Lecture 1p

The Rank-Nullity Theorem

(pages 230-232)

Since Range(L) and Null(L) are subspaces of $\mathbb{W}$ and $\mathbb{V}$ (respectively), we can try to find a basis for them.

Example: Determine a basis for the range and nullspace of the linear mapping $L : \mathbb{R}^2 \to M(2,2)$ defined by $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$

The range of L is matrices of the form $\begin{bmatrix} a & b \\ b & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. And so we see that $\mathcal{B} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$ is a spanning set for Range(L). Moreover, as the matrices in $\mathcal{B}$ are clearly not a scalar multiple of each other, $\mathcal{B}$ is linearly independent too. So we have found that $\mathcal{B}$ is a basis for the range of L.

The nullspace of L consists of all vectors $\begin{bmatrix} a \\ b \end{bmatrix}$ whose entries a and b satisfy $\begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Setting the entries equal to each other, we see that this means that $a = 0$ and $b = 0$. So the only element of the nullspace is the vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. That is, Null($L$) $= \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$, and the basis for this is the empty set.

Example: Determine a basis for the range and nullspace of the linear mapping $L : P_3 \to \mathbb{R}^2$ defined by $L(a + bx + cx^2 + dx^3) = \begin{bmatrix} a + 2b + c + 2d \\ 3a + 4b - c - 2d \end{bmatrix}$.

The range of L is all vectors of the form $\begin{bmatrix} a + 2b + c + 2d \\ 3a + 4b - c - 2d \end{bmatrix} = a\begin{bmatrix} 1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 2 \\ 4 \end{bmatrix} + c\begin{bmatrix} 1 \\ -1 \end{bmatrix} + d\begin{bmatrix} 2 \\ -2 \end{bmatrix}$. And so we see that $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}\right\}$ is a spanning set for the range of L. But is it linearly independent? To check, we need to look for solutions to the equation

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 2 \\ 4 \end{bmatrix} + c\begin{bmatrix} 1 \\ -1 \end{bmatrix} + d\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} a + 2b + c + 2d \\ 3a + 4b - c - 2d \end{bmatrix}$$

Setting the entries equal to each other, we see that this is equivalent to the following system:

$$\begin{array}{rrrrl} a & +2b & +c & +2d & = 0 \\ 3a & +4b & -c & -2d & = 0 \end{array}$$

We solve this homogeneous system by row reducing its coefficient matrix:

$$\left[\begin{array}{rrrr} 1 & 2 & 1 & 2 \\ 3 & 4 & -1 & -2 \end{array}\right] \; R_2 - 3R_1 \sim \left[\begin{array}{rrrr} 1 & 2 & 1 & 2 \\ 0 & -2 & -4 & -8 \end{array}\right]$$

From the row echelon form, we see that the solution to our system will have two parameters. So $\mathcal{B}$ is linearly dependent. However, since $\mathcal{B}$ is a spanning set, we know that we can remove the dependent members in order to end up with a basis. To that end, we will continue our row reduction to find the solution to the homogeneous system, and use the solution to write elements of $\mathcal{B}$ as a linear combination of the other elements.

$$\left[\begin{array}{rrrr} 1 & 2 & 1 & 2 \\ 0 & -2 & -4 & -8 \end{array}\right] \; (-1/2)R_2 \sim \left[\begin{array}{rrrr} 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & 4 \end{array}\right] \; R_1 - 2R_2$$
$$\sim \left[\begin{array}{rrrr} 1 & 0 & -3 & -6 \\ 0 & 1 & 2 & 4 \end{array}\right]$$

So we have that our system is equivalent to the system

$$\begin{array}{ccccc} a & & -3c & -6d & = 0 \\ & b & +2c & +4d & = 0 \end{array}$$

Replacing the variable c with the parameter s and the variable d with the parameter t, we see that the general solution to our system is

$$\left[\begin{array}{c} a \\ b \\ c \\ d \end{array}\right] = \left[\begin{array}{c} 3s+6t \\ -2s-4t \\ s \\ t \end{array}\right] = s\left[\begin{array}{r} 3 \\ -2 \\ 1 \\ 0 \end{array}\right] + t\left[\begin{array}{r} 6 \\ -4 \\ 0 \\ 1 \end{array}\right]$$

If we set $s = -1$ and $t = 0$ we get that $-3\left[\begin{array}{r} 1 \\ 3 \end{array}\right] + 2\left[\begin{array}{r} 2 \\ 4 \end{array}\right] - \left[\begin{array}{r} 1 \\ -1 \end{array}\right] + 0\left[\begin{array}{r} 2 \\ -2 \end{array}\right] = \left[\begin{array}{r} 0 \\ 0 \end{array}\right]$, so we have that $\left[\begin{array}{r} 1 \\ -1 \end{array}\right] = -3\left[\begin{array}{r} 1 \\ 3 \end{array}\right] + 2\left[\begin{array}{r} 2 \\ 4 \end{array}\right]$. Meanwhile, setting $s = 0$ and $t = -1$ gives us $-6\left[\begin{array}{r} 1 \\ 3 \end{array}\right] + 4\left[\begin{array}{r} 2 \\ 4 \end{array}\right] + 0\left[\begin{array}{r} 1 \\ -1 \end{array}\right] - 1\left[\begin{array}{r} 2 \\ -2 \end{array}\right] = \left[\begin{array}{r} 0 \\ 0 \end{array}\right]$, so we have that $\left[\begin{array}{r} 2 \\ -2 \end{array}\right] = -6\left[\begin{array}{r} 1 \\ 3 \end{array}\right] + 4\left[\begin{array}{r} 2 \\ 4 \end{array}\right]$. And so we can remove both $\left[\begin{array}{r} 1 \\ -1 \end{array}\right]$ and $\left[\begin{array}{r} 2 \\ -2 \end{array}\right]$ from $\mathcal{B}$ to get the set $\mathcal{B}_1 = \left\{\left[\begin{array}{r} 1 \\ 3 \end{array}\right], \left[\begin{array}{r} 2 \\ 4 \end{array}\right]\right\}$, which is still a spanning set for the range of L. And since the vectors in $\mathcal{B}_1$ are not scalar multiples of each other, $\mathcal{B}_1$ is also linearly independent, and so we see that $\mathcal{B}_1$ is a basis for the range of L.

The nullspace of L consists of all polynomials $a+bx+cx^2+dx^3$ whose coefficients a, b, c, and d satisfy $\left[\begin{array}{c} a+2b+c+2d \\ 3a+4b-c-2d \end{array}\right] = \left[\begin{array}{r} 0 \\ 0 \end{array}\right]$. Luckily, we have already solved this equation, in our efforts to find the range of L. The solution is

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = s \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 6 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

Plugging this solution in, we see that the nullspace of L consists of all polynomials of the form $s(3 - 2x + x^2) + t(6 - 4x + x^3)$. And so we see that $\mathcal{C} = \{3 - 2x + x^2, 6 - 4x + x^3\}$ is a spanning set for the nullspace of L. And since the polynomials are not scalar multiples of each other, $\mathcal{C}$ is clearly linearly independent as well. And so we have that $\mathcal{C}$ is a basis for the nullspace of L.

Once we find a basis for the range or the nullspace for a linear mapping, we can calculate its dimension. It turns out that the dimension of these special subspaces is something we want to concern ourselves with, so we give them their own special name.

Definition: Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{R}$. The **rank of a linear mapping** $L : \mathbb{V} \to \mathbb{W}$ is the dimension of the range of L:

$$\text{rank}(L) = \dim(\text{Range}(L))$$

Definition: Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{R}$. The **nullity of a linear mapping** $L : \mathbb{V} \to \mathbb{W}$ is the dimension of the nullspace of L:

$$\text{nullity}(L) = \dim(\text{Null}(L))$$

Example: In our first example, we found that the set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis for the range of L. Since this basis has two matrices in it, we see that the rank of L is 2. We then found that the empty set is the basis for the nullspace of L, and since the empty set has zero entries in it, the nullity of L is 0.

In our second example, we found that the set $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$ is a basis for the range of L. Since this set has two elements, the rank of L is 2. We then found that the set $\{3 - 2x + x^2, 6 - 4x + x^3\}$ is a basis for the nullspace of L. Since this set has two elements, the nullity of L is 2.

Note that in our first example, we have that $\text{rank}(L) + \text{nullity}(L) = 2 + 0 = 2 = \dim \mathbb{R}^2$, and that in our second example we have that $\text{rank}(L) + \text{nullity}(L) = 2 + 2 = 4 = \dim P_3$. That is, in both of these examples, the sum of the rank and the nullity equals the dimension of the domain. As you may be guessing, this is not a coincidence.

Theorem 4.5.2 (The Rank-Nullity Theorem): Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{R}$ with dim $\mathbb{V} = n$, and let $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping. Then,

$$\text{rank}(L) + \text{nullity}(L) = n$$

Proof of the Rank-Nullity Theorem: In fact, what we are going to show, is that the rank of L equals dim $\mathbb{V} - \text{nullity}(L)$, by finding a basis for the range of L with $n - \text{nullity}(L)$ elements in it. To that end, let's let say that the nullity of L is k, and that $\{\mathbf{v}_1, \ldots \mathbf{v}_k\}$ is a basis for the nullspace of L. Since the nullspace of L is a subspace of $\mathbb{V}$, we can expand the linearly independent set $\{\mathbf{v}_1, \ldots \mathbf{v}_k\}$ into a basis for $\mathbb{V}$. Let's let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ be such a basis.

First, let's consider the possibility that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ may have already been a spanning set for $\mathbb{V}$. Then $k = n$ and no additional $\mathbf{u}_i$ vectors are necessary. But in this case, that would mean that every vector is mapped by L to the zero vector, which means that the only vector in the range of L is the zero vector. Thus, the range of L is the single element set $\{\mathbf{0}_{\mathbb{W}}\}$, which has the empty set for a basis. And so we see that the rank of L is 0. This means that $\text{rank}(L) + \text{nullity}(L) = 0 + n = n$, as desired.

Otherwise, $k < n$, and at least one $\mathbf{u}_i$ is necessary. Now, let's let $\mathbf{w}$ be any vector in the range of L. Then there is some vector $\mathbf{x} \in \mathbb{V}$ such that $L(\mathbf{x}) = \mathbf{w}$. But, since $\mathbf{x}$ is in $\mathbb{V}$, we know that it can be written as a linear combination of our basis elements. That is, we have scalars t_1 through t_n such that

$$\mathbf{x} = t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k + t_{k+1}\mathbf{u}_{k+1} + \cdots + t_n\mathbf{u}_n$$

And since $\mathbf{w} = L(\mathbf{x})$, we have

$$\begin{aligned} \mathbf{w} &= L(t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k + t_{k+1}\mathbf{u}_{k+1} + \cdots + t_n\mathbf{u}_n) \\ &= t_1L(\mathbf{v}_1) + \cdots + t_kL(\mathbf{v}_k) + t_{k+1}L(\mathbf{u}_{k+1}) + \cdots + t_nL(\mathbf{u}_n) \\ &= t_1(0) + \cdots + t_k(0) + t_{k+1}L(\mathbf{u}_{k+1}) + \cdots + t_nL(\mathbf{u}_n) \\ &= t_{k+1}L(\mathbf{u}_{k+1}) + \cdots + t_nL(\mathbf{u}_n) \end{aligned}$$

And so we see that every element $\mathbf{w}$ in the range of L can be written as a linear combination of the vectors $L(\mathbf{u}_{k+1}), \ldots, L(\mathbf{u}_n)$. Which means that $\{L(\mathbf{u}_{k+1}), \ldots, L(\mathbf{u}_n)\}$ is a spanning set for the range of L. But is it linearly independent? Well, suppose that we have scalars $s_{k+1}, \ldots, s_n$, such that

$$s_{k+1}L(\mathbf{u}_{k+1}) + \cdots + s_nL(\mathbf{u}_n) = \mathbf{0}_{\mathbb{W}}$$

Then, using the linearity of L, we have that

$$L(s_{k+1}\mathbf{u}_{k+1} + \cdots + s_n\mathbf{u}_n) = \mathbf{0}_{\mathbb{W}}$$

This means that $s_{k+1}\mathbf{u}_{k+1} + \cdots + s_n\mathbf{u}_n$ is in the nullspace of L. Which means that we can write $s_{k+1}\mathbf{u}_{k+1} + \cdots + s_n\mathbf{u}_n$ as a linear combination of the elements of our basis for the nullspace of L: $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. So there are scalars $s_1, \ldots, s_k$ such that

$$s_{k+1}\mathbf{u}_{k+1} + \cdots + s_n\mathbf{u}_n = s_1\mathbf{v}_1 + \cdots + s_k\mathbf{v}_k$$

But this means that

$$-s_1\mathbf{v}_1 - \cdots - s_k\mathbf{v}_k + s_{k+1}\mathbf{u}_{k+1} + \cdots + s_n\mathbf{u}_n = \mathbf{0}_{\mathbb{V}}$$

Since the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is a basis for $\mathbb{V}$, it is linearly independent, so the only solution to this equation is $s_1, \ldots, s_n = 0$. Which means that $s_{k+1}, \ldots, s_n = 0$ is the only solution to $s_{k+1}L(\mathbf{u}_{k+1}) + \cdots + s_nL(\mathbf{u}_n) = \mathbf{0}_{\mathbb{W}}$. And so we see that the set $\{L(\mathbf{u}_{k+1}), \ldots, L(\mathbf{u}_n)\}$ is linearly independent, and thus is a basis for the range of V. And this means the rank of L is $n-k$, and we have shown that $\text{rank}(L) + \text{nullity}(L) = (n-k) + k = n$.