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Generating weakly 4-connected matroids ☆

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Abstract

We prove that, if M is a weakly 4-connected matroid with $|E(M)| \geqslant 7$ and neither M nor M^* is isomorphic to the cycle matroid of a ladder, then M has a proper minor M' such that M' is weakly 4-connected and $|E(M')| \geqslant |E(M)| - 2$ unless M is some 12-element matroid with a special structure.

Keywords: Wheels and Whirls theorem; 4-Connectivity

1. Introduction

A matroid M is said to be 4-*connected up to separators of size* l if M is 3-connected and, for each 3-separation (X, Y) of M, either $|X| \leqslant l$ or $|Y| \leqslant l$. Thus a matroid M is internally 4-connected if it is 4-connected up to separators of size 3. A matroid M is *weakly* 4-*connected* if M is 4-connected up to separators of size 4.

Theorem 1.1 *(Main theorem). Let M be a weakly* 4*-connected matroid with $|E(M)| \geqslant 7$. Then either*

- *there exists $e \in E(M)$ such that $M \setminus e$ or M/e is weakly* 4*-connected,*
- *M has a* 4*-element* 3*-separating set A with elements $c, d \in A$ such that $M \setminus d/c$ is weakly* 4*-connected,*
- *M or M^* is isomorphic to the cycle matroid of a ladder, or*

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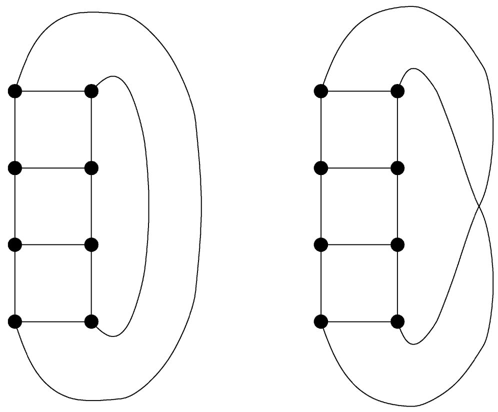

Fig. 1. A planar ladder and a Möbius ladder.

- $|E(M)| = 12$ *and* M *is a trident.*

We postpone the definition of "tridents" until Section 4 (see Definition 4.4).

There are two types of *ladders*, namely, *planar ladders* and *Möbius ladders*; see Fig. 1. A *planar ladder* is obtained from two disjoint circuits $(u_1, u_2, \dots, u_n, u_1)$ and $(v_1, v_2, \dots, v_n, v_1)$ by adding the matching $\{u_1v_1, \dots, u_nv_n\}$. A *Möbius ladder* is obtained from a circuit $(u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n, u_1)$ by adding the matching $\{u_1v_1, \dots, u_nv_n\}$.

Theorem 1.1 is analogous to Tutte's Wheels and Whirls theorem [8]:

Theorem 1.2 *(Wheels and Whirls theorem). Let* M *be a* 3*-connected matroid with at least one element. If* M *is neither a wheel nor a whirl, then* M *has a element* e *such that either* $M \setminus e$ *or* M/e *is* 3*-connected.*

The Wheels and Whirls theorem is stated here in a "top–down" way, however, it is perhaps more natural to think of it as a way of constructing 3-connected matroids: *any* 3*-connected matroid with at least* 4 *elements can be built from a wheel or a whirl by a sequence of single element extensions and coextensions so that each of the intermediate matroids is* 3*-connected.* Similarly, Theorem 1.1 can be viewed as an inductive construction of weakly 4-connected matroids. Such constructions have also been found for other variations of 4-connectivity. Geelen and Whittle [2] construct "sequentially 4-connected" matroids, and Hall [4] constructs matroids that are 4-connected up to separators of size 5. For binary matroids, internal 4-connectivity is certainly the most natural variant of 4-connectivity, and it would be particularly useful to have an inductive construction for this class. Unfortunately, even for internally 4-connected graphs, it is not possible to obtain a simple inductive construction; see Johnson and Thomas [5].

Seymour's decomposition theorem for regular matroids [7] is equivalent to the assertion that: *every weakly* 4*-connected regular matroid is either graphic, cographic, or is isomorphic to* R_{10}. Theorem 1.1 suggests a reasonably natural line of proof for the decomposition theorem. (We are not suggesting that this will be any easier than Seymour's approach.) However, similar ideas could be used in proving new decomposition results; for example, one could consider the class of binary matroids with no AG(3, 2)-minor.

We assume the reader is familiar with matroid theory. Our notation and terminology will follow Oxley [6] with one exception: We use $\mathrm{si}(M)$ (respectively $\mathrm{co}(M)$) to denote the simplification (respectively cosimplification) of a matroid M.

2. Preliminaries

In this section, we present some basic lemmas on separations that will be used in later sections.

Let $M = (E, r)$ be a matroid where r is the rank function. For $A \subseteq E$, we let $\lambda_M(A) = r(A) + r(E \setminus A) - r(M)$. We refer to λ_M as the *connectivity function* of M. Tutte [8] proved that the connectivity function is *submodular*; that is, if $X, Y \subseteq E(M)$, then

$$\lambda_M(X) + \lambda_M(Y) \geqslant \lambda_M(X \cap Y) + \lambda_M(X \cup Y).$$

For sets $X, Y \subseteq E(M)$, we let $\sqcap_M(X, Y) = r_M(X) + r_M(Y) - r_M(X \cup Y)$. If $\sqcap_M(X, Y) = 0$ then we say that X and Y are *skew*. The next three identities follow directly from the definitions.

Lemma 2.1. *Let M be a matroid and let (A, B, C) be a partition of $E(M)$. Then*

$$\lambda_M(A) = \sqcap_M(A, B) + \sqcap_{M^*}(A, C).$$

Lemma 2.2. *Let M be a matroid and let A and B be disjoint subsets of $E(M)$. Then*

$$\lambda_M(A \cup B) = \lambda_M(A) + \lambda_M(B) - \sqcap_M(A, B) - \sqcap_{M^*}(A, B).$$

Lemma 2.3. *Let M be a matroid and let A and B be disjoint subsets of $E(M)$. Then*

$$\lambda_{M/B}(A) = \lambda_M(A) - \sqcap_M(A, B).$$

A set $A \subseteq E$ is said to be *k-separating* if $\lambda_M(A) \leqslant k - 1$; when equality holds we say that A is *exactly k-separating*. Thus a partition (X, Y) of E is a *k-separation* if X is k-separating and $|X|, |Y| \geqslant k$. The next lemma is an easy consequence of submodularity.

Lemma 2.4. *Let X and Y be k-separating sets of a matroid M. If $X \cap Y$ is not $(k-1)$-separating in M, then $X \cup Y$ is k-separating in M.*

A 3-separation (X, Y) of a matroid M is called a *meaty* 3-separation if $|X|, |Y| \geqslant 5$. Thus M is weakly 4-connected if and only if M is 3-connected and has no meaty 3-separations. A sequence $e_1, e_2, \ldots, e_i$ of distinct elements in $E(M)$ is called a *fan* if $\{e_1, e_2, e_3\}, \{e_2, e_3, e_4\}, \ldots, \{e_{i-2}, e_{i-1}, e_i\}$ are alternately triangles and triads.

The *coclosure* of a set $X \subseteq E(M)$ is the closure of X in M^*. Clearly, an element $x \in E(M) \setminus X$ belongs to the coclosure of X if and only if x does not belong to the closure of $E(M) \setminus (X \cup \{x\})$. A set $X \subseteq E(M)$ is *coclosed* if the coclosure of X is the set X itself. We say X is *fully closed* if X is both closed and coclosed.

Let (A, B) be a k-separation of the matroid M. An element $x \in E(M)$ is in the *guts* of (A, B) if x belongs to the closure of both A and B. Dually, x is in the *coguts* of (A, B) if x belongs to the coclosure of both A and B. The next lemma follows easily from definitions.

Lemma 2.5. *Let (A, B) be an exact k-separation of matroid M and let $x \in B$. Then*

- *$A \cup \{x\}$ is exactly k-separating if x belongs to either the guts or the coguts of (A, B), but not both.*

- *$A \cup \{x\}$ is exactly $(k-1)$-separating if x belongs to both the guts and the coguts of (A, B).*
- *$A \cup \{x\}$ is exactly $(k+1)$-separating if x belongs to neither the guts nor the coguts of (A, B).*

Let x be an element of the matroid M and let (A, B) be a k-separation of $M \setminus x$. Then x *blocks* (A, B) if neither $(A \cup \{x\}, B)$ nor $(A, B \cup \{x\})$ is a k-separation of M. Now let (A, B) be a k-separation of M/x. Then x *coblocks* (A, B) if neither $(A \cup \{x\}, B)$ nor $(A, B \cup \{x\})$ is a k-separation of M. The following lemma also follows easily from definitions.

Lemma 2.6. *Let M be a matroid and let $\{A, B, \{x\}\}$ be a partition of $E(M)$. Then*

- *if (A, B) is an exact k-separation of $M \setminus x$, then x blocks (A, B) if and only if x is not a coloop of M, $x \notin \mathrm{cl}_M(A)$, and $x \notin \mathrm{cl}_M(B)$,*
- *if (A, B) is an exact k-separation of M/x, then x coblocks (A, B) if and only if x is not a loop, $x \in \mathrm{cl}_M(A)$, and $x \in \mathrm{cl}_M(B)$.*

For sets X_1, X_2, Y_1, and Y_2, the pairs (X_1, Y_1) and (X_2, Y_2) are said to *cross* if all the four sets $X_1 \cap X_2$, $X_1 \cap Y_2$, $Y_1 \cap X_2$, and $Y_1 \cap Y_2$ are non-empty. The next lemma is due to Coullard [1], see also [6, Lemma 8.4.7].

Lemma 2.7. *Let e be an element of a 3-connected matroid M. Now, let (X_d, Y_d) be a 3-separation of $M \setminus e$ that is blocked by e, and let (X_c, Y_c) be a 3-separation of M/e that is coblocked by e. Then (X_d, Y_d) and (X_c, Y_c) cross. Moreover,*

- *one of $X_d \cap X_c$ and $Y_d \cap Y_c$ is 3-separating in M, and*
- *one of $X_d \cap Y_c$ and $Y_d \cap X_c$ is 3-separating in M.*

The next lemma is due to Geelen and Whittle [2, Lemma 4.2]; it is an easy consequence of Lemma 2.7.

Lemma 2.8. *Let M be a 4-connected matroid and x be an element of M. Then at least one of $M \setminus x$ and M/x is weakly 4-connected.*

The next lemma can be found in Geelen and Zhou [3, Lemma 3.6].

Lemma 2.9. *Let (A, B) be a 3-separation of a 3-connected matroid M where A is coclosed and $|A| \geqslant 4$. If $e \in A$ is in the guts of the separation (A, B), then $M \setminus e$ is 3-connected.*

Lemma 2.10. *If e is an element of an internally 4-connected matroid M with $|E(M)| \geqslant 7$, then either $M \setminus e$ or M/e is 3-connected.*

Proof. Since M is internally 4-connected, $\mathrm{si}(M/e)$ and $\mathrm{co}(M \setminus e)$ are both 3-connected. Moreover e cannot be in both a triangle and a triad, and hence one of $M \setminus e$ and M/e is 3-connected. □

Lemma 2.11. *Let M be an internally 4-connected matroid and let T_1 and T_2 be two disjoint triangles of M. If $\sqcap_M(T_1, T_2) = 1$, then $M \setminus e$ is internally 4-connected for every $e \in T_1 \cup T_2$.*

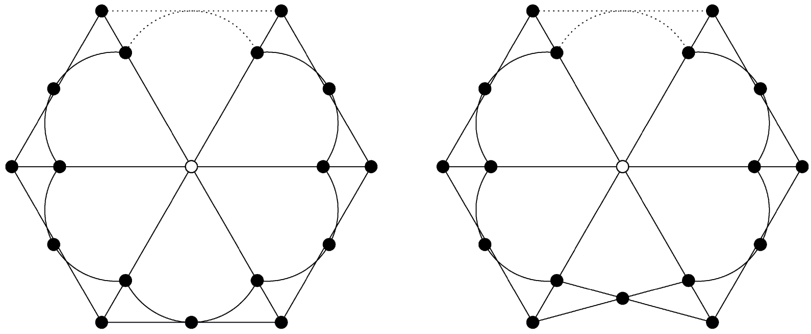

Fig. 2. Duals of ladders.

Proof. By symmetry assume $e \in T_1$. Let (X, Y) be a 3-separation of $M \setminus e$ with $|X| \geqslant 4$ and $|Y| \geqslant 4$. Then $e \notin \mathrm{cl}_M(X)$ and $e \notin \mathrm{cl}_M(Y)$. Therefore $|X \cap T_1| = |Y \cap T_1| = 1$. Either X or Y contains at least two elements of T_2. By symmetry, we may assume $|X \cap T_2| \geqslant 2$. So $e \in \mathrm{cl}_M(X)$, a contradiction. □

Lemma 2.12. *Let M be an internally 4-connected matroid and let T be a triangle and T^* be a triad of M. If $\sqcap_M(T, T^*) \neq 0$, then $M \setminus e$ is weakly 4-connected for every $e \in T$.*

Proof. Let $e \in T$ and let (X, Y) be a meaty 3-separation of $M \setminus e$ with $|X \cap T^*| \geqslant 2$. Let $X' := X \cup T^*$ and $Y' := Y - T^*$. Then (X', Y') is a 3-separation in $M \setminus e$ and $|X'|, |Y'| \geqslant 4$. Since M is internally 4-connected, $e \notin \mathrm{cl}_M(X')$ and $e \notin \mathrm{cl}_M(Y')$. Since $e \notin \mathrm{cl}_M(Y')$, there exists an element $f \in T \cap X'$. Since M is internally 4-connected, $f \notin \mathrm{cl}_M(T^*)$ and, since $\sqcap_M(T, T^*) \neq 0$, we have $T \subseteq \mathrm{cl}_M(T^* \cup \{f\})$, contrary to the fact the $e \notin \mathrm{cl}_M(X')$. □

3. Internally 4-connected matroids

In this section we prove Theorem 1.1 for internally 4-connected matroids. By Lemma 2.8, me may assume that M is not 4-connected, so, by duality, we may assume that M has a triangle T. In this section we prove the following theorem.

Theorem 3.1. *Let M be an internally 4-connected matroid with $|E(M)| \geqslant 7$. If M has a triangle, then either*

- *there exists $e \in E(M)$ such that $M \setminus e$ is weakly 4-connected,*
- *M^* is isomorphic to the cycle matroid of a ladder, or*
- *$|E(M)| \leqslant 13$ and there exists $e \in E(M)$ such that M/e is weakly 4-connected.*

The duals of cycle matroids of ladders have a nice geometric structure; see Fig. 2. (Note that each "rung" of a ladder is in two traids; so the rungs correspond to the elements in Fig. 2 that are in two triangles.)

The key result of this section describes the local obstruction to deleting an element of T. The obstruction is a "rotor," which is defined formally below; see Fig. 3.

Theorem 3.2. *Let T be a triangle of an internally 4-connected matroid M. Then either*

- *there is an element $e \in T$ such that $M \setminus e$ is weakly 4-connected,*

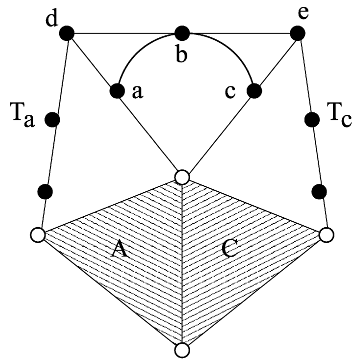

Fig. 3. A rotor with central triangle $\{a, b, c\}$.

- *T is the central triangle of a rotor,*
- *$|E(M)| \leqslant 15$ and there is an element $x \in E(M)$ such that $M \setminus x$ is weakly 4-connected,*
- *$|E(M)| \leqslant 13$, $r(M^*) < r(M)$, and either M has a triad or there is an element $e \in E(M)$ such that M/e is weakly 4-connected.*

Definition 3.3. We say that $(a, b, c, d, e, T_a, T_c, A, C)$ is a *rotor* of M if a, b, c, d, and e are distinct elements of M, T_a, T_c, and $\{a, b, c\}$ are disjoint triangles of M with $d \in T_a$ and $e \in T_c$, and (A, C) is a proper partition of $E(M) - (T_a \cup T_c \cup \{a, b, c\})$ such that

- $T_a \cup \{b, e\}$ is 3-separating in $M \setminus a$, $T_a \cup \{a\} \cup A$ is 3-separating in $M \setminus b$, and $T_c \cup \{b, d\}$ is 3-separating in $M \setminus c$; and
- T_a is 2-separating in $M \setminus a, b$ and T_c is 2-separating in $M \setminus b, c$.

We call $\{a, b, c\}$ the *central triangle* of the rotor.

Lemma 3.4. *Let $T = \{a, b, c\}$ be a triangle in an internally 4-connected matroid. If (A, B) is a meaty 3-separation of $M \setminus c$ with $a \in A$, then $b \in B$. Moreover, neither a nor b is in the guts or coguts of (A, B).*

Proof. Since M is internally 4-connected, c blocks (A, B). Therefore $c \notin \mathrm{cl}_M(A)$, and hence $b \in B$. If b were in the guts or coguts (A, B), then $(A \cup \{b\}, B \setminus \{b\})$ is a 3-separation of $M \setminus c$, and hence $(A \cup \{b, c\}, B \setminus \{b\})$ is a 3-separation of M, contradicting the fact M is internally 4-connected. Therefore b is not in the guts or coguts of (A, B) and, by symmetry, neither is a. $\square$

We will now work toward a proof of Theorem 3.2. Note that, if $|E(M)| \leqslant 10$, then Theorem 3.2 is an immediate consequence of Lemma 2.10. The results below are subject to the following hypothesis.

Hypothesis 3.5. *Let M be an internally 4-connected matroid with at least 11 elements, let $T = \{a, b, c\}$ be a triangle in M, and let (A_b, A_c), (B_a, B_c), and (C_a, C_b) be meaty 3-separations of $M \setminus a$, $M \setminus b$, and $M \setminus c$, respectively, where a is in B_a and C_a, b is in A_b and C_b, and c is in A_c and B_c.*

We begin by considering interactions between the 3-separations (A_b, A_c) in $M \setminus a$ and (B_a, B_c) in $M \setminus b$. Evidently similar results hold for any two of a, b, and c.

3.5.1. $\lambda_M(A_b \cap B_c) = \lambda_{M\setminus a,b}(A_b \cap B_c)$.

Proof. By Lemma 3.4, a is not in the coguts of (B_a, B_c) in $M \setminus b$ and, hence, $a \in \mathrm{cl}_M(B_a - \{a\})$. Therefore $\lambda_{M\setminus b}(A_b \cap B_c) = \lambda_{M\setminus a,b}(A_b \cap B_c)$. Then, since $b \in \mathrm{cl}_M(\{a, c\})$, we have $\lambda_M(A_b \cap B_c) = \lambda_{M\setminus b}(A_b \cap B_c)$. Hence $\lambda_M(A_b \cap B_c) = \lambda_{M\setminus a,b}(A_b \cap B_c)$. □

A similar argument proves the following result.

3.5.2. $\lambda_M(A_c \cap B_c) = \lambda_{M\setminus a,b}(A_c \cap B_c)$.

3.5.3. *If $|A_b \cap B_c| \geqslant 2$, then $A_c \cap B_a$ is 3-separating in M and $|A_c \cap B_a| \leqslant 3$.*

Proof. If $|A_b \cap B_c| \geqslant 2$, then, by 3.5.1, $\lambda_{M\setminus a,b}(A_b \cap B_c) = \lambda_M(A_b \cap B_c) \geqslant 2$. Then, by submodularity, $\lambda_{M\setminus a,b}(A_c \cap B_a) \leqslant 2$. Now, by swapping the roles of a and b in 3.5.1, $\lambda_M(A_c \cap B_a) = \lambda_{M\setminus a,b}(A_c \cap B_a) \leqslant 2$. That is, $A_c \cap B_a$ is 3-separating in M, and, hence, $|A_c \cap B_a| \leqslant 3$, as required. □

The following two results are proved similarly.

3.5.4. *If $|A_c \cap B_c| \geqslant 2$, then $A_b \cap B_a$ is 3-separating in $M \setminus a, b$.*

3.5.5. *If $A_b \cap B_a$ is not 2-separating in $M \setminus a, b$, then $A_c \cap B_c$ is 3-separating in M, thus $|A_c \cap B_c| \leqslant 3$.*

Note that $M \setminus a, b$ need not be 3-connected (for example, in a rotor $M \setminus a, b$ is not 3-connected).

3.5.6. *Suppose that $A_b \cap B_a$ is 2-separating in $M \setminus a, b$ and that $|A_b \cap B_a| \geqslant 2$. Then $|A_b \cap B_a| \leqslant 3$ and both of $A_b \cap B_a \cap C_a$ and $A_b \cap B_a \cap C_b$ are non-empty. Moreover, if $|A_b \cap B_a| = 3$ and $|A_b \cap B_a \cap C_b| = 1$, then $A_b \cap B_a$ is a triangle of M, $|C_b| = 5$, and $\lambda_M(B_a \cap C_b) \geqslant \min(3, |B_a \cap C_b|)$.*

Proof. Note that $A_b \cap B_a$ is 3-separating in $M \setminus a$, and, hence, also in M. Since M is internally 4-connected, $|A_b \cap B_a| \leqslant 3$.

Suppose that $A_b \cap B_a \cap C_b = \emptyset$. Then $A_b \cap B_a \subseteq C_a$. Note that b coblocks the 2-separation of $M \setminus a, b$ determined by $A_b \cap B_a$. Hence $b \in \mathrm{cl}^*_{M\setminus a}(A_b \cap B_a)$. This implies $b \in \mathrm{cl}^*_M(C_a)$, contradicting Lemma 3.4.

Now suppose that $|A_b \cap B_a| = 3$ and that $A_b \cap B_a \cap C_b = \{p\}$. Note that $A_b \cap B_a$ is 3-separating in M, therefore it is either a triangle or a triad of M. Now if $A_b \cap B_a$ is a triad, we would have $b \in \mathrm{cl}^*_M((A_b \cap B_a) \cup \{a\}) = \mathrm{cl}^*_M((A_b \cap B_a \cap C_a) \cup \{a\})$. Then b is in the coguts of (C_a, C_b), contrary to Lemma 3.4, and, hence, $A_b \cap B_a$ is a triangle. Now p is in the guts of (C_a, C_b) and b is in the coguts of $(C_a \cup \{p\}, C_b \setminus \{p\})$, so $C_b \setminus \{b, p\}$ is 3-separating in M and, hence, $|C_b| \leqslant 5$. Finally suppose that $B_a \cap C_b$ is 3-separating in M and that $|B_a \cap C_b| = 3$. Since $p \in B_a \cap C_b$ is in a triangle and M is internally 4-connected, p is not in a triad. Therefore $B_a \cap C_b$

is a triangle Note that there exists a cocircuit C^* of M with $a, b \in C^* \subseteq (A_b \cap B_a) \cup \{a, b\}$. However, $(B_a \cap C_b) \cap ((A_b \cap B_a) \cup \{a, b\}) = \{p\}$. Since a circuit and a cocircuit cannot meet in a singleton, $p \notin C^*$. But then $C^* \cap C_b = \{b\}$, contrary to Lemma 3.4. □

3.5.7. $|A_b \cap B_a|, |A_b \cap B_c|, |A_c \cap B_a| \geqslant 1$ *and* $|A_c \cap B_c| \geqslant 2$.

Proof. Suppose that $A_b \cap B_a = \emptyset$. Since $|A_b| \geqslant 5$ and $|B_a| \geqslant 5$, $|A_b \cap B_c|, |B_a \cap A_c| \geqslant 4$, contrary to 3.5.3. Similarly if $A_c \cap B_c = \{c\}$, then $|A_b \cap B_c|, |B_a \cap A_c| \geqslant 4$, contradicting 3.5.3.

Now suppose that $A_b \cap B_c = \emptyset$. Then $|A_b \cap B_a|, |A_c \cap B_c| \geqslant 4$. By Lemma 3.5.6, $A_b \cap B_a$ is not 2-separating in $M \setminus a, b$ and this gives a contradiction to 3.5.5. Thus $A_b \cap B_c \neq \emptyset$ and, by symmetry, $A_c \cap B_a \neq \emptyset$. □

3.5.8. *If* $|A_b \cap B_c| = 1$ *or* $|A_c \cap B_a| = 1$, *then* $A_b \cap B_a$ *is a triangle of* M *and is* 2*-separating in* $M \setminus a, b$.

Proof. By symmetry we may assume that $|A_b \cap B_c| = 1$. Since $|A_b|, |B_c| \geqslant 5$, we have $|A_b \cap B_a| \geqslant 3$ and $|A_c \cap B_c| \geqslant 4$. Then, by 3.5.4, $A_b \cap B_a$ is 2-separating in $M \setminus a, b$. Now the result follows from 3.5.6. □

3.5.9. *If* $A_c \cap B_c$ *is* 3*-separating in* M *and has size* 3, *then* $|A_c \cap B_c \cap C_a| = 1$ *and* $|A_c \cap B_c \cap C_b| = 1$.

Proof. Since c is in the triangle $\{a, b, c\}$ and $A_c \cap B_c$ is 3-separating, $A_c \cap B_c$ is a triangle of M. If $|A_c \cap B_c \cap C_a| = 2$, then $c \in \mathrm{cl}_M(C_a)$, thus c does not block (C_a, C_b), a contradiction. □

In order to keep track of symmetries we introduce the following notation.

$$\begin{aligned} S^0 &:= A_b \cap B_c \cap C_a, & S^1 &:= A_c \cap B_a \cap C_b, \\ S^0_a &:= A_b \cap B_a \cap C_a, & S^1_a &:= A_c \cap B_a \cap C_a, \\ S^0_b &:= A_b \cap B_c \cap C_b, & S^1_b &:= A_b \cap B_a \cap C_b, \\ S^0_c &:= A_c \cap B_c \cap C_a, & S^1_c &:= A_c \cap B_c \cap C_b. \end{aligned}$$

The above eight sets together with the set $\{a, b, c\}$ partition $E(M)$. Figure 4 reveals the various symmetries induced by permutations of $\{a, b, c\}$.

The results below are subject to the following hypotheses.

Hypothesis 3.6. *In addition to Hypothesis* 3.5 *we assume that* T *is not the central triangle in a rotor and that if* $|E(M)| \geqslant 15$ *then* $M \setminus e$ *is not weakly* 4*-connected for any* $e \in E(M)$.

3.6.1. $|S^0_a| \leqslant 1$ *or* $|S^1_c| \leqslant 1$.

Proof. Suppose that $|S^0_a|, |S^1_c| \geqslant 2$. By 3.5.9, $\lambda_M(A_c \cap B_c) \geqslant 3$. Then, by 3.5.5 and 3.5.6, $A_b \cap B_a$ is a triangle and is 2-separating in $M \setminus a, b$. Thus $|S^0_a| = 2$ and $|S^1_b| = 1$. By symmetry, $|S^1_c| = 2$ and $|S^0_b| = 1$. Moreover, by 3.5.6, $|C_b| = 5$ and, hence, S^1 is empty. However $|B_a| \geqslant 5$, so $|S^1_a| > 0$. By symmetry, $B_c \cap C_b$ is a triangle, $\lambda_{M \setminus b,c}(B_c \cap C_b) = 1$, $S^0 = \emptyset$, and $S^0_c \neq \emptyset$. Therefore we get a rotor $(a, b, c, d, e, A_b \cap B_a, B_c \cap C_b, S^1_a, S^0_c)$ where $d \in S^1_b$ and $e \in S^0_b$, contrary to Hypothesis 3.6. □

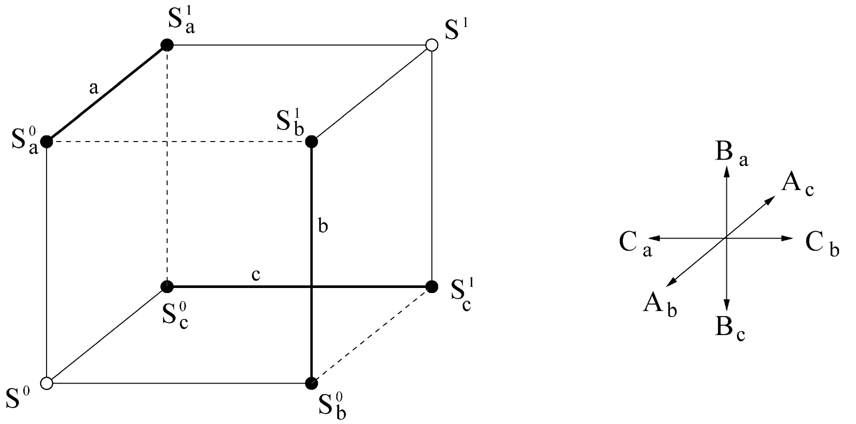

Fig. 4. Three crossing separations.

3.6.2. *There exists $\alpha \in \{a, b, c\}$ such that $|S_\alpha^0| \leqslant 1$.*

Proof. Suppose that $|S_\alpha^0| \geqslant 2$ for each $\alpha \in \{a, b, c\}$. By 3.5.9, $\lambda_M(A_c \cap B_c) \geqslant 3$. Let $T'_a = B_c \cap C_b$, $T'_b = A_c \cap C_a$, and $T'_c = A_b \cap B_a$. Then, by 3.5.5 and 3.5.6, T'_c is a triangle and is 2-separating in $M \setminus a, b$. By symmetry, T'_b and T'_a are also triangles and $\lambda_{M\setminus a,c}(T'_b) = \lambda_{M\setminus b,c}(T'_a) = 1$. From 3.5.6 we also see that $|S_a^0| = 2$, $|S_b^1| = 1$, and $|C_b| = 5$. By symmetry, $|S_\alpha^0| = 2$ and $|S_\alpha^1| = 1$ for each $\alpha \in \{a, b, c\}$. Since $|C_b| = 5$, the set S^1 is empty. Let $e_a \in S_a^1$, $e_b \in S_b^1$, and $e_c \in S_c^1$.

Since M is internally 4-connected and $|E(M)| > 9$, $M \setminus T$ is connected. Moreover each of T'_a, T'_b, and T'_c is 2-separating in $M \setminus T$. Moreover, by Lemma 2.11, the triangles T'_a, T'_b and T'_c are pair-wise skew. It follows that $r_M(T'_a \cup T'_b \cup T'_c) \in \{5, 6\}$.

If $T'_a \cap \mathrm{cl}_M(T'_c \cup \{b\}) \neq \emptyset$, then we obtain a rotor $(a, b, c, e_b, d, T'_c, T'_a, \{e_a\}, S^0 \cup S_c^0)$ where $d \in T'_a \cap \mathrm{cl}_M(T'_c \cup \{b\})$. Therefore we may assume that $T'_a \cap \mathrm{cl}_M(T'_c \cup \{b\})$ is empty. By symmetry we may also assume that $T'_b \cap \mathrm{cl}_M(T'_a \cup \{c\})$ and $T'_c \cap \mathrm{cl}_M(T'_b \cup \{a\})$ are empty.

Suppose that $r_M(T'_a \cup T'_b \cup T'_c) = 6$. Since a is not in the coguts of the 3-separation (B_a, B_c) in $M \setminus b$, we see that $a \in \mathrm{cl}_M(T'_c \cup \{e_a\})$. Similarly $b \in \mathrm{cl}_M(T'_a \cup \{e_b\})$ and $c \in \mathrm{cl}_M(T'_b \cup \{e_c\})$. Therefore $b, c \in \mathrm{cl}_M(T'_b \cup T'_a \cup \{e_b\})$ and, hence, $a \in \mathrm{cl}_M(T'_b \cup T'_a \cup \{e_b\})$. Note that the flats $\mathrm{cl}_M(T'_c \cup \{e_a\})$ and $\mathrm{cl}_M(T'_b \cup T'_a \cup \{e_b\})$ are modular, so $\{a, e_a, e_b\}$ is a triangle. However, this is contrary to the fact that $T'_c \cap \mathrm{cl}_M(T'_b \cup \{a\})$ is empty. Therefore $r_M(T'_a \cup T'_b \cup T'_c) = 5$. Now S^0 is 3-separating in $M \setminus T$ and, hence, also in M. So $|S^0| \leqslant 3$ and $12 \leqslant |E(M)| \leqslant 15$.

Finally we will show that there exists $e \in T'_c$ such that $M \setminus e$ is weakly 4-connected. Since $a \in \mathrm{cl}^*_{M\setminus b}(T'_c)$ and M is internally 4-connected, there exists $e \in T'_c$ such that $\{a, b\} \cup (T'_c \setminus \{e\})$ is a cocircuit of $M \setminus e$. Let (X, Y) be a meaty 3-separation of $M \setminus e$ with $a \in X$. Note that X and Y each contain one element of T'_c; let $f \in X \cap T'_c$ and $g \in Y \cap T'_c$. By Lemma 3.4, g is not in the coguts of the 3-separation (X, Y) in $M \setminus e$. However $\{a, b, f, g\}$ is a cocircuit of $M \setminus e$ and $a, f \in X$, so $b \in Y$. Since $T'_c \cap \mathrm{cl}_M(T'_b \cup \{a\}) = \emptyset$, for each $x \in T'_c$, we have $T'_c \subseteq \mathrm{cl}_M(T'_b \cup \{a, x\})$. In particular, $e \subseteq \mathrm{cl}_M(T'_b \cup \{a, x\})$. Then, since $a, f \in X$ and $e \notin \mathrm{cl}_M(X)$, we have $|T'_b \cap Y| \geqslant 2$. Now, since $e \in \mathrm{cl}_M(T'_b \cup \{a, g\})$ but $e \notin \mathrm{cl}_M(Y)$, we see that $a \notin \mathrm{cl}_M(Y)$. However $b \in Y$, so we have $c \in X$. Now $b \in \mathrm{cl}_M(X)$, $g \in \mathrm{cl}^*_{M\setminus e}(X \cup \{b\})$, and $e \in \mathrm{cl}_M(X \cup \{b, g\})$, so $Y - \{b, g\}$ is 3-separating in M and, hence, $|Y| \leqslant 5$. Similarly $f \in \mathrm{cl}_{M\setminus e}(Y \cup \{c\}) \cap \mathrm{cl}^*_{M\setminus e}(Y \cup \{c\})$, so $Y \cup \{c, f\}$ is 3-separating in $M \setminus e$. Moreover, $e \in \mathrm{cl}_M(Y \cup \{c, f\})$ so $X - \{c, f\}$ is 3-separating in M and, hence, $|X| \leqslant 5$; contrary to the fact that $|E(M)| \geqslant 12$. □

3.6.3. $|A_c \cap B_a| \geqslant 2$.

Proof. By 3.5.7, we have $|A_c \cap B_a| \geqslant 1$; suppose that $|A_c \cap B_a| = 1$. Since $|A_c| \geqslant 5$ and $|B_a| \geqslant 5$, we have $|A_b \cap B_a| \geqslant 3$ and $|A_c \cap B_c| \geqslant 4$. Then, by 3.5.5 and 3.5.6, $A_b \cap B_a$ is a triangle in M and is 2-separating in $M \setminus a, b$, and $|S_a^0|, |S_b^1| \geqslant 1$. We consider the following two cases separately.

Case 1. $|S_a^0| = 1$ and $|S_b^1| = 2$.

Since $|C_a| \geqslant 5$ and $|S_a^1| \leqslant |A_c \cap B_a| = 1$, we have $|B_c \cap C_a| \geqslant 2$. Then, by 3.5.3 and symmetry, $B_a \cap C_b$ is 3-separating. However S_b^1 is a 2-element set that is in the triangle $A_b \cap B_a$, so, since M is internally 4-connected, we have $|S^1| = 0$. Then, since $|A_c \cap B_a| = 1$, we have $|S_a^1| = 1$. By 3.6.1 and symmetry, we may assume that $|S_c^0| \leqslant 1$. Since $|C_a| \geqslant 5$, we have $S^0 \neq \emptyset$.

Claim. $B_a \cap C_a$ is a triangle of M.

Proof. Suppose that $B_a \cap C_a$ is not a triangle. Since $A_b \cap B_a$ is a triangle, $B_a \cap C_a$ is not a triad and, hence, $\lambda_M(B_a \cap C_a) \geqslant 3$. Then, by 3.5.5 and symmetry, $\lambda_{M \setminus b,c}(B_c \cap C_b) = 1$. Now since $|A_c| \geqslant 5$, we have $|S_c^1| \geqslant 2$. Then, by 3.5.6 and symmetry, $|S_b^0| = 1$ and $|A_b| = 5$, which is not possible. □

By 3.5.6, we have $|C_a| = 5$ and, hence, $|S^0| + |S_c^0| = 2$. This gives the following two subcases.

Case 1.1. $|S^0| = 2$ and $|S_c^0| = 0$.

Since $|A_c| \geqslant 5$, we have $|S_c^1| \geqslant 3$. By 3.5.3 and symmetry, $A_b \cap C_a$ is 3-separating in M, hence is a triangle of M. Now, by 3.5.6, the triangle $A_b \cap B_a$ is 2-separating in $M \setminus a, b$. However, since $A_b \cap C_a$ is a triangle, S_b^1 is 2-separating in $M \setminus a, b$. Then, since $B_a \cap C_a$ and $\{a, b, c\}$ are triangles, S_b^1 is 2-separating in M, contradicting the fact that M is 3-connected.

Case 1.2. $|S^0| = |S_c^0| = 1$.

Let w, x, y, and z denote the elements in S_a^0, S_a^1, S_c^0, and S^0 respectively. Note that, if $\{x, y, z\}$ is a triangle, then we have a rotor $(b, a, c, w, z, A_b \cap B_a, \{x, y, z\}, S_b^0, S_c^1)$. Therefore we may assume that $\{x, y, z\}$ is not a triangle.

By 3.5.9 and symmetry, $A_b \cap C_b$ is not 3-separating in M. Then, by 3.5.5, 3.5.6, and symmetry, $\{x, y\}$ is 2-separating in $M \setminus a, c$. Thus $\{x, y, a\}$ is a triad in $M \setminus c$. Then, since $B_a \cap C_a = \{w, a, x\}$ is a triangle, the sets $\{a, x, y\}$, $\{a, w, x, y\}$ and $C_a = \{a, w, x, y, z\}$ are all 3-separating in $M \setminus c$. Thus z is in the guts or the coguts of the 3-separation (C_a, C_b) in $M \setminus c$.

First suppose that z is in the coguts of (C_a, C_b) in $M \setminus c$. Therefore there is a cocircuit $C^* \subseteq C_a$ containing z. Since $A_b \cap B_a$ is a circuit, $w \notin C^*$. Now $\{x, y\}$ is 2-separating in $M \setminus a, c$, and $z \in \mathrm{cl}^*_{M \setminus a,c}(\{x, y\})$. Therefore $\{x, y, z\}$ is 2-separating in $M \setminus a, c$ and, hence, $\{x, y, z\}$ is 3-separating in M. Since x is in a triangle and M is internally 4-connected, $\{x, y, z\}$ is a triangle, contrary to our assumption above. Therefore z is in the guts of (C_a, C_b). So $z \in \mathrm{cl}_M(\{a, w, x, y\})$ and, since $\{w, a, x\}$ is a triangle, $z \in \mathrm{cl}_M(\{a, x, y\})$. However $\{x, y, z\}$ is not a triangle, so there is a circuit $C \subseteq \{a, x, y, z\}$ of M containing a and z. Recall that $A_b \cap B_a$ is 2-separating in $M \setminus a, b$.

Then, since C and T are circuits, $A_b \cap B_a$ is 2-separating in M, contrary to the fact that M is 3-connected.

Case 2. $|S_a^0| = 2$ and $|S_b^1| = 1$.

By 3.6.1, we may assume that $|S_c^1| \leqslant 1$. Therefore, since $|A_c \cap B_c| \geqslant 4$, we have $|S_c^0| \geqslant 2$. Then, by 3.6.2, we may assume that $|S_b^0| \leqslant 1$. However $|C_b| \geqslant 5$, so we have $|S_b^0| = 1$, $|S_c^1| = 1$, and $|S^1| = 1$. Since $|A_c \cap B_a| = 1$, we have $|S_a^1| = 0$. Note that $|A_c \cap C_b| = 2$, so, by 3.5.3 and symmetry, $A_b \cap C_a$ is 3-separating. However S_a^0 is a 2-element set that is in a triangle $A_b \cap B_a$, so, since M is internally 4-connected, we have $|S^0| = 0$. Thus $|A_b \cap B_c| = 1$ and, up to symmetry, we are back in Case 1. □

3.6.4. $|S^0| = |S^1| = 1$, $|S_\alpha^i| \in \{1, 2\}$ *for each* $i \in \{0, 1\}$ *and* $\alpha \in \{a, b, c\}$, *and* $|S_a^i| + |S_b^i| + |S_c^i| \leqslant 4$ *for* $i \in \{0, 1\}$.

Proof. Consider the following claims.

Claim 1. For each $i \in \{0, 1\}$ and $\alpha \in \{a, b, c\}$ the set $S^i \cup S_\alpha^i$ is 3-separating and $2 \leqslant |S^i| + |S_\alpha^i| \leqslant 3$.

Proof. By 3.5.7, 3.6.3 and symmetry, $\alpha \in \{a, b, c\}$ we have $|S^i| + |S_\alpha^i| \geqslant 2$. So the claim follows from 3.5.3 and symmetry. □

Claim 2. For each $i \in \{0, 1\}$, no two of the sets S_a^i, S_b^i, and S_c^i can have size equal to 2.

Proof. Up to symmetry we may assume that $|S_a^0| = |S_b^0| = 2$. By 3.6.2, $|S_c^0| \leqslant 1$ and, hence, by Claim 1, $|S^0| = 1$ and $|S_c^0| = 1$. Now, by 3.5.9 and 3.5.6, $S_b^0 \cup S_c^1$ is a triangle and $|S_c^1| = 1$. However S^0 and S_c^1 are now in $\mathrm{cl}_M(S_b^0) \cup \mathrm{cl}_M^*(S_b^0)$, contradicting the fact that M is internally 4-connected. □

Claim 3. $|S^0| = |S^1| = 1$.

Proof. By 3.6.2, Claim 1, and symmetry neither S^0 nor S^1 is empty. Up to symmetry we may assume that $|S^0| \geqslant 2$. Then, by Claim 1, each of S_a^0, S_b^0, and S_c^0 is contained in $\mathrm{cl}_M(S^0) \cup \mathrm{cl}_M^*(S^0)$. Then, since M is internally 4-connected, $|S^0| + |S_a^0| + |S_b^0| + |S_c^0| \leqslant 3$. However $|E(M)| \geqslant 11$ so $|S^1| + |S_a^1| + |S_b^1| + |S_c^1| \geqslant 5$. By symmetry this means that $|S^1| = 1$. Then, by Claims 1 and 2 and symmetry, we may assume that $|S_a^1| = 2$, $|S_b^1| = 1$, and $|S_c^1| = 1$. By 3.6.1 and the fact that $|C_b| \geqslant 5$, we have $|S_b^0| = 1$. Then, since $|S^0| + |S_a^0| + |S_b^0| + |S_c^0| \leqslant 3$ and $|S^0| \geqslant 2$, we have $|S^0| = 2$, $|S_a^0| = 0$, and $|S_c^0| = 0$.

Let x, y, and z be the elements in S_b^0, S_b^1, and S_c^1 respectively. Since $|S_a^1| = 2$, $|S_a^0| = 0$, and $|S_c^0| = 0$, by 3.5.5, 3.5.6, and 3.5.9, we see that $\{x, b, y\}$ is a triangle and that $\{x, z\}$ is 2-separating in $M \setminus b, c$. Thus $\{x, z\}$ is a series-pair in $M \setminus b, c$. However, since $\{x, b, y\}$ is a triangle and M is internally 4-connected, $S^0 \cup \{x\}$ is also a triangle, contradicting the fact that $\{x, z\}$ is a series-pair in $M \setminus b, c$. □

The result follows by Claims 1, 2, and 3. □

By 3.6.4, $11 \leqslant |E(M)| \leqslant 13$. Therefore the proof of Theorem 3.2 is reduced to a finite case analysis. Let $e_0 \in S^0$, $e_1 \in S^1$, and $\alpha_i \in S^i_\alpha$ for $\alpha \in \{a,b,c\}$ and $i \in \{0,1\}$. Now let $T_a := B_a \cap C_a$, $T_b := A_b \cap C_b$, $T_c := A_c \cap B_c$, $C^*_a := (B_c \cap C_b) \cup \{b,c\}$, $C^*_b := (A_c \cap C_a) \cup \{a,c\}$, and $C^*_c := (A_b \cap B_a) \cup \{a,b\}$.

3.6.5. *For each $\alpha \in \{a,b,c\}$, either T_α is a triangle or C^*_α is a 4-element cocircuit of M.*

Proof. This is an immediate consequence of 3.6.4, 3.5.4 and 3.5.5. □

3.6.6. *At most one of T_a, T_b, and T_c is a triangle.*

Proof. We start with two easy claims.

Claim 1. If T_a and T_c are both triangles of M, then $r_M(C_a) = 3$ and $r_M(A_c) = 3$.

Proof. By 3.5.9, each of S^0_a, S^1_a, S^0_c and S^1_c has size 1. Suppose that $r_M(C_a) > 3$. Then c_0 is in the coguts of the 3-separation (C_a, C_b) in $M \setminus c$. However $\{c, c_0, c_1\}$ is a triangle, which is contrary to Lemma 3.4. Thus $r_M(C_a) = 3$ and, by symmetry, $r_M(A_c) = 3$. □

Claim 2. T_a, T_b, and T_c cannot all be triangles.

Proof. Suppose that T_a, T_b, and T_c are all triangles. Now consider any partition (X,Y) of $E(M) - \{e_0\}$. By symmetry we may assume that T_a and T_c are both contained in $\mathrm{cl}_M(X)$. Then, by Claim 1, $e_0 \in \mathrm{cl}_M(X)$. It follows that $M \setminus e_0$ is internally 4-connected, contrary to Hypothesis 3.6. □

Suppose that T_a and T_c are triangles of M. By Claim 2, T_b is not a triangle and, by 3.6.5, C^*_b is a 4-element cocircuit of M.

Claim 3. Either $\sqcap_M(T_a, B_a \cap C_b) \neq 0$ or $\sqcap_M(T_c, A_b \cap B_c) \neq 0$.

Proof. Suppose that $\sqcap_M(T_a, B_a \cap C_b) = 0$ and $\sqcap_M(T_c, A_b \cap B_c) = 0$. Then

$$\begin{aligned}
\lambda_{M\setminus b}(T_a \cup T_c) &\leqslant r_M(T_a) + r_M(T_c) + r_M(B_a \cap C_b) + r_M(A_b \cap B_c) - r(M) \\
&= r_M\big(T_a \cup (B_a \cap C_b)\big) + r_M\big(T_c \cup (A_b \cap B_c)\big) - r(M) \\
&= \lambda_{M\setminus b}(B_a) \\
&= 2.
\end{aligned}$$

However $b \in \mathrm{cl}_M(T_a \cup T_c)$, so $T_a \cup T_c \cup \{b\}$ is 3-separating in M, contradicting the fact that M is internally 4-connected. □

By symmetry we may assume that $\sqcap_M(T_c, A_b \cap B_c) \neq 0$. Then, by Lemmas 2.11 and 2.12, $A_b \cap B_c$ is not a triangle or a triad, so, by 3.5.3, $|S^0_b| = 1$. Since $\sqcap_M(T_c, \{e_0, b_0\}) \neq 0$ and C^*_b is a cocircuit, $\{e_0, b_0, c_1\}$ is a triangle. Since T_a, T_c, and $\{a,b,c\}$ are triangles, $a_0, c_1, b \in \mathrm{cl}_M(C^*_b)$. Moreover, by Claim 1, $e_0, e_1 \in \mathrm{cl}_M(C^*)$ and, since $\{e_0, b_0, c_1\}$ is a triangle, $b_0 \in \mathrm{cl}_M(C^*_b)$. Thus $E(M) - S^1_b \subseteq \mathrm{cl}_M(C^*_b)$. However $|S^1_b| \leqslant 2$ and M is 3-connected, so $E(M) \subseteq \mathrm{cl}_M(C^*_b)$. Therefore $r(M) = 4$ and $r_M(E(M) - C^*_b) = 3$. Now $b \notin \mathrm{cl}_M(B_a)$ and, hence, $r_M(B_a - C^*_b) \leqslant 2$. It follows that $|S_b^1| = 1$ and that $\{a_0, b_1, e_1\}$ is a triangle. However the triangles $\{a_0, b_1, e_1\}$ and $\{e_0, b_0, c_1\}$ are not skew, contrary to Lemma 2.11. □

3.6.7. If one of T_a, T_b, and T_c is a triangle, then $r(M^*) < r(M)$ and either M has a triad or there exists $e \in E(M)$ such that M/e is internally 4-connected.

Proof. Suppose that T_b is a triangle. By 3.6.6, T_a and T_c are not triangles and hence, by 3.6.5, C_a^* and C_c^* are both 4-element cocircuits. Now since T_b and $\{a, b, c\}$ are circuits and C_a^* and C_c^* are cocircuits, $\{a, b_0, b_1, c\}$ is a circuit.

Claim 1. If $\{a_0, b_1, e_0\}$ is not a triangle, then $\lambda_{M\backslash a,a_0,e_0}(T_b) = 0$.

Proof. Note that $r_M(T_b \cup \{a_0, e_0\}) \in \{3, 4\}$. If $r_M(T_b \cup \{a_0, e_0\}) = 3$, then, since C_a^* is a cocircuit, $\{a_0, b_1, e_0\}$ is a triangle. Henceforth we may assume that $r_M(T_b \cup \{a_0, e_0\}) = 4$. Note that $T_b \cup \{a_0\}$ and $T_b \cup \{a_0, e_0\}$ are both 3-separating in $M \backslash a$. Then, since $r_M(T_b \cup \{a_0, e_0\}) = 4$, we have $e_0 \in \mathrm{cl}^*_{M\backslash a}(T_b \cup \{a_0\})$. Since C_c^* is a cocircuit, $a_0 \in \mathrm{cl}^*_{M\backslash a}(T_b)$ and, hence, $a_0, e_0 \in \mathrm{cl}^*_{M\backslash a}(T_b)$. Moreover, since a is not in a triad, $\{a_0, e_0\}$ is not a series-pair in $M \backslash a$ and, hence, $\lambda_{M\backslash a,a_0,e_0}(T_b) = 0$. □

Claim 2. Neither $\{a_0, b_1, e_0\}$ nor $\{b_0, c_1, e_1\}$ is a triangle.

Proof. Note that $\{a_0, b_1, e_0\}$ and $\{b_0, c_1, e_1\}$ cannot both be triangles since, otherwise $\{a, b, c\}$ would be in a rotor (where $\{a_0, b_1, e_0\}$ and $\{b_0, c_1, e_1\}$ play the roles of T_a and T_b in Definition 3.3), contradicting Hypothesis 3.6. By symmetry we may assume that $\{a_0, b_1, e_0\}$ is not a triangle. Then, by Claim 1, $\lambda_{M\backslash a,a_0,e_0}(T_b) = 0$ and, hence, $\{b_0, c_1, e_1\}$ is not a triangle either. □

Claim 3. If $|S_c^0| = 2$, then $B_c \cap C_a$ is a triad.

Proof. By 3.5.3, $B_c \cap C_a$ is either a triangle or a triad. However, by Claims 1 and 2, $e_0 \in \mathrm{cl}^*_{M\backslash a}(T_b)$ and, hence, $B_c \cap C_a$ cannot be a triangle. □

Claim 4. $r(M^*) < r(M)$.

Proof. Suppose that $r(M^*) \geqslant r(M)$. Since $|E(M)| \leqslant 13$, $r(M) \leqslant 6$. By Claims 1 and 2, $\lambda_{M\backslash a,a_0,e_0}(T_b) = 0$. Then $a_0, e_0 \in \mathrm{cl}^*_M(T_b \cup \{a\})$. By symmetry, $c_1, e_1 \in \mathrm{cl}^*_M(T_b \cup \{c\})$. Therefore $a_0, c_1, e_0, e_1 \in \mathrm{cl}^*_M(T_b \cup \{a, c\})$. Let $Z = A_c \cap C_a$. Now $r(M \backslash Z) \leqslant 6$, so there exists $f \in \{a_0, c_1, e_0, e_1\}$ such that f is not a coloop of $M \backslash Z$. Since $\lambda_M(T_b \cup \{a, c\}) = 3$ and $a_0, c_1, e_0, e_1 \in \mathrm{cl}^*_M(T_b \cup \{a, c\})$, we have $\lambda_M(Z \cup \{f\}) \leqslant 3$. Moreover f is spanned by $(T_b \cup \{a_0, c_1, e_0, e_1\}) - \{f\}$ in both M and M^*, so $\lambda_M(Z) \leqslant 2$ and $\lambda_M(Z \cup \{f\}) = 3$. Since $\lambda_M(Z \cup \{f\}) = \lambda_M(T_b \cup \{a, c\})$ and $\{a_0, c_1, e_0, e_1\} - \{f\} \subseteq \mathrm{cl}^*_M(T_b \cup \{a, c\})$, the set $\{a_0, c_1, e_0, e_1\} - \{f\}$ contains co-loops of $M \backslash (Z \cup \{f\})$ and, hence, $r(M) = 6$. Then, since $r(M) \leqslant r(M^*)$, $|E(M)| \geqslant 12$ and, hence, $|Z| \geqslant 3$. However $\lambda_M(Z) = 2$, so Z is either a triangle or a triad of M. By symmetry we may assume that $|S_c^0| = 2$. Then, by Claim 3, $B_c \cap C_a$ is a triad, contradicting the fact that M is internally 4-connected. □

By Claim 4, we may assume that M has no triads. Then, by Claim 3 and symmetry, $|S_c^0| = |S_a^1| = 1$ and, hence, $|E(M)| = 11$. Let $W = E(M) - (T_b \cup \{a, c\})$. Since $\lambda_M(T_b \cup \{a, c\}) = 3$, we have $r_M(W) = r(M) \geqslant 6$. Thus W is an independent set in M. Since M has no triads, for any 3-element subset W' of W, we have $W \subseteq \mathrm{cl}^*_M(W')$. It follows easily that M/w is internally 4-connected for any $w \in W$. □

3.6.8. *If none of T_a, T_b, and T_c is a triangle, then $r(M^*) > r(M)$ and either M has a triad or there exists $e \in E(M)$ such that M/e is weakly* 4*-connected.*

Proof. By 3.6.5, C^*_a, C^*_b, and C^*_c are all 4-element cocircuits. Thus, by 3.6.4, $|S^i_\alpha| = 1$ for all $\alpha \in \{a, b, c\}$ and $i \in \{0, 1\}$ and, hence, $|E(M)| = 11$.

We will state the following claims for $M \setminus a$; analogous results hold for $M \setminus b$ and $M \setminus c$.

Claim 1. $r_M(A_b) \leqslant 4$ and, if $r_M(A_b) = 3$, then $\{e_0, a_0, b_1\}$ is a triangle.

Proof. If $r_M(A_b) = 5$, then, contrary to Lemma 3.4, b is in the coguts of the 3-separation (A_b, A_c) in $M \setminus a$. Thus $r_M(A_b) \leqslant 4$; suppose that $r_M(A_b) = 3$. Then, since C^*_a is a cocircuit, $r_M(\{e_0, a_0, b_1\}) = 2$ and, hence, $\{e_0, a_0, b_1\}$ is a triangle. □

Claim 2. $r(M) = 6$ and $r_M(A_b) = r_M(A_c) = 4$.

Proof. Note that (A_b, A_c) is a 3-separation in $M \setminus a$ and, by Claim 1 and symmetry, $r_M(A_b), r_M(A_c) \leqslant 4$. Thus $r(M) \leqslant 6$ and $r(M) = 6$ if and only if $r_M(A_b) = r_M(A_c) = 4$. Suppose that $r(M) < 6$. By symmetry we may assume that $r_M(A_b) = 3$, so, by Claim 1, $\{e_0, a_0, b_1\}$ is a triangle. Now $\{a_0, b_1, e_1\}$ cannot be a triangle, so, by Claim 1 and symmetry, $r_M(B_a) = 4$. Then $r_M(B_c) = 3$ and, by Claim 1 and symmetry, $\{b_0, c_1, e_0\}$ is a triangle. Similarly, $\{a_1, c_0, e_0\}$ is also a triangle. Since C^*_a, C^*_b, and C^*_c are all cocircuits, $\{a_0, b_1\}$, $\{b_0, c_1\}$, and $\{a_1, c_0\}$ are all series-pairs in $M \setminus a, b, c$ and, hence, e_1 is a co-loop of $M \setminus a, b, c$. However this implies that $\{a, b, c, e_1\}$ is 3-separating in M, which is a contradiction. □

By Claim 2, $r(M) > r(M^*)$ so we may assume that M has no triads.

Claim 3. $A_b \setminus \{b\}$ and $A_c \setminus \{c\}$ are both cocircuits.

Proof. By Claim 2, $A_b \setminus \{b\}$ and $A_c \setminus \{c\}$ both contain cocircuits. Then, since M has no triads, $A_b \setminus \{b\}$ and $A_c \setminus \{c\}$ are cocircuits. □

We may assume that M/e_0 is not weakly 4-connected and, hence, there is a meaty 3-separation (X', Y') of M/e_0 with $|X' \cap \{a, b, c\}| \geqslant 2$. Let $X = X' \cup \{a, b, c\}$ and $Y = Y - \{a, b, c\}$. Since M is internally 4-connected, $e_0 \notin \mathrm{cl}^*_M(X)$ and $e_0 \notin \mathrm{cl}^*_M(Y)$. Now, since $C_a - \{a\}$, $A_b - \{b\}$, and $B_c - \{c\}$ are all cocircuits, X must have a non-empty intersection with at least two of the pairs $\{a_0, b_1\}$, $\{b_0, c_1\}$, and $\{a_1, c_0\}$. By symmetry we may assume that $X \cap \{a_0, b_1\} \neq \emptyset$ and $X \cap \{b_0, c_1\} \neq \emptyset$. Therefore, since C^*_c and C^*_a are cocircuits, $a_0, b_1, b_0, c_1 \in \mathrm{cl}^*_M(X)$. However $A_b - \{b\}$ is a cocircuit, so $e_0 \in \mathrm{cl}^*_M(X)$, which is a contradiction. □

The results above prove Theorem 3.2. We will now use Theorem 3.2 to prove Theorem 3.1. The following results are subject to the following hypothesis.

Hypothesis 3.7. *Let M be an internally* 4*-connected matriod with a triangle such that*

- *there does not exist an element $e \in E(M)$ such that $M \setminus e$ is weakly* 4*-connected, and*

- *if* $|E(M)| \leqslant 13$*, then there is no element* $e \in E(M)$ *such that* M/e *is weakly* 4*-connected,*
- *if* $|E(M)| \leqslant 13$ *and* $r(M) > r(M^*)$*, then* M *has no triads.*

Suppose that M satisfies Hypothesis 3.7 and T is a triangle of M. By Lemma 2.10, $M \setminus e$ is 3-connected for each $e \in T$. Hence, $|E(M)| \geqslant 11$. Moreover, by Theorem 3.2, each triangle of M is the central triangle of a rotor. The next three results show that rotors have additional properties that are implicitly suggested by Fig. 3.

3.7.1. *If* $(a, b, c, d, e, T_a, T_c, A, C)$ *is a rotor in* M*, then* $\{b, d, e\}$ *is a triangle and* $\lambda_M(A \cup C) \leqslant 3$*. Moreover, if* $|E(M)| \geqslant 13$*, then* $\lambda_M(A \cup C) = 3$.

Proof. By Lemma 2.11, T_a and T_c are skew. Now $T_a \cup \{b\}$ and $T_a \cup \{b, e\}$ are both 3-separating in $M \setminus a$, so either $e \in \mathrm{cl}_M(T_a \cup \{b\})$ or $e \in \mathrm{cl}^*_{M\setminus a}(T_a \cup \{b\})$. Since e is in the triangle T_c, the latter is not possible and, hence, $e \in \mathrm{cl}_M(T_a \cup \{b\})$. Therefore $b \in \mathrm{cl}_M(T_a \cup \{e\})$ and, by symmetry, $b \in \mathrm{cl}_M(T_c \cup \{d\})$. Since $r_M(T_a \cup T_c) = 4$, the flats $\mathrm{cl}_M(T_a \cup \{e\})$ and $\mathrm{cl}_M(T_c \cup \{d\})$ are modular and, hence, $\{b, d, e\}$ is a triangle.

Since $\{b, d, e\}$ and $\{a, b, c\}$ are circuits, $\{a, c, d, e\}$ is a circuit. Now $T_a \cup \{a\}$ and $T_c \cup \{c\}$ are both 3-separating in $M \setminus b$ and, since $\{a, c, d, e\}$ is a circuit, $\sqcap_M(T_a \cup \{a\}, T_c \cup \{c\}) \geqslant 1$. Hence $\lambda_M(A \cup C) = \lambda_{M\setminus b}(T_a \cup T_c \cup \{a, c\}) \leqslant 3$. If $|E(M)| \geqslant 13$, then $|A \cup C| \geqslant 4$ and, hence, we must have $\lambda_M(A \cup C) = 3$. □

3.7.2. *If* $(a, b, c, d, e, T_a, T_c, A, C)$ *is a rotor in* M*, then there exists a cocircuit* C^* *with* $a, b, d \in C^* \subseteq T_a \cup \{a, b\}$.

Proof. Since T_a is 2-separating in $M \setminus a, b$, there exists a cocircuit $C^* \subseteq T_a \cup \{a, b\}$ with $a, b \in C^*$ and $|C^* \cap T_a| \geqslant 2$. Suppose that $d \notin C^*$. Then $(T_a - \{d\}) \cup \{a\}$ is a triad in $M \setminus b$. Therefore, since $\{b, d, e\}$ is a triangle, $(T_a - \{d\}) \cup \{a\}$ is a triad in M. However T_a is a triangle, which contradicts the fact that M is internally 4-connected. □

3.7.3. *If* $(a, b, c, d, e, T_a, T_c, A, C)$ *is a rotor in* M*, then* $r_M(T_a \cup T_c \cup \{a, b, c\}) = 5$.

Proof. By 3.7.1, $r_M(T_a \cup T_c \cup \{a, b, c\}) \leqslant 5$. By Lemma 2.11, T_a, T_c, and $\{a, b, c\}$ are pair-wise skew, and, hence, $r_M(T_a \cup T_c \cup \{a, b, c\}) \geqslant 4$. Suppose that $r_M(T_a \cup T_c \cup \{a, b, c\}) = 4$. Note that $T_a \cup \{a\}$ and $T_c \cup \{c\}$ are both 3-separating in $M \setminus b$ and $\sqcap_M(T_a \cup \{a\}, T_c \cup \{c\}) = 2$, so $\lambda_M(A \cup C) = 2$. Hence $|A \cup C| \leqslant 3$ and $|E(M)| \leqslant 12$. Let $x \in A \cup C$. If $r_M(A \cup C) = 2$ then we let $N = M \setminus x$, and if $r_M(A \cup C) = 3$ then we let $N = M/x$. Note that $r(N) = 4$ and any two of T_a, T_c, and $\{a, b, c\}$ span N. Therefore N is *vertically* 4-connected. (That is, N is 3-connected and for each 3-separation (X, Y) of N either $r_N(X) = 2$ or $r_N(Y) = 2$.) However, by Hypothesis 3.7, N is not internally 4-connected, so $N = M/x$ and, hence, $A \cup C$ is a triad of M. By Lemma 2.12, $A \cup C$ is skew to every triangle of M and, hence, M/x is internally 4-connected, contradicting Hypothesis 3.7. □

A *double-fan of length* k consists of sequences $(a_0, \ldots, a_k)$, $(b_0, \ldots, b_k)$, and $(c_1, \ldots, c_k)$ of distinct elements of M such that $\{a_{i-1}, c_i, a_i\}$ and $\{b_{i-1}, c_i, b_i\}$ are triangles for each $i \in \{1, \ldots, k\}$ and $\{c_i, a_i, b_i, c_{i+1}\}$ is a cocircuit for each $i \in \{1, \ldots, k-1\}$; see Fig. 5.

Much of the difficulty in the remainder of the section stems from the case that $|E(M)| = 12$. The root of the difficulty is the fact that the cube can be seen as a ladder in several different ways.

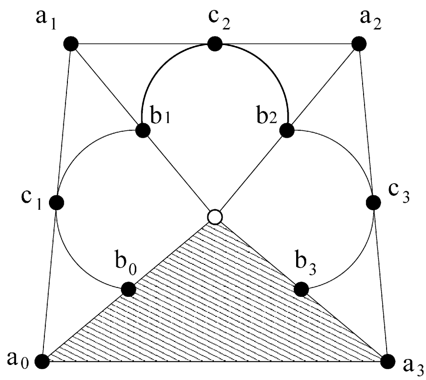

Fig. 5. A double-fan of length 3.

3.7.4. *If T is a triangle of M, then there exists a double-fan* $(a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3, c_1, c_2, c_3)$ *of length* 3 *in* M *such that* $T = \{b_1, b_2, c_2\}$.

Proof. The triangle T is the central triangle of a rotor $(a, b, c, d, e, T_a, T_c, A, C)$. By 3.7.1, $\{b, d, e\}$ is a triangle and, hence, $\{b, d, e\}$ is the central triangle of a rotor $(a', b', c', d', e', T'_{a'}, T'_{c'}, A', C')$ with $\{b, d, e\} = \{a', b', c'\}$. By symmetry we may assume that $b' \in \{b, d\}$.

Claim 1. There exists $x \in T_a - \{d\}$ and $y \in A \cup C$ such that $\{a, x, y\}$ is a triangle and $\{a, b, d, x\}$ is a cocircuit.

Proof. By 3.7.2, there is a cocircuit C_1 with $a, b, d \in C_1 \subseteq T_a \cup \{a, b\}$. By 3.7.1, $\{b', d', e'\}$ is a triangle. Now $b' \in \{b, d\} \subseteq C_1$ and, hence, there exists $w \in \{d', e'\} \cap C_1$. By symmetry we may assume that $w = d'$. Now $d' \in T'_{a'}$, $b, d \notin T'_{a'}$, and $d' \in C_1$. Therefore $T'_{a'} = \{a, x, y\}$ for some $x \in T_a - \{d\}$ and $y \in A \cup C$.

Note that $\lambda_{M\backslash b}(T_a \cup \{a\}) = \lambda_{M\backslash b}(T_a \cup \{a, y\}) = 2$. Therefore $y \in \mathrm{cl}_M((A \cup C \cup T_c \cup \{c\}) - \{y\})$. However $T'_{a'}$ is 2-separating in $M \setminus b, d$. So $\{a, x\}$ is a series-pair in $M \setminus b, d$. Therefore, since b and d are in triangles and M is internally 4-connected, $\{a, b, d, x\}$ is a cocircuit of M. □

If $b' = b$ then, by the claim and symmetry, we obtain the required double-fan. Therefore we may assume that $b' = d$, $a' = b$, $c' = e$. By Claim 1, there exists $x \in T_a - \{d\}$ and $y \in A \cup C$ such that $\{a, x, y\}$ is a triangle and $\{a, b, d, x\}$ is a cocircuit. Let $z \in T_a - \{d, x\}$. Note that $T'_{a'} = \{a, x, y\}$.

Claim 2. There exist distinct elements $x', z' \in T_c$ and an element $w \in A \cup C$ such that $\{b, c, e, x'\}$ is a cocircuit, $\{d, e, z, z'\}$ is a cocircuit, and $\{z, w, z'\}$ is a triangle.

Proof. By 3.7.2 and symmetry, there is a cocircuit C^* such that $b', c' \in C^* \subseteq T'_{c'} \cup \{b', c'\}$. If C^* contains a triangle, then that triangle is $T'_{c'}$. Since T_a and T_c are triangles that intersect C^* and neither of these triangles is $T'_{c'}$, $|C^* \cap T_a| = 2$ and $|C^* \cap T_c| = 2$. Since C^* is disjoint from $T'_{a'}$, we have $C^* \cap T_a = \{d, z\}$. Let $z' \in (C^* \cap T_c) - \{c'\}$ and let $x' \in T_c - \{c', z'\}$. Note that $T'_{c'} = \mathrm{cl}_M(\{z, z'\})$; let $w \in T'_{c'} - \{z, z'\}$. Since $r_M(T_a \cup T_c \cup \{a, c\}) = 5$, $w \in A \cup C$. Now, since $z' \in \mathrm{cl}_M(z, w)$, the set $\{b, c, e, x'\}$ is a cocircuit of M. Note that $\{a, b, d, x\}$ and $\{b, c, e, x'\}$ are both 4-element cocircuits, and this would also be the case if $\{a, b, c\}$ were in a double-fan of length 3. Then, by symmetry the same holds for any rotor. In particular, $|C^*| = 4$ and, hence, $C^* = \{d, e, z, z'\}$. □

Claim 3. $|E(M)| = 12$ and $r(M) = 5$.

Proof. Let $Z = T_a \cup T_c \cup \{a, b, c\}$. Then $r_M(Z) = 5$ and, since $\{a, b, d, x\}$, $\{b, c, e, x'\}$ and $\{b, e, z, z'\}$ are cocircuits, $r_M(A \cup C) \leqslant r(M) - 3$. Therefore $\lambda_M(A \cup C) \leqslant 2$. So $|A \cup C| \leqslant 3$ and, hence, $|E(M)| \leqslant 12$. Now $r_M(T_a \cup \{a, b, c, y\}) = 4$, so there is a cocircuit $C^* \subseteq \{x', z'\} \cup ((A \cup C) - \{y\})$. Since $|C^*| \geqslant 3$, $C^* \cap \{x', z'\}$ is non-empty. However, x' and z' span a triangle, so, since M is internally 4-connected, $|C^*| = 4$ and, hence, $|E(M)| = 12$. Moreover, since $C^* = \{x', z'\} \cup ((A \cup C) - \{y\})$ is a cocircuit, so $r(M) = 5$. □

Claim 4. If $v \in E(M)$ is in a triangle, then there is a triad $\{t_1, t_2, t_3\}$ in $M \setminus v$ such that $\mathrm{cl}_M(\{t_1, t_2\})$ and $\mathrm{cl}_M(\{t_1, t_3\})$ are both triangles.

Proof. This holds for each $v \in \{a, b, c\}$ (whether or not $\{a, b, c\}$ is in a double-fan of length 3). Then, by symmetry, it holds for any triangle. □

Let $y' \in (A \cup C) - \{w, y\}$. We may assume that $\{c, x', y'\}$ is not a triangle since otherwise we would have the required double-fan. Then, since $\{d, e, z, z'\}$ is a cocircuit, $y' \notin \mathrm{cl}_M(\{e, c, x'\})$. Therefore $\{x, z, y, w, y'\}$ is a cocircuit of M. Now $r_M(\{w, y, y', z, z'\}) = 3$ and $\{d, e, z, z'\}$ is a cocircuit, so $\{w, y, y'\}$ is a triangle. By Claim 4, there is a triad $T^* = \{t_1, t_2, t_3\}$ is $M \setminus y$ such that $\mathrm{cl}_M(\{t_1, t_2\})$ and $\mathrm{cl}_M(\{t_1, t_3\})$ are both triangles. Note that $\{y, t_1, t_2, t_3\}$ is a cocircuit and, hence, $T^* \cap \{w, y'\}$ and $T^* \cap \{a, x\}$ are both non-empty. No triangle in M intersects both $\{a, x\}$ and $\{w, y'\}$ and, hence, we may assume that $t_2 \in \{a, x\}$ and $t_3 \in \{w, y'\}$. Now there is a triangle through each of $\{t_1, t_2\}$ and $\{t_1, t_3\}$. It follows that $t_1 = z$, $t_2 = x$, and $t_3 = w$. Then $\{x, y, z, w\}$ is a cocircuit, contradicting the fact that $\{w, x, y, y', z\}$ is a cocircuit. □

Proof of Theorem 3.1. Consider a double-fan of length $k \geqslant 3$. By 3.7.4, there is a double-fan of length 3 centered around the triangle $\{a_{k-1}, c_k, a_k\}$. Therefore there is an element $c_{k+1} \in E(M)$ such that $\{c_k, a_k, b_k, c_{k+1}\}$ is a 4-element cocircuit and there exist triangles $\{a_k, c_{k+1}, a_{k+1}\}$ and $\{b_k, c_{k+1}, b_{k+1}\}$. Either this gives a longer double-fan, or we have $\{a_0, b_0\} = \{a_{k+1}, b_{k+1}\}$, in which case M^* is the cycle matroid of a ladder. □

4. Weakly 4-connected matroids

In this section we complete the proof of Theorem 1.1. It remains to consider a weakly 4-connected matroid M containing a 4-element 3-separating set A.

Lemma 4.1. *Let M be a weakly 4-connected matroid and let A be a 4-element 3-separating set of M. If $r_M(A) = 2$, then $M \setminus x$ is weakly 4-connected for each $x \in A$.*

Proof. Suppose that (X, Y) is 3-separation of $M \setminus x$. By symmetry, we may assume that $|X \cap A| \geqslant 2$. So $x \in \mathrm{cl}_M(X)$ and, hence, $(X \cup \{x\}, Y)$ is a 3-separation of M. Since M is weakly 4-connected, either $|X| \leqslant 3$ or $|Y| \leqslant 4$. Therefore $M \setminus x$ is weakly 4-connected. □

By Lemma 4.1, we may assume that $r_M(A) \geqslant 3$ and, dually, that $r^*_M(A) \geqslant 3$. However, since $\lambda_M(A) = 2$ and $\lambda_M(A) = r_M(A) + r^*_M(A) - |A|$, we have $r_M(A) = r^*_M(A) = 3$.

Lemma 4.2. *Let M be a weakly 4-connected matroid, let A be a 4-element 3-separating set with $r_M(A) = 3$, and let $c \in A$ such that M/c is simple and $c \notin \mathrm{cl}_M(E(M) - A)$. Then M/c is 4-connected up to separators of size 5. Moreover, if X is a 5-element 3-separating set of M/c, then $|X \cap A| = 1$.*

Proof. Let $T = A - \{c\}$; note that T is a triangle in M/c. We start by proving that M/c is 3-connected. Consider a 2-separation (X, Y) of M/c. Since M/c is simple, $|X|, |Y| \geqslant 3$. By symmetry we may assume that $|X \cap A| \geqslant 2$. Thus $T \subseteq \mathrm{cl}_{M/c}(X)$ and, hence $X \cup T$ is 2-separating in M/c. However $c \in \mathrm{cl}^*_M(T)$, so $X \cup T \cup \{c\}$ is 2-separating in M, contradicting the fact that M is 3-connected. Thus M/c is in fact 3-connected.

Now consider a meaty 3-separation (U, V) of M/c. By symmetry, we may assume that $|U \cap T| \geqslant 2$. Then $U \cup T$ is 3-separating in M/c and, since $c \in \mathrm{cl}^*_M(T)$, $U \cup T \cup \{c\}$ is 3-separating in M. Therefore $|V| = 5$ and $|V \cap A| = 1$. □

We are now ready to prove the main result of this section.

Theorem 4.3. *Let M is a weakly 4-connected matroid and let A be a 4-element 3-separating set. Then either*

- *there exists $e \in A$ such that $M \setminus e$ or M/e is weakly 4-connected,*
- *there exist $e, f \in A$ such that $M/e \setminus f$ is weakly 4-connected, or*
- *$|E(M)| = 12$ and there is a partition of $E(M)$ into three 4-element 3-separating sets.*

Proof. By Lemma 4.1, we may assume that $r_M(A) = 3$. Since A is closed, it is straightforward to see that there exists an element $c \in A$ such that M/c is simple and $c \notin \mathrm{cl}_M(E(M) - A)$. Dually, there exists an element $d \in A$ such that $M \setminus d$ is cosimple and $d \notin \mathrm{cl}^*_M(E(M) - A)$. Let $T = A - \{c\}$; note that T is a triangle in M/c.

Since M is weakly 4-connected, A is co-closed in M and, hence, T is co-closed in M/c. Then, by Tutte's Triangle Lemma (see Oxley [6, Lemma 8.4.9]), there exists $e \in T$ such that $M/c \setminus e$ is 3-connected. We may assume that $M/c \setminus e$ is not weakly 4-connected and, hence, $|E(M)| \geqslant 12$.

4.3.1. *Let $B \subseteq E(M) - A$ be a 3-separating set in M with at most 4 elements. If $\sqcap_M(B, A) + \sqcap_{M^*}(B, A) \geqslant 2$, then $|E(M)| = 12$ and there exists a partition of $E(M)$ into three 4-element 3-separating sets.*

Proof. If $\sqcap_M(B, A) + \sqcap_{M^*}(B, A) \geqslant 2$, then, by Lemma 2.2, $\lambda_M(A \cup B) \leqslant \lambda_M(A) + \lambda_M(B) - 2 \leqslant 2$. Thus, $A \cup B$ is 3-separating. However, since $|E(M)| \geqslant 12$ and M is weakly 4-connected, we conclude that $|E(M)| = 12$ and that both B and $E(M) - (A \cup B)$ are 4-element 3-separating sets in M. □

Henceforth we may assume that:

4.3.2. *For each 3-separating set $B \subseteq E(M) - A$ with at most 4 elements, we have*

$$\sqcap_M(B, A) + \sqcap_{M^*}(B, A) \leqslant 1.$$

4.3.3. *Let $P, P^* \subseteq E(M) - A$ be 3-separating sets of M with 3 or 4 elements. If $\sqcap_M(P, A) = 1$ and $\sqcap_{M^*}(P^*, A) = 1$, then $|P \cap P^*| \leqslant 1$ and $\sqcap_M(P - P^*, A) = 1$.*

Proof. First suppose that $|P \cap P^*| \geqslant 2$. Then $P \cup P^*$ is 3-separating. Since M is weakly 4-connected, $|P \cup P^*| \leqslant 4$. However, $\sqcap_M(P \cup P^*, A) + \sqcap_{M^*}(P \cup P^*, A) \geqslant \sqcap_M(P, A) + \sqcap_{M^*}(P^*, A) = 2$; contrary to 4.3.2. Hence $|P \cap P^*| \leqslant 1$.

Since $\sqcap_M(P, A) = 1$, there is a circuit $C \subseteq A \cup P$ such that $C \cap P \neq \emptyset$ and $C \cap A \neq \emptyset$. If $C \cap P^* = \emptyset$, then $\sqcap_M(P - P^*, A) = 1$, as required. Thus, we may assume that there exists an element $e \in C \cap P^*$; note that $P \cap P^* = \{e\}$. Since $e \in C$, we see that e is in the guts of the 3-separation $(P^*, E(M) - P^*)$. Then, since M is 3-connected, $e \in \mathrm{cl}_M(P^* - \{e\})$. It follows that e is in the guts of $(P, E(M) - P)$ and, hence, that $e \in \mathrm{cl}_M(P - \{e\})$. Therefore $\sqcap_M(P - P^*, A) = \sqcap_M(P, A) = 1$. □

We may assume that neither M/c nor $M \setminus d$ is weakly 4-connected. Then, by Lemma 4.2 and duality, there exist two 4-element 3-separating sets $Q, Q^* \subseteq E(M) - A$ and elements $f, f^* \in A$ such that $f \in \mathrm{cl}_{M/c}(Q)$ and $f^* \in \mathrm{cl}^*_{M\setminus d}(Q^*)$.

4.3.4. *Let $P \subseteq E(M) - A$ be a 3-separating set of M with 3 or 4 elements. If $\sqcap_M(P, A) = 1$, then $T \cap \mathrm{cl}_{M/c}(P) = \{f\}$.*

Proof. Let $N = M \setminus Q^*/c$. By the dual of Lemma 2.3, A is 2-separating in $M \setminus Q^*$, so T is 2-separating in N. By 4.3.3, $\sqcap_M(Q - Q^*, A) = 1$ and $\sqcap_M(P - Q^*, A) = 1$. Then, since $c \notin \mathrm{cl}_M(E(M) - A)$, $\sqcap_N(Q - Q^*, T) = 1$ and $\sqcap_N(P - Q^*, T) = 1$. Since $\sqcap_{M/c}(Q, \{f\}) = \sqcap_{M/c}(Q, T)$, we see that $\sqcap_{M/c,f}(Q, T - \{f\}) = 0$ and, hence, that $\sqcap_{N/f}(Q - Q^*, T - \{f\}) = 0$. Then, since $\sqcap_N(Q - Q^*, T) = 1$, we see that $f \in \mathrm{cl}_N(Q - Q^*)$. Thus f is in the guts of the 2-separation $(T, E(N) - T)$ in N. Then, since $\sqcap_N(P - Q^*, T) = 1$, we see that $T \cap \mathrm{cl}_N(P - Q^*) = \{f\}$. Thus, $T \cap \mathrm{cl}_{M/c}(P - Q^*) = \{f\}$. Then, since $\sqcap_M(P, A) = 1$, we have $T \cap \mathrm{cl}_{M/c}(P) = \{f\}$. □

By Lemmas 4.2 and 4.3, each 5-element 3-separating set in M/c contains f. In the remainder of the proof we will show that $M/c \setminus f$ is weakly 4-connected.

Suppose that $M/c \setminus f$ is not 3-connected. Then f is in the coguts of a 3-separation (X, Y) in M/c. Since $f \notin \mathrm{cl}_{M/c}(X - \{f\})$ and $f \notin \mathrm{cl}_{M/c}(Y - \{f\})$, $X - \{f\}$ and $Y - \{f\}$ each contain one element of T. Now note that $X - T$ and $Y - T$ are both 3-separating in M/c. However, $c \in \mathrm{cl}^*_M(T)$, so $X - T$ and $Y - T$ are both 3-separating in M. Since M is weakly 4-connected and $|E(M)| \geqslant 12$, $|X - T| = 4$ and $|Y - T| = 4$. Thus we may assume that $M/c \setminus f$ is 3-connected.

Finally, we suppose that $M/c \setminus f$ is not weakly 4-connected; let (X, Y) be a meaty 3-separation in $M/c \setminus f$. By symmetry we may assume that $|Y \cap Q| \geqslant 2$. Note that this 3-separation is blocked by f in M/c. Therefore, X and Y must each contain one element of T. Suppose that $X \cap T = \{x\}$. Since $|Y \cap Q| \geqslant 2$, we see that $X - Q$ is 3-separating in $M/c \setminus f$. However $f \in \mathrm{cl}_{M/c}(Q)$, so $X - Q$ is 3-separating in M/c. Now $T \cap (X - Q) = \{x\}$, so $(X - Q) - \{x\}$ is 3-separating in M/c. Since $c \in \mathrm{cl}^*_M(T)$, $(X - Q) - \{x\}$ is 3-separating in M. Since $|X| \geqslant 5$ and $|Q \cap Y| \geqslant 2$, we have $|(X - Q) - \{x\}| \geqslant 2$. However, x is in the guts of the 3-separation $(X - Q, Y \cup Q)$ in M/c. So, by 4.3.3, $|(X - Q) - \{x\}| = 2$. Therefore, $|X| = 5$ and $|X \cap Q| = |Y \cap Q| = 2$. Now, by the symmetry between X and Y, $|Y| = 5$ and, hence, $|E(M)| = 12$. Now consider the 3-separating set Q^*. By 4.3.3, $|Q \cap Q^*| \leqslant 1$. Therefore Q^* meets either $X - Q$ or $Y - Q$ in two elements; by symmetry we may assume that $|(X - Q) \cap Q^*| = 2$. But then $\sqcap_{M/c}(Q^*, T) \geqslant 1$ and, hence, $\sqcap_{M^*}(Q^*, A) + \sqcap_M(Q^*, A) \geqslant 2$. Then, by 4.3.1, M has the required structure. □

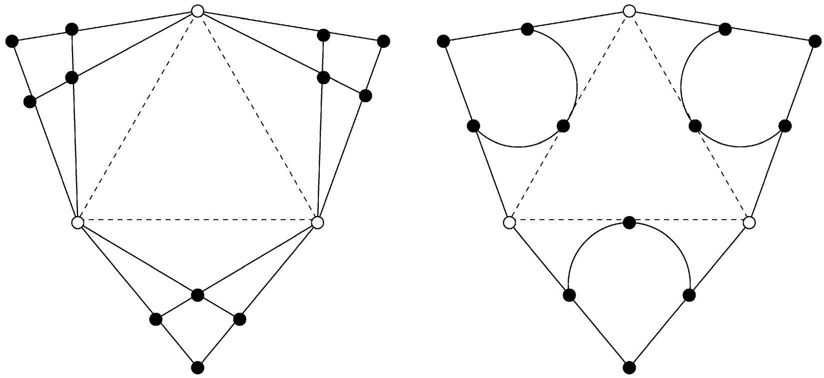

Fig. 6. Tridents.

Definition 4.4 *(Tridents).* A *trident* is a weakly 4-connected rank-6 matroid M on 12 elements such that $E(M)$ can be partitioned into three 4-element rank-3 3-separating sets.

Figure 6 shows two tridents M with the property that they do not contain a weakly 4-connected minor on 10 or 11 elements. Not all tridents have this property, however, there are 4 non-isomorphic binary tridents and each of these does have the property.

Proof of Theorem 1.1. By Theorems 3.1 and 4.3, we may assume that $|E(M)| = 12$ and that there is a partition (A, B, C) of $E(M)$ into three 4-element 3-separating sets. By Lemma 4.1, we may assume that each of A, B, and C has rank 3. Up to duality, we may assume that $r(M) \leqslant 6$. Moreover, we may assume that M is not a trident and, hence, that $r(M) \leqslant 5$. Since A, B, and C are rank-3 3-separating sets, $r(M) = 5$. Therefore $r_M(A \cup B) = 4$ and, hence, $\sqcap_M(A, B) = 2$. Similarly, $\sqcap_M(A, C) = 2$. Then, by Lemma 2.2, $\sqcap_{M^*}(A, B) = 0$ and $\sqcap_{M^*}(A, C) = 0$.

Consider any 4-element 3-separating set Q. Up to symmetry, we may assume that $|Q \cap C| \geqslant 2$. Then $Q \cup C$ is 3-separating. However, M is weakly 4-connected, so $Q = C$. Thus A, B, and C are the only 4-element 3-separating sets in M. Therefore, there is no 4-element 3-separating set $Q \subseteq E(M) - A$ such that $\sqcap_{M^*}(A, Q) \geqslant 1$. Then it follows easily from the dual of Lemma 4.2, that there exists $d \in A$ such that $M \setminus d$ is weakly 4-connected. $\square$

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