Lecture 1t

Volume of a Parallelepiped

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If we take three linearly independent vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$, they define a three dimensional object known as a parallelepiped, which is what happens when you stretch a parallelogram into three dimensions. We can calculate the volume of this parallelepiped by computing the **scalar triple product** of $\vec{w}$, $\vec{v}$, and $\vec{u}$: $\vec{w} \cdot (\vec{u} \times \vec{v})$. As we always take volume to be a positive value, we will in fact look at the absolute value of the scalar triple product.

Summary: The volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$ and $\vec{w}$ is $|\vec{w} \cdot (\vec{u} \times \vec{v})|$.

Example: Find the volume of the parallelepiped determined by $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} -4 \\ 3 \\ 2 \end{bmatrix}$.

First, we need to compute $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} (1)(3) - (4)(1) \\ (4)(2) - (1)(3) \\ (1)(1) - (1)(2) \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -1 \end{bmatrix}$.

Then we need to compute $\begin{bmatrix} -4 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 5 \\ -1 \end{bmatrix} = (-4)(-1)+(3)(5)+(2)(-1) = 17$.

So the volume of the parallelepiped determined by $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} -4 \\ 3 \\ 2 \end{bmatrix}$ is 17.

Example: Note that a rectangular box is a type of parallelepiped, and that this calculation matches the known formula of height×width×length for the volume of a box. For example, if we want to find that volume of a box of height 2, width 3 and length 5, we could think of this as the parallelepiped determined by the vectors $\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$. Thus, we would find the volume by first calculating

$$\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} (0)(0) - (0)(5) \\ (0)(0) - (3)(0) \\ (3)(5) - (0)(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 15 \end{bmatrix}$$

And then we would calculate $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 15 \end{bmatrix} = (0)(0) + (0)(0) + (2)(15) = 30.$