcal{F}/e := \bigl\{F - \{e\}\colon\ e \in F \in \mathcal{F}\bigr\}.$$

It is easy to verify that $\Lambda(S) \setminus a_e/b_e = \Lambda(S \setminus a_e)$, dually, and $\Lambda(S)/a_e \setminus b_e = \Lambda(S/e)$. Moreover, for a matroid M and $e \in E(M)$, $S(M \setminus e) = S(M) \setminus e$ and $S(M/e) = S(M)/e$. This shows that if N is a minor of M, then $\psi(N)$ is a minor of $\psi(M)$. The converse requires one additional observation. Suppose that $\Lambda(S_1)$ is a minor of $\Lambda(S_2)$ and $e \in E(S_1) - E(S_2)$. We need to show that $\Lambda(S_1)$ is a minor of $\Lambda(S_2 \setminus e)$ or of $\Lambda(S_2 \setminus e)$. By duality we may assume that $\Lambda(S_1)$ is a minor of $\Lambda(S_2) \setminus a_e$. If $\Lambda(S_1)$ is not a minor of $\Lambda(S_2 \setminus e)$, then $\Lambda(S_1)$ is a minor of $\Lambda(S_2) \setminus \{a_e, b_e\}$. However, this cannot be the case since the other legs become series pairs in $\Lambda(S_2) \setminus \{a_i, b_i\}$.

For spikes of rank at least 5, the only 4-element circuits are the union of two legs and, hence, the legs are invariant under isomorphism. This implies the following important strengthening of Theorem 3.1: if N and M are matroids with at least 5 elements, then N is *isomorphic to* a minor of M if and only if $\psi(N)$ is *isomorphic to* a minor of $\psi(M)$.

Spikes, particularly representable spikes, appear frequently in the matroid literature. For $n \geqslant 4$, if a rank-n spike is representable, then it admits a representation of the form $[I_n | J_n + D]$, where I_n is the $n \times n$ identity, J_n is the $n \times n$ all-ones matrix, and D is an $n \times n$ diagonal matrix. One of the earliest papers containing such representations is a classical paper of Lazarson [9]; see also Oxley [10, page 208].

Theorem 3.2. *(See Lazarson [9].) There is an infinite number of spikes that are excluded minors for the class of $\mathbb{R}$-representable matroids.*

The *rank-n free spike*, denoted Λ_n, is the spike $\Lambda(S)$ where $E(S) = \{1, \ldots, n\}$ and $\mathcal{F}(S) = \emptyset$. Suppose that we are given a $2n$-element matroid M, via a rank oracle, and a partition $(\{a_1, b_1\}, \ldots, \{a_n, b_n\})$ of $E(M)$. We can readily test whether M is a spike, but to verify that M is a free spike would require us to test the rank of every transversal of $(\{a_1, b_1\}, \ldots, \{a_n, b_n\})$. Therefore, *recognizing Λ_n requires 2^n oracle calls.* That is, the problem of recognizing free spikes is intractable for matroids given by rank oracles. Hliněný [8] proved that recognizing free spikes is also NP-hard. Specifically, he proved that it is NP-hard to test, for a rational matrix A, whether $M(A)$ is a free spike. It is straightforward to explicitly construct a representation of the rank-n free spike, indeed, $[I_n | J_n + nI_n]$ is a representation. This proves the following theorem.

Theorem 3.3 *(Equivalence-testing is hard). Testing $M(A_1) = M(A_2)$, given two rational matrices A_1 and A_2 with the same column indices, is NP-hard, even when $M(A_1)$ is a free spike.*

Hliněný also observed that, for all $n \geqslant 5$, a rank-n spike is free if and only if it does not contain $M(C_5+e)$-minor. Here C_5+e is the unique simple graph obtained by adding an edge to the circuit C_5.

Theorem 3.4 *(Minor-testing is hard). Testing for an $M(C_5+e)$-minor is computationally intractible under the rank oracle model and also NP-hard, even in the class of spikes.*

A sequence $(M_1, M_2, \ldots)$ of matroids is an *antichain* if, for each $i < j$, M_i is not isomorphic to a minor of M_j. Let $S_n = (E, \mathcal{F})$ where $E = \{1, \ldots, n\}$ and $\mathcal{F} = \{\emptyset, E\}$. Now let $\Lambda_n^= := \Lambda(S_n)$. Note that $\Lambda(S_n)$ has two dependent transversals but, for any $i \in \{1, \ldots, n\}$, $\Lambda(S_n \setminus e)$ and $\Lambda(S_n/e)$ have just one dependent transversal. It follows that $(\Lambda_5^=, \Lambda_6^=, \ldots)$ is an antichain. Moreover, it is easy to show that each of these matroids is $\mathbb{R}$-representable.

Theorem 3.5 *(Well-quasi-ordering fails). There is an infinite anti-chain of $\mathbb{R}$-representable spikes.*

4. Representability over the reals

When Whitney [21] introduced matroids in 1935, he posed the problem of characterizing the class of representable matroids. To make the problem more concrete, it seems natural to consider representability over a given field although it is not quite clear what Whitney intended. I will focus on the reals, but much of the discussion is relevant to any infinite field. The problem of testing $\mathbb{R}$-representability is decidable, which is an answer, albeit weak, to Whitney's question. (The decidability follows easily from the Real Nullstellenstatz; see Sturmfels [16, page 94]. Testing whether a matroid is representable over some field is also decidable; see Vámos [17].) Given his definition of a matroid, Whitney had probably hoped for a finite system of axioms involving the rank-function. This does not look promising; Vámos [18] showed that *there is no sentence in the first order language of matroid theory characterizing representability over the reals*.

Vámos' negative result is tantamount to the existence of infinitely many excluded minors for the class of $\mathbb{R}$-representable matroids. However, the fact that there are infinitely many excluded minors does not preclude the possibility of finding them all. It follows from decidability that the excluded minors are recursively enumerable, but it would be more satisfying to learn something about their structure. It would be particularly nice if the following problem had an affirmative answer. (For a definition of branch-width, see [4].)

Problem 4.1. Is there an integer n such that all excluded minors for the class of $\mathbb{R}$-representable matroids have branch-width $\leqslant n$?

The following problem is a little weaker, but it is of central interest to this paper.

Problem 4.2. Is there an integer k such that no excluded minor for the class of $\mathbb{R}$-representable matroids contains a $U_{k,2k}$-minor?

A negative answer to Problem 4.2 would immediately give a negative answer to Problem 4.1. I think that the above two problems are, in fact, equivalent.

Conjecture 4.3. *For every integer k there is an integer n such that, if M is an excluded minor for the class of $\mathbb{R}$-representable matroids and M has no $U_{k,2k}$-minor, then M has branch-width at most n.*

I suspect that the answer to both Problems 4.1 and 4.2 is negative.

Conjecture 4.4. *For any $\mathbb{R}$-representable matroid N, there is an excluded minor M for the class of $\mathbb{R}$-representable matroids such that M contains an N-minor.*

Conjecture 4.4 has received very little attention, and may be easy.

Seymour [15] proved that there is no polynomial-time algorithm for determining whether or not a matroid, given by its rank oracle, is binary. His proof used spikes; there is a unique binary spike of each rank. For each $n \geqslant 3$, let $S_n = (E, \mathcal{F})$ where $E = \{1, \dots, n\}$ and $\mathcal{F}$ contains all odd subsets of E. Then $\Lambda(S_n)$ is, up to isomorphism, the unique binary rank-n spike. There are 2^{n-1} dependent transversals, and relaxing any one of these yields a non-binary spike. Therefore to verify that a rank-n spike is binary we need at least 2^{n-1} oracle calls. It is easy to prove similar results for all other fields.

Certifying non-representability seems to be easier. If Rota's Conjecture holds, then, for any finite field $\mathbb{F}$, it takes only $O(1)$ oracle calls to certify non-representability over $\mathbb{F}$. Geelen, Gerards, and Whittle [5] have proved that, for any finite field $\mathbb{F}$ of prime order, certifying non-$\mathbb{F}$-representability requires only $O(n^2)$ rank evaluations for matroids on n elements.

Problem 4.5. Can non-$\mathbb{R}$-representability be certified using only a polynomial number of rank-evaluations?

In my opinion, Problem 4.5 is the most natural interpretation of Whitney's question. I have no intuition as to what the answer might be; I would like to believe that the answer is yes. The following conjecture would be an interesting start.

Conjecture 4.6. *Let k be a fixed positive integer. For matroids of branch-width k, non-$\mathbb{R}$-representability can be certified using only a polynomial number of rank-evaluations.*

5. Intertwining

A matroid M is an *intertwine* of two matroids N_1 and N_2 if M contains an N_1- and an N_2-minor but no proper minor of M does. Problems involving intertwines are often quite difficult. For example, much of the work in proving Seymour's Decomposition Theorem for regular matroids [14] is in finding the 3-connected regular intertwines of $M(K_{3,3})$ and $M(K_{3,3})^*$.

Recall that when we say that M has an N-minor we mean that M has a minor that is *isomorphic* to N. There is a stronger notion of intertwine in which we do not allow isomorphism. Most of the discussion here also applies to the stronger notion.

Obviously no one intertwine can contain another as a minor. So the following conjecture would follow from the Well-Quasi-Ordering Conjecture.

Conjecture 5.1. *For any finite field $\mathbb{F}$, there are, up to isomorphism, only finitely many $\mathbb{F}$-representable intertwines for any two matroids.*

Several people (Brylawski, Robertson, and Welsh) independently asked whether the number of intertwines of two matroids is always finite; see Oxley [10, page 471]. Dirk Vertigan [20] gave a construction showing that the answer is no. Subject to surprisingly mild conditions on the matroids N_1 and N_2, Vertigan constructs infinitely many intertwines of N_1 and N_2. Moreover, he constructs intertwines that contain arbitrary uniform matroids as minors. Below we give a weaker construction using spikes.

Theorem 5.2 *(Finite intertwining fails). Let $\Lambda(S_1)$ and $\Lambda(S_2)$ be spikes of rank at least 5. If each element of $\Lambda(S_1)$ and $\Lambda(S_2)$ is in a dependent transversal and neither $\Lambda(S_1)$ nor $\Lambda(S_2)$ is isomorphic to a minor of the other, then there exist arbitrarily large spikes that are intertwines of $\Lambda(S_1)$ and $\Lambda(S_2)$.*

Proof. We may assume that $E(S_1)$ and $E(S_2)$ are disjoint. Let X be a set disjoint from $E(S_1) \cup E(S_2)$. Now define a set-system $S = (E, \mathcal{F})$ such that $E = E(S_1) \cup E(S_2) \cup X$ and $\mathcal{F} = \mathcal{F}(S_1) \cup \{F \cup X : F \in \mathcal{F}(S_2)\}$. Note that, for $e \in X$, $\mathcal{F} \setminus e = \mathcal{F}(S_1)$. For any $f \in E(S) - (E(S_1) \cup \{e\})$, the element b_f is in no dependent transversal of $\Lambda(S \setminus e)$. However every element of $\Lambda(S_2)$ is in a dependent transversal and $\Lambda(S_2)$ is not isomorphic to a minor of $\Lambda(S_1)$. Therefore $\Lambda(S_2)$ is not isomorphic to a minor of

$\Lambda(S \setminus e)$. Similarly, $\Lambda(S_1)$ is not isomorphic to a minor of $\Lambda(S/e)$. Therefore, if $\Lambda(S')$ is a minor of $\Lambda(S)$ that is an intertwine of $\Lambda(S_1)$ and $\Lambda(S_2)$, then $X \subseteq E(S')$. $\square$

With a little extra work, one can show that the construction of $\Lambda(S)$ from $\Lambda(S_1)$ and $\Lambda(S_2)$ preserves representation over $\mathbb{R}$ (or over any other infinite field).

The negative results leave little room for hope, but there are at least two avenues left to explore.

Conjecture 5.3. *For every pair of matroids N_1 and N_2 and every positive integer k there is an integer n such that, if M is an intertwine of N_1 and N_2 and M does not contain a $U_{k,2k}$-minor, then M has branch-width at most n.*

Conjecture 5.3, together with results in [4], implies Conjecture 5.1.

I cannot quite bring myself to pose the following problem as a conjecture, but it would be nice if it were true. I do not even know the answer for $U_{2,4}$, but I would certainly conjecture that it is true in this case.

Problem 5.4. For any positive integers a and b and any matroid M, are there, up to isomorphism, only finitely many intertwines of M and $U_{a,b}$?

6. The basic classes

Let V be a basis of a matroid M. If each of the fundamental circuits of M, with respect to the basis V, have size at most 3, then we call M a *framed Dowling matroid with joint set V* and we call the matroid $M \setminus V$ a *Dowling matroid*. From a geometric point of view, these matroids are very natural; each element is on a line spanned by two of the joints. It is easy to check that the class is also minor-closed and that $U_{3,7}$ is not a Dowling matroid. The class of Dowling matroids is not self dual; for example, $U_{5,7}$ is not Dowling, but its dual, $U_{2,7}$, clearly is.

The class of Dowling matroids contains two well-known classes, namely, graphic matroids and bicircular matroids. Note that, every graphic matroid admits a representation over GF(2) with at most two 1s per column; by appending in an identity matrix to such a representation we obtain a framed Dowling matroid. Hence, graphic matroids are Dowling. There is also a partial converse; every binary framed Dowling matroid is graphic. However, there are binary Dowling matroids that are non-graphic. For example, R_{10} is Dowling but non-graphic.

Let $G = (V, E)$ be a graph with n vertices. We define a matroid $B^+(G)$ in ground set $V \cup E$ by describing its geometric representation in $\mathbb{R}^n$. We embed V as a basis in $\mathbb{R}^n$. Then for each non-loop edge $e = uv$ in G, we place a point e freely on the line spanned by u and v. If e is a loop of G on a vertex v, then we place e in parallel with v. The *bicircular matroid of* G, denoted $B(G)$, is defined as $B^+(G) \setminus V$. Note that $B^+(G)$ is a framed Dowling matroid and $B(G)$ is a Dowling matroid.

The following conjecture of Johnson, Robertson, and Seymour (personal communication, 1999) would be a significant step towards solving Problem 1.1.

Conjecture 6.1. *For any integer k there is an integer n such that if M has branch-width $\geqslant n$, then either M or M^* contains one of the following minors: $U_{k,2k}$, the cycle matroid of a $k \times k$ grid, or the bicircular matroid of the $k \times k$ grid.*

A matroid is *round* if it does not contain two disjoint cocircuits. The following theorem, see [2], may help in proving Theorem 6.1.

Theorem 6.2. *For any integer k there exists an integer n such that, if M is a round matroid of rank at least n, then M contains a minor isomorphic to $U_{k,2k}$, $M(K_k)$, or $B(K_k)$.*

Here is a miscellany of natural open problems involving Dowling matroids, although these are only peripherally related to Problem 1.1.

- Find the excluded minors for the class of Dowling matroids.
- Can we recognize Dowling matroids in polynomial time?

As we saw earlier, R_{10} is binary and Dowling, but it cannot be extended to a binary framed Dowling matroid. For a field $\mathbb{F}$, we will call a matroid a *Dowling matroid over* $\mathbb{F}$ if it can be extended to an $\mathbb{F}$-representable framed Dowling matroid or, equivalently, it admits a representation over $\mathbb{F}$ with at most two non-zero entries in each column.

- Find the excluded minors for the class of Dowling matroids over $\mathbb{R}$.
- Can we recognize Dowling matroids over $\mathbb{R}$ in polynomial time.

The class of Dowling matroids over $\mathbb{R}$ contains the class of Dowling matroids over GF(3); starting with GF(3) is already of considerable interest.

There is another rich class of matroids that avoid a $U_{a,b}$-minor. If $U_{a,b}$ is not representable over a finite field $\mathbb{F}$, then no $\mathbb{F}$-representable matroid contains a $U_{a,b}$-minor. Which begs the question:

Problem 6.3. Over which fields is $U_{a,b}$ representable?

This is a famously hard problem in geometry; for more information see Oxley [10, page 206]. Fortunately, the answer is not so important to us, which we explain by analogy with graphs. For any given graph H, if Σ is a surface in which H does not embed, then no graph that embeds in Σ contains an H-minor. The Graph Minors Structure Theorem shows that this is the basic class of graphs with no H-minor. In the proof of the Graph Minors Structure Theorem we do not need to know which surfaces H does not embed in, we only need to know that the number of such surfaces is finite.

7. Lifting and projection

Let N_1 and N_2 be matroids on a common ground set. If there is a matroid M and an element $e \in E(M)$ such that $N_1 = M \setminus e$ and $N_2 = M/e$, then we say that N_2 is an *elementary projection* of N_1 and that N_1 is an *elementary lift* of N_2.

Our interest in these operations is that they enable us to construct new classes of matroids avoiding a uniform matroid from the basic classes.

Theorem 7.1. *Let $\mathcal{M}$ be a minor-closed class of matroids, let $\mathcal{M}^+$ be the set of all matroids obtained by taking elementary lifts of matroids in $\mathcal{M}$, and let $\mathcal{M}^-$ be the set of all matroids obtained by elementary projection from matroids in $\mathcal{M}$. Then $\mathcal{M}^+$ and $\mathcal{M}^-$ are minor-closed. Moreover, if $U_{a,b} \notin \mathcal{M}$, then neither $\mathcal{M}^+$ nor $\mathcal{M}^-$ contains all uniform matroids.*

Note that, if N_2 is an elementary projection of N_1, then N_2^* is an elementary lift of N_1^*. Therefore it suffices to verify Lemma 7.1 for $\mathcal{M}^+$. It is easy to show that $\mathcal{M}^+$ is minor-closed. The fact that it does not contain all uniform matroids follows from the following lemma.

Lemma 7.2. *For any positive integer k there is an integer n such that if M is an elementary projection of $U_{n,2n}$, then M has a $U_{k,2k}$-minor.*

I do not have a nice proof of Lemma 7.2, but it is not difficult to prove Lemma 7.2 as a corollary of Theorem 6.2. It would be nice to know how large n needs to be as a function of k.

Lifting and projection are quite nice operations from a macroscopic point of view since the rank of any set changes by at most one. However, this point of view hides the complexity that can be created. For example, every paving matroid is an elementary lift of a uniform matroid and every spike is an elementary projection of $M(K_{2,n})$.

8. An extremal problem

Combining various results of Geelen, Kabell, Kung, and Whittle [1,3,7] yields the following theorem.

Theorem 8.1 *(Growth Rate Theorem). If $\mathcal{M}$ is a minor-closed class of matroids that does not contain all simple rank-2 matroids, then either*

(1) *there exists $c \in \mathbb{R}$ such that $|E(M)| \leqslant cr(M)$ for all simple matroids $M \in \mathcal{M}$,*
(2) *$\mathcal{M}$ contains all graphic matroids and there exists $c \in \mathbb{R}$ such that $|E(M)| \leqslant cr(M)^2$ for all simple matroids $M \in \mathcal{M}$, or*
(3) *there is a prime-power q and $c \in \mathbb{R}$ such that $\mathcal{M}$ contains all GF(q)-representable matroids and $|E(M)| \leqslant cq^{r(M)}$ for all simple matroids $M \in \mathcal{M}$.*

For a matroid M and integer t we let $\epsilon_t(M)$ denote the minimum number of rank-t flats that are required to cover all elements of M. Thus $\epsilon_1(M)$ is the number of points (rank-one flats) in M. (For fixed $t > 1$, can we compute $\epsilon_t(M)$ efficiently?)

Conjecture 8.2. *Let $\mathcal{M}$ be a minor-closed class of matroids that does not contain all uniform matroids and let t be minimum such that M does not contain all uniform matroids of rank $t+1$. Then either*

(1) *there exists $c \in \mathbb{R}$ such that $\epsilon_t(M) \leqslant cr(M)$ for all simple matroids $M \in \mathcal{M}$,*
(2) *$\mathcal{M}$ contains either all graphic matroids or all bicircular matroids and there exists $c \in \mathbb{R}$ such that $\epsilon_t(M) \leqslant cr(M)^2$ for all simple matroids $M \in \mathcal{M}$, or*
(3) *there is a prime-power q and $c \in \mathbb{R}$ such that $\mathcal{M}$ contains all GF(q)-representable matroids and $\epsilon_t(M) \leqslant cq^{r(M)}$ for all simple matroids $M \in \mathcal{M}$.*

9. Incompatible pairs of minors

Considering the proposed structure for matroids with no $U_{a,b}$-minor naturally gives rise to numerous conjectures. The highly connected pieces of a matroid with no $U_{a,b}$-minor should belong to certain basic classes which precludes the possibility of certain pairs of minors coexisting. Let N_1 and N_2 be minors of a matroid M. A pair (X, Y) of subsets of $E(M)$ *separates* N_1 from N_2 if $E(N_1) \subseteq X$, $E(N_2) \subseteq Y$, and $X \cup Y = E(M)$. We say that N_1 and N_2 are *k-interconnected* in M if for each pair (X, Y) separating N_1 from N_2, we have $r_M(X) + r_M(Y) - r(M) \geqslant k$.

Conjecture 9.1. *Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be finite fields of different characteristic and let k be a positive integer. Then there is an integer n such that, if M is a matroid with two n-interconnected minors $PG(n, \mathbb{F}_1)$ and $PG(n, \mathbb{F}_2)$, then M contains a $U_{k,2k}$-minor.*

The graph G_n^+ is constructed by adding an apex vertex to an $n \times n$ grid.

Conjecture 9.2. *For any positive integer k, there is an integer n such that, if M is a matroid with two n-interconnected minors $B(G_n^+)$ and $B(G_n^+)^*$, then M contains a $U_{k,2k}$-minor.*

Conjecture 9.3. *For any positive integer k, there is an integer n such that, if M is a matroid with two n-interconnected minors $B(G_n^+)$ and $M(G_n^+)^*$, then M contains a $U_{k,2k}$-minor.*

Conjecture 9.4. *For any positive integer k, there is an integer n such that, if M is a matroid with two n-interconnected minors $M(G_n^+)$ and $M(G_n^+)^*$, then either M contains a $U_{k,2k}$-minor or M contains a minor isomorphic to a rank-k projective geometry over some finite field.*

The above conjectures remain open when G_n^+ is replaced by K_n. These conjectures were inspired by the following conjecture of Robertson and Seymour.

Conjecture 9.5. *For any positive integer k there is an integer n such that, if M is a binary matroid with two n-interconnected minors $M(K_n)$ and $M(K_n)^*$, then M contains a $PG(k,2)$-minor.*

The above conjectures are frustrating because they involve intertwining. The following conjecture would be of considerable help in proving Conjecture 9.5.

Conjecture 9.6. *For any positive integer k, there is an integer n such that, if M is a binary matroid with an $M(K_n)$-minor and an $M(K_n)^*$-minor, then M has an $M(K_k) \oplus M(K_k)^*$-minor.*

Similar conjectures can be made relating to Conjectures 9.1–9.4.

10. Round matroids

In an attempt to simplify Problem 1.1, one could restrict his/her attention to highly connected matroids. This creates as many problems as it solves, since there are no useful inductive tools for k-connected matroids. There is, however, a connectivity notion that is easy to work with. Recall that a matroid is *round* if it does not contain two disjoint cocircuits. Thus round matroids have infinite vertical connectivity. Note that, if M is round, then M/e is round for all $e \in E(M)$. The following problem looks tractable and would be of considerable interest.

Problem 10.1. Let a and b be positive integers. Find a qualitative structure theorem for the class of round matroids with no $U_{a,b}$-minor.

Theorem 6.2 shows that if M is a round matroid with sufficiently large rank, then M contains a minor isomorphic to either $M(K_n)$, $B(K_n)$, or $U_{n,2n}$. That motivates the following conjecture.

Conjecture 10.2. *For any positive integer k, there is an integer n such that, if M is a round matroid with a $B(G_n^+)^*$-minor, then M contains a $U_{k,2k}$-minor.*

Problems 1.1 and 10.1 may be easier for the class of $\mathbb{R}$-representable matroids; this would be particularly interesting if Problem 4.2 has a positive answer. Note that projective planes over finite fields are not $\mathbb{R}$-representable. Therefore there should be just two basic classes, Dowling matroids and their duals.

Conjecture 10.3. *For any positive integer k, there is an integer n such that, if M is a round $\mathbb{R}$-representable matroid with an $M(G_n^+)^*$-minor, then M contains a $U_{k,2k}$-minor.*

For $\mathbb{R}$-representable matroids it may even be possible to find some exact characterizations.

Problem 10.4. Find a structural characterization for the class of round $\mathbb{R}$-representable matroids with no $U_{3,6}$-minor.

It may be possible to find a structural characterization of all $\mathbb{R}$-representable matroids with no $U_{3,6}$-minor, which may help to solve the following conjecture. (Real representability is required in the conjecture, Jeff Kahn showed that exists an infinite antichain of spikes none of which contains a $U_{2,5}$-minor; see Oxley [10, page 471].)

Conjecture 10.5. *If $(M_1, M_2, \ldots)$ is an infinite sequence of $\mathbb{R}$-representable matroids with no $U_{3,6}$-minor, then there exist integers $j > i \geqslant 1$ such that M_j contains a minor isomorphic to M_i.*

The following problems should be a good warm up for Problems 1.1 and 10.1.

- Let $\mathbb{F}$ be a field over which $U_{a,b}$ is not representable. Give a qualitative structure theorem for the matroids with no $U_{a,b}$-minor that have a spanning $\mathrm{PG}(k, \mathbb{F})$-restriction.
- Give a qualitative structure theorem for the matroids with no $U_{a,b}$-minor that have a spanning $\mathrm{B}(K_n)$-restriction.
- Give a qualitative structure theorem for the matroids with no $U_{a,b}$-minor that have a spanning $\mathrm{M}(K_n)$-restriction.

Acknowledgments

I have been working with Bert Gerards and Geoff Whittle for so long now that I would not claim to have independent ideas on this topic.

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