Lecture 3x

Finding Inverses Using Elementary Matrices

(pages 178-9)

In the previous lecture, we learned that for every matrix A, there is a sequence of elementary matrices $E_1, \ldots, E_k$ such that $E_k \cdots E_1 A$ is the reduced row echelon form of A. But what if the reduced row echelon form of A is I? Then we have that $E_k \cdots E_1 A = I$. But this means that $(E_k \cdots E_1)$ is A^{-1}.

Example: Let $A = \begin{bmatrix} 2 & 4 \\ 5 & 8 \end{bmatrix}$, and consider the following row reduction of A to I:

$$\begin{bmatrix} 2 & 4 \\ 5 & 8 \end{bmatrix} \; (1/2)R_1 \sim \begin{bmatrix} 1 & 2 \\ 5 & 8 \end{bmatrix} \; R_2 - 5R_1 \sim \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \; (-1/2)R_2$$
$$\sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \; R_1 - 2R_2 \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then we can construct a sequence of elementary matrices $E_4, \ldots, E_1$ such that $E_4 \cdots E_1 A = I$ as follows:

The first row operation is $(1/2)R_1$, so $E_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$.

The second row operation is $R_2 - 5R_1$, so $E_2 = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$.

The third row operation is $(-1/2)R_2$, so $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$.

The fourth row operation is $R_1 - 2R_2$, so $E_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$.

Then $A^{-1} = E_4E_3E_2E_1$, which we can calculate as follows:

$$E_2E_1 = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ -5/2 & 1 \end{bmatrix}$$

$$E_3E_2E_1 = E_3(E_2E_1) = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ -5/2 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 5/4 & -1/2 \end{bmatrix}$$

$$A^{-1} = E_4E_3E_2E_1 = E_4(E_3E_2E_1) = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 5/4 & -1/2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 5/4 & -1/2 \end{bmatrix}.$$

We verify our calculation by looking at the product AA^{-1}:

$$AA^{-1} = \begin{bmatrix} 2 & 4 \\ 5 & 8 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 5/4 & -1/2 \end{bmatrix} = \begin{bmatrix} -4+5 & 2-2 \\ -10+10 & 5-4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example: Let's look at a different row reduction of A to I:

$$\begin{bmatrix} 2 & 4 \\ 5 & 8 \end{bmatrix} \; R_2 - 2R_1 \sim \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix} \; R_1 \updownarrow R_2 \sim \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} \; R_2 - 2R_1$$
$$\sim \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \; (1/4)R_2 \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then we can construct a sequence of elementary matrices $E_4, \ldots, E_1$ such that $E_4 \cdots E_1 A = I$ as follows:

The first row operation is $R_2 - 2R_1$, so $E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$.

The second row operation is $R_1 \updownarrow R_2$, so $E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

The third row operation is $R_2 - 2R_1$, so $E_3 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$.

The fourth row operation is $(1/4)R_2$, so $E_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}$.

Then $A^{-1} = E_4E_3E_2E_1$, which we can calculate as follows:

$$E_2E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$E_3E_2E_1 = E_3(E_2E_1) = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 5 & -2 \end{bmatrix}$$

$$A^{-1} = E_4E_3E_2E_1 = E_4(E_3E_2E_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 5/4 & -1/2 \end{bmatrix}.$$

This is the same value for A^{-1} that we got on the previous example, which is good, since we've already shown that the inverse of a matrix is unique! So the point of this example is to emphasize that it while a different collection of row reductions steps will lead to a different sequence of matrices $E_1, \ldots, E_k$, no matter which sequence we use, we still end up with our unique matrix A^{-1}.

But what about the inverse of an elementary matrix? As they are row equivalent to I, we know that the are invertible. And to get from an elementary matrix E to I, you simply need to undo the row operation you did to get from I to E in the first place. As this will be a single row operation, it turns out that the inverse of an elementary matrix is itself an elementary matrix. And the best way to find the inverse is to think in terms of row operations.

Example: $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$, since the way we undo multiplying row 1 by 2 is to multiply row 1 by 1/2. In general, the inverse of the operation sR_i

is $(1/s)R_i$.

Example: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, since to undo switching rows 1 and 2, we simply need to switch rows 1 and 2 again. In general, the inverse of the operation $R_i \updownarrow R_j$ is$R_i \updownarrow R_j$. (That is, switching rows is its own inverse.)

Example: $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, since the way we undo adding four times row 3 to row 1 is to subtract four times row 3 from row 1. In general, the inverse of the operation $R_i + sR_j$ is $R_i - sR_j$.

Example: Let $A = \begin{bmatrix} 2 & 4 \\ 5 & 8 \end{bmatrix}$, as in our previous examples. Then not only can we write A^{-1} as a product of elementary matrices, but we can also write A as a product of elementary matrices. Since $A^{-1} = E_4E_3E_2E_1$, we have $A = (A^{-1})^{-1} = (E_4E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$. (REMEMBER: the order of multiplication switches when we distribute the inverse.) And since we just saw that the inverse of an elementary matrix is itself an elementary matrix, we know that $E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$ is a product of elementary matrices. Specifically, we get that

$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (if we use the first row reduction),
or

$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ (if we use the second row reduction).

Generalizing the procedure in this example, we get the following theorem:

<u>Theorem 3.6.3</u>: If an $n \times n$ matrix A has rank n, then it may be represented as a product of elementary matrices.

Note: When asked to "write A as a product of elementary matrices", you are expected to write out the matrices, and not simply describe them using row operations, or leave them as E^{-1} even if you have already written out E.

There is one result that I would like to point out, that is missing from the textbook:

<u>Course Author's Theorem</u>: If A is row equivalent to B, then there is an invertible matrix P such that $PA = B$.

<u>Proof</u>: If A is row equivalent to B, then there is a sequence of elementary row

operations from A to B. If $E_1, \dots, E_k$ are the elementary matrices for these row operations, then we have that $E_k \cdots E_1 A = B$. So, if we let $P = E_k \cdots E_1$, then P is invertible, as the elementary matrices are invertible and the product of invertible matrices is invertible. And so P is an invertible matrix such that $PA = B$.

As this theorem doesn't appear in the text, we won't really use it for anything, but the text has placed a lot of emphasis on there being a sequence of elementary matrices leading to the reduced row echelon form, and I wanted to point out this fact, which holds for *any* matrix row equivalent to A.