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Finding small stabilizers for unstable graphs

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Abstract An undirected graph $G = (V, E)$ is *stable* if the cardinality of a maximum matching equals the size of a minimum fractional vertex cover. We call a set of edges $F \subseteq E$ a *stabilizer* if its removal from G yields a stable graph. In this paper we study the following natural edge-deletion question: given a graph $G = (V, E)$, can we find a minimum-cardinality stabilizer?

Stable graphs play an important role in cooperative game theory. In the classic *matching game* introduced by Shapley and Shubik [19] we are given an undirected graph $G = (V, E)$ where vertices represent players, and we define the *value* of each subset $S \subseteq V$ as the cardinality of a maximum matching in the subgraph induced by S. The *core* of such a game contains all *fair* allocations of the *value* of V among the players, and is well-known to be non-empty iff graph G is *stable*. The stabilizer problem addresses the question of how to modify the graph to ensure that the core is non-empty.

We show that this problem is vertex-cover hard. We then prove that there is a minimum-cardinality stabilizer that avoids some maximum matching of G. We use this insight to give efficient approximation algorithms for sparse graphs and for regular graphs.

Keywords Matchings · Game Theory · Network Bargaining

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1 Introduction

Given an undirected graph $G = (V, E)$, a subset of edges $M \subseteq E$ is a *matching* if every vertex $v \in V$ is incident to at most one edge in M. Dually, a subset of vertices $U \subseteq V$ is called *vertex cover* if every edge has at least one endpoint in U. The corresponding optimization problems of finding a matching and a vertex cover of largest and smallest size, respectively, have a rich history in the field of Combinatorial Optimization. Relaxing canonical integer programming formulations for these problems yields the following primal-dual pair of linear programs:

$$\nu_f(G) := \max\{\mathbf{1}^T x : x(\delta(v)) \leq 1 \ \forall v \in V,\ x \geq 0\} \tag{P}$$

where $\delta(v)$ denotes the set of edges incident to v, and

$$\tau_f(G) := \min\{\mathbf{1}^T y : y_u + y_v \geq 1 \ \forall uv \in E,\ y \geq 0\}. \tag{D}$$

We will henceforth refer to feasible solutions of (P) and (D) as *fractional* matchings and vertex covers, respectively. An application of duality theory easily yields

$$\nu(G) \leq \nu_f(G) = \tau_f(G) \leq \tau(G)$$

where $\nu(G)$ and $\tau(G)$ denote the size of a maximum matching and a minimum vertex cover, respectively.

In this paper, we study graphs G with the property $\nu(G) = \tau_f(G)$. We denote the family of graphs satisfying this property to be *stable* graphs. Stable graphs subsume the well-studied class of *König-Egerváry* (KEG) graphs (e.g., see [20,13,14,15]) for which $\nu(G) = \tau(G)$. Stable graphs arise quite naturally in the study of cooperative *matching games* introduced by Shapley and Shubik in their seminal paper [19]. An instance of this game is associated with an undirected graph $G = (V, E)$ where vertices represent players. We define the *value* of each subset $S \subseteq V$ as the cardinality of a maximum matching in the subgraph $G[S]$ induced by S, and the *core* of the game consists of all *stable* allocations of total value $\nu(G)$ among the vertices in V in which no coalition of vertices has an incentive to deviate. This is formally defined as

$$\mathrm{core}(G) \ := \ \left\{ y \in \mathbb{R}^V_+ \ : \ \sum_{v \in S} y_v \geq \nu(G[S]) \ \forall S \subseteq V,\ \sum_{v \in V} y_v = \nu(G) \right\}.$$

It is well-known (e.g., see [7]) that core(G) is non-empty iff G is stable.

Matching games in turn are closely related to *network bargaining*, a natural, recent generalization of Nash's famous bargaining solution [16] to networks due to Kleinberg and Tardos [11]. Here, we are given an undirected graph $G = (V, E)$ whose vertices correspond to players, and whose edges correspond to potential unit-value deals between the incident players. Each player is allowed to engage in at most one deal with one of its neighbors. Hence, a permissible outcome is naturally associated with a matching M among the vertices of G, as well as an allocation $y \in \mathbb{R}^V_+$ of $|M|$ among M's endpoints. Kleinberg and

Tardos define an allocation to be *stable* if $y_u + y_v \geq 1$ for all $uv \in E$. The authors further define an *outside option* α_u for each vertex $u \in V$ as

$$\alpha_u := \max\{1 - y_v \, : \, uv \in \delta(u) \setminus M\},$$

and say that an outcome (M, y) is *balanced* if for every edge $uv \in M$, the surplus $1 - \alpha_u - \alpha_v$ is split evenly among u and v. The main result in [11] is that an instance of network bargaining has a stable outcome iff it has a balanced one. One now realizes (see also [5]) that a stable outcome exists iff the core of the underlying matching game instance is non-empty and hence iff G is stable.

In this paper, we focus on *unstable* instances of the matching game, where the core is empty. Our motivating goal is to establish strategies for *stabilizing* such instances in the *least intrusive way*; i.e., we would like to alter the input graph in few places and ideally maintain the value of the grand coalition formed by the set of vertices V in the process. The following natural edge-deletion *stabilizer problem* formalizes this: given a graph $G = (V, E)$, find a minimum-cardinality *stabilizer*, where a stabilizer is an edge set $F \subseteq E$ such that the subgraph $G \setminus F := (V, E \setminus F)$ is stable.

Stable graphs form a proper superclass of KEGs which in turn form a superclass of bipartite graphs. Readers familiar with the literature of bipartite graphs would immediately recognize that the stabilizer problem closely resembles the optimization problems of deleting the minimum number of edges to convert a given graph into a KEG or a bipartite graph, both of which have been well studied (e.g., see [1,15]). The investigation of structural properties of unstable graphs has a long history (e.g., see [21,3,17]), but there are few algorithmic results on how to convert an unstable graph to a stable graph. Biró et al. [6] recently studied the minimum stabilizer problem in the weighted setting, where maximum-weight matchings are considered instead of maximum-cardinality matchings. The authors showed that the problem is NP-hard in this case, and leave the complexity of the question in the unweighted setting open.

1.1 Our results

We first show that removing a minimum stabilizer from a given graph G does not reduce the cardinality of the maximum matching. Hence the value of the grand coalition of the associated matching game remains the same.

Theorem 1 *For every minimum stabilizer F, we have $\nu(G \setminus F) = \nu(G)$.*

The proof of this theorem is algorithmic: given any stabilizer F, we can efficiently find a maximum matching M in G and a stabilizer F' such that $F' \subseteq F$ and $M \cap F' = \emptyset$. The last equality implies that M is still a maximum matching in $G \setminus F'$. The result motivates the following intermediate *M-stabilizer* problem: given a maximum matching M, find a minimum-cardinality stabilizer F_M

that is disjoint from M. In the network bargaining setting, this question asks how to convert a specific maximum matching into one with a stable allocation through minimal edge deletions in the underlying network. Biró et al. [6] previously showed that this problem is NP-hard. We strengthen the hardness result and complement it with a tight algorithmic counterpart.

Theorem 2 *The M-stabilizer problem is NP-hard, and no efficient $(2-\varepsilon)$-approximation algorithm exists for any $\varepsilon > 0$ assuming the Unique Games Conjecture [10]. Furthermore, the M-stabilizer problem admits an efficient 2-approximation algorithm.*

The hardness proof employs an approximation preserving reduction from vertex cover. The approximation algorithm uses linear programming, and one shows that a suitable linear programming relaxation for the problem has a half-integral optimal solution.

Turning to the stabilizer problem, Theorems 1 and 2 suggest that the crux of the hardness of the stabilizer problem lies in finding the *right* maximum matching that survives the removal of a minimum stabilizer. Once such a matching is found one could indeed *simply* apply our 2-approximation algorithm for the M-stabilizer problem. However, not every maximum matching survives the removal of a minimum stabilizer. In fact, for two different maximum matchings M and M', the cardinality of F_M and $F_{M'}$ can differ by a factor of $\Omega(|V|)$ even on a planar factor-critical graph (Section 4.1). In Section 5.1, we present an approximation algorithm whose approximation factor depends on the *sparsity* of the graph. We say that a graph $G = (V, E)$ is *ω-sparse* if $|E(S)| \leq \omega\,|S|$ for all vertex subsets $S \subseteq V$.

Theorem 3 *There exists an efficient $O(\omega)$-approximation algorithm for the stabilizer problem, where ω is the sparsity of the input graph.*

We note that the above result implies a constant factor approximation algorithm for graphs with constant sparsity, e.g., *planar graphs.* We do not know whether a constant factor approximation algorithm can be developed for arbitrary graphs. However, we give a 2-approximation algorithm for regular graphs (graphs where all vertex degrees are equal). In the network bargaining setting, this gives a 2-approximation algorithm to stabilize networks in which every player has the same number of potential deals to make.

Theorem 4 *There exists an efficient 2-approximation algorithm for the stabilizer problem in regular graphs.*

The analysis of our algorithm combines some classic results about matchings and vertex covers such as the structure of basic solutions of (P) and (D) and the Edmonds-Gallai decomposition.

Regarding hardness, we extend the hardness result obtained for M-stabilizers answering the open question in [6]. Interestingly, our hardness result holds for *factor-critical* graphs (see next subsection for the definition).

Theorem 5 *The stabilizer problem is NP-hard. Furthermore, no efficient* $(2-\varepsilon)$*-approximation algorithm exists for any* $\varepsilon > 0$ *assuming the Unique Games Conjecture [10].*

Organization. We begin by proving the relation between maximum matchings and minimum stabilizer (Theorem 1) in Section 2. We show in Section 3 certain properties of the Edmonds-Gallai decomposition of graphs and derive a lower bound for the size of the minimum stabilizer. Next, we address the M-stabilizer problem in Section 4 and prove the hardness and approximation result for this problem (Theorem 2). We present the approximation algorithms (Theorems 3 and 4) in Section 5. In this section, we also observe that our lower bound to prove the 2-approximation result in Theorem 4 uses a linear program, and that a natural integer programming formulation of the stabilizer problem can be obtained by adding more constraints to this linear program. However, we show an example that exhibits an integrality gap of $\Omega(n)$ for this linear program. We show the hardness result for the stabilizer problem (Theorem 5) in Section 6.

1.2 Related work

The problem of removing vertices or edges from a graph in order to attain a certain graph property is natural, and thus not surprisingly, its variants have been studied extensively. Much of the work on deletion problems addresses *monotone* graph properties (e.g., see [22,2]) that are invariant under edge-removal or vertex-removal. Crucially, graph stability is not a monotone property as one easily verifies: the triangle is not stable, and adding a single pendant edge to one of its vertices yields a stable graph.

Our work is closely related to that of Mishra et al. [15] on edge-deletion problems to attain the König-Egerváry graph property. Similar to stability, KEG is not a monotone property. Mishra et al. showed that it is NP-hard to approximate the corresponding edge-deletion problem within a factor of 2.88. Assuming the Unique Games Conjecture, no constant-factor approximation algorithm may exist for the problem. We note that the reduction used in [15] will likely not be helpful for proving hardness of the stabilizer problem as the graphs constructed in the reduction are stable. On the positive side, the authors show that, for a given graph $G = (V, E)$ one can efficiently find a KEG (and hence stable) subgraph with at least $3|E|/5$ edges.

The recent paper by Könemann et al. [12] addressed the related, NP-hard problem of finding a minimum-cardinality *blocking set* in an input graph $G = (V, E)$. Here one wants to find a set of edges $F \subseteq E$ such that $G \setminus F$ has a fractional vertex cover of size at most $\nu(G)$. Importantly, the resulting graph $G \setminus F$ is not required to be stable; indeed, the cardinality of a minimum blocking set can differ from the cardinality of a minimum stabilizer by a factor of $\Omega(|V|)$.

1.3 Preliminaries

Given an undirected graph G and a matching M in G, a path is called *M-alternating* if it alternates edges from M and those from $E \setminus M$. An odd cycle of length $2k+1$ in which exactly k edges are in M is called an *M-blossom*. An *M-flower* consists of an M-blossom and an even-length M-alternating path from a vertex in the blossom to a vertex that is exposed by M. For a subset of vertices $S \subseteq V$, we use $E(S)$ to denote the set of edges in the graph induced by S and $G[S]$ to denote the subgraph induced by S. A graph $G = (V, E)$ is called *factor-critical* if for all $v \in V$, $G[V \setminus \{v\}]$ has a perfect matching; i.e., a matching that does not expose any vertex. We will use the following odd-ear decomposition characterization of factor-critical graphs.

Lemma 1 ([25]) *A graph H is factor-critical if and only if the edges of the graph can be decomposed into a sequence of disjoint subgraphs $C, P_1, \ldots, P_t$, where (i) C is an odd cycle, (ii) $P_1, \ldots, P_t$ are odd-length paths or cycles, (iii) each path P_i has both end vertices in $C \cup \left(\cup_{j=1}^{i-1} P_j\right)$ and no interior vertices in $C \cup \left(\cup_{j=1}^{i-1} P_j\right)$, and (iv) each cycle P_i has exactly one end vertex in $C \cup \left(\cup_{j=1}^{i-1} P_j\right)$ and no interior vertices in $C \cup \left(\cup_{j=1}^{i-1} P_j\right)$.*

A vertex v is called *inessential* for G if there exists a maximum matching M that exposes v, and *essential* otherwise. In this paper, we will also use the following characterization of stable graphs.

Theorem 6 ([11]) *The following are equivalent: (i) G is stable, (ii) The set of inessential vertices of G forms a stable set, (iii) G contains no M-flower for any maximum matching M. Moreover, if G is not stable, then G contains an M-flower for every maximum matching M.*

Given a graph G, the Edmonds-Gallai decomposition is a partition of its vertex set into three parts $B(G)$, $C(G)$, $D(G)$, where $B(G)$ is the set of inessential vertices, the set $C(G)$ consists of the neighbors of $B(G)$ and $D(G) = V \setminus (B(G) \cup C(G))$. We list several useful properties of this decomposition.

Theorem 7 ([18]) *Given a graph G, its Edmonds-Gallai decomposition $B(G), C(G), D(G)$ can be computed in polynomial time. Further, we have the following properties.*

1. *Each component of $G[B(G)]$ is factor-critical.*
2. *Every maximum matching M in G exposes at most one vertex in each component K of $G[B(G)]$.*
3. *If U is a non-trivial factor-critical component in $G[B(G)]$ (i.e., a factor-critical component with more than one vertex), then $\nu(G \setminus E(U)) < \nu(G)$.*

Finally, we will need the following two well-known classic results on the structure of fractional and integral matchings.

Theorem 8 *[4] Every basic feasible solution to (P) has components equal to* $0, 1$ *or* $\frac{1}{2}$*, and the edges with half integral components induce vertex disjoint cycles.*

Theorem 9 *[3, 21] Let* $\hat{x}$ *be a maximum fractional matching in a graph* G *having half integral fractional components for a minimum number of odd cycles* $C_1, \ldots, C_q$*. Let* $\hat{M} := \{e \in E : \hat{x}_e = 1\}$ *and* M_i *be a maximum matching in* C_i*. Then* $M = \hat{M} \cup M_1 \cup \cdots \cup M_q$ *is a maximum matching in* G*. Moreover, such* $\hat{x}$ *and* M *can be found in time polynomial in the number of vertices.*

2 Maximum matchings and minimum stabilizers

In this section, we show that the deletion of any minimum stabilizer does not decrease the size of the maximum matching in the graph.

Proof (of Theorem 1) Let F be a minimum stabilizer. Find a maximum matching M in G such that $|M \cap F|$ is minimum. Suppose $|M \cap F| \neq 0$.

Consider $G' := G \setminus (F \setminus M)$, the graph obtained by removing all the edges of $F \setminus M$ from G. Clearly M is still a maximum matching in G'. However, since F is minimum, G' is not stable. By Theorem 6, this implies that there exists an M-flower in G' starting at an M-exposed vertex w.

Suppose the M-flower contains an edge $uv \in F$. Then, $uv \in M$, since all other edges from F have been removed in G'. Therefore, we can find an even M-alternating path P from w to either u or v. Switching along the edges of this path, we obtain another maximum matching $M' = M \Delta P$ in G with $|F \cap M'| < |F \cap M|$, a contradiction.

It follows that the M-flower does not contain any edge from F, and therefore the M-flower also exists in $G \setminus F$. However, since $G \setminus F$ is stable, this implies that $M \setminus F$ is not a maximum matching in $G \setminus F$. Apply Edmonds' maximum matching algorithm on the graph $G \setminus F$ initialized with the matching $M \setminus F$, and construct an $M \setminus F$-alternating tree starting with the exposed vertex w (we refer to [8] for the terminology we use in the description of Edmonds' algorithm). There are two possibilities: either we find an augmenting path P or a frustrated tree rooted at w. In the first case, the path P starts with w and ends with a $M \setminus F$-exposed vertex, say w'. However, such a path cannot exist in G because M is a maximum matching, and therefore w' must have been incident to an edge $f \in M \cap F$. Also, note that the path P is in $G \setminus F$. Hence, $P + f$ is an even M-alternating path in G containing exactly one edge in $M \cap F$. Switching along the edges of this path, we obtain another maximum matching $M' = M \Delta P$ in G with $|F \cap M'| < |F \cap M|$, a contradiction. The only remaining possibility is that we find a frustrated tree T rooted at w. Let $G[T] = (V_T, E_T)$ be the graph induced by all vertices in the frustrated tree T (after expanding pseudonodes). In this case, $M \cap E_T$ is a maximum matching in $G[T]$, and the M-flower is contained in E_T. However, if we continue Edmonds' algorithm, it would remove the vertices of the frustrated tree, and continue running in the resulting subgraph to find a maximum matching. Therefore it ends by computing a maximum matching M^* in $G \setminus F$ with $M^* \cap E_T = M \cap E_T$. Therefore, we have a M^*-flower in $G \setminus F$, again a contradiction. □

We remark here that the above proof is algorithmic, therefore given a stabilizer F, we can find in polynomial time a maximum matching M in G and another stabilizer $F' \subseteq F$ such that $M \cap F' = \emptyset$. The first step of computing a maximum matching M in G with minimum intersection with F can be done by assigning a cost of one to the edges in F, zero to the rest of the edges, and computing a min-cost matching in G of cardinality $\nu(G)$.

3 Edmonds-Gallai decomposition and a lower bound

In this section, we state a consequence of the Edmonds-Gallai decomposition theorem. This will be used to derive a lower bound for the cardinality of a stabilizer. The consequence of interest to us is stated in the proposition below.

Proposition 1 *Let M be a maximum matching in G that also matches the maximum possible number of isolated vertices in $G[B(G)]$. Let k be the number of non-trivial factor-critical components with at least one vertex exposed by M. Then $k = 2(\nu_f(G) - \nu(G))$.*

The proof relies crucially on the following result of Pulleyblank.

Theorem 10 (Theorem 4 in [17]) *Let x be a fractional matching with $x(e) \in \{0, 1/2, 1\}$ for all $e \in E$. Suppose x satisfies the following:*

1. *whenever $x(e) = \frac{1}{2}$ for some edge $e \in E$ then $e \in E(B(G))$,*
2. *the edges $\{e : x(e) = 1\}$ induce a perfect matching on $G[D(G)]$,*
3. *for each vertex $v \in C(G)$, there exists a vertex $u \in B(G)$ such that $x(uv) = 1$,*
4. *each component $K = (V_K, E_K)$ of $G[B(G)]$ is such that $E_K \cap Support(x)$ contains at most one cycle; if K contains such a cycle, then x induces a fractional perfect matching (each vertex has fractional degree one) of K; if not, then x induces a perfect (integer) matching in $K \setminus u$ for some vertex $u \in V_K$ and if K is non-trivial, then there exists $v \in C(G)$ such that $x(uv) = 1$, and*
5. *let $S \subseteq B(G)$ be the set of isolated vertices in $G[B(G)]$, and let $N(S) \subseteq C(G)$ be their neighbors, then x induces a maximum integer matching on $G[S \cup N(S)]$.*

Then x is a maximum fractional matching in G.

Proof (Proof of Proposition 1) Let $U_1, \ldots, U_k$ denote the non-trivial factor-critical components of $B(G)$ that have at least one vertex exposed by M. Since M is a maximum matching in G, by Theorem 7, M exposes exactly one vertex in $U_1, \ldots, U_k$.

An odd cycle C in G is said to be *separated* by a maximum matching N if $N \cap \delta(C) = \emptyset$. We consider the following quantities.

$$\begin{aligned}
\sigma(G,N) &:= \text{Maximum number of vertex-disjoint odd cycles separated by } N,\\
\sigma(G) &:= \max_{\text{max matchings } N \text{ in } G} \sigma(G,N),\\
\gamma(G,x) &:= \text{Number of vertex-disjoint odd cycles in the support of fractional}\\
&\quad\ \text{matching } x,\\
\gamma(G) &:= \min_{\text{max fractional matchings } x \text{ in } G} \gamma(G,x).
\end{aligned}$$

By a result of Balas [3], we know that $2(\nu_f(G) - \nu(G)) = \sigma(G) = \gamma(G)$. Next we will show that $\sigma(G) \geq k$ and $\gamma(G) \leq k$ to complete the proof. For each U_i, let $(C_i, P_1^i, \ldots, P_{t_i}^i)$ be an odd-ear decomposition of U_i given by Lemma 1, where C_i is the starting odd cycle and odd-length paths/cycles $P_1^i, \ldots, P_{t_i}^i$ are added in sequence to obtain U_i.

Consider the matching M_i in U_i obtained as follows: pick a maximum matching in C_i and for each odd-length P_j^i, pick every even edge in P_j^i. Such a matching M_i is a maximum matching in U_i since U_i is factor-critical and all but one vertex in U_i is matched by M_i. Now for each $i = 1, \ldots, k$, replace the edges of M in U_i by M_i. Let N denote the resulting matching. Since $|N| = |M|$, the matching N is still a maximum matching in G. Moreover, the isolated vertices in $G[B(G)]$ that are matched by M are also matched by N. Thus, N also matches the maximum possible number of isolated vertices in $G[B(G)]$. The number of non-trivial factor-critical components with at least one vertex exposed by N is still k. Since the cycles $C_i,\ i = 1, \ldots, k$ are separated by N, we have that $\sigma(G,N) \geq k$. Hence, $\sigma(G) \geq \sigma(G,N) \geq k$.

In order to show that $\gamma(G) \leq k$, we will identify a maximum fractional matching x with $\gamma(G,x) = k$. For each component U_i, we use the same odd-ear decomposition $(C_i, P_1^i, \ldots, P_{t_i}^i)$ of U_i to obtain a fractional *perfect* matching x_i in U_i: set $x_i(e) = 1/2$ for each $e \in C_i$ and $x_i(e) = 1$ for each even edge e in P_j^i. Now take

$$x(e) = \begin{cases} x_i(e) & \text{if } e \in U_i \text{ for } i \in \{1, \ldots, k\},\\ 1 & \text{if } e \in M \setminus \left(\cup_{i=1}^k U_i\right),\\ 0 & \text{if } e \notin M \cup \left(\cup_{i=1}^k U_i\right). \end{cases}$$

Since each $x_i, i = 1, \ldots, k$ has exactly one odd cycle in its support and these are the only odd cycles in the support of x, we have that $\gamma(G,x) = k$. It remains to verify that x is a maximum fractional matching in G. We verify the conditions in Theorem 10.

1. Since $x(e) = 1/2$ only for edges $e \in C_i \subseteq U_i \subseteq G[B(G)]$, the first condition holds.
2. Since M is a maximum matching in G, it follows that M induces a perfect matching in $G[D(G)]$. Therefore, x also induces a perfect matching in $G[D(G)]$.

3. Since M is a maximum matching, it follows that for each vertex $v \in C(G)$, there exists a vertex $u \in B(G)$ such that $uv \in M$. Hence, $x(uv) = 1$.
4. Let K be a component of $G[B(G)]$. If $K = U_i$ for some $i = 1, \dots, k$, then $E_K \cap \text{Support}(x)$ contains exactly one cycle, namely C_i. Moreover, x restricted to U_i is a fractional perfect matching x_i in U_i. If $K \neq U_i$ for all $i = 1, \dots, k$, then $E_K \cap \text{Support}(x)$ does not contain a cycle. Moreover, x induces a perfect (integer) matching in $K \setminus u$ for some vertex $u \in K$, since M induces a perfect (integer) matching in $K \setminus u$ for some vertex $u \in K$. If K is non-trivial, since $K \neq U_i$ for all $i = 1, \dots, k$, the matching M does not have an exposed vertex in K. This implies that M matches u to a vertex $v \in C(G)$. Hence, $x(uv) = 1$.
5. Since M matches the maximum possible number of isolated vertices in $G[B(G)]$, it follows that M induces a maximum matching on $G[S \cup N(S)]$. Now, x restricted to $G[S \cup N(S)]$ is the indicator vector of M restricted to $G[S \cup N(S)]$. Consequently, x induces a maximum matching on $G[S \cup N(S)]$. □

We next prove a lower bound on the cardinality of a stabilizer.

Theorem 11 *For every stabilizer F, we have $|F| \geq 2(\nu_f(G) - \nu(G))$.*

Proof Let $B(G), C(G), D(G)$ denote the Edmonds-Gallai decomposition and let M be a maximum matching in G that also matches the maximum possible number of isolated vertices in $G[B(G)]$. Let $U_1, \dots, U_k$ denote the non-trivial components in $G[B(G)]$ with at least one vertex exposed by M. Let F be a minimum stabilizer and $H = G \setminus F$. For each component $U_1, \dots, U_k$, at least one vertex $v_i \in U_i$ becomes essential in H. Suppose not, then all vertices of some U_i are inessential in H. Theorem 6.(ii) now implies that F contains all edges in $G[U_i]$, and thus, by Theorem 7 we have that $\nu(H) < \nu(G)$, a contradiction to Theorem 1.

Pick a maximum matching N in H. Then, N will cover all these vertices $v_1, \dots, v_k$ that are essential in H. Since $G[U_i]$ is factor-critical and M matches all but one vertex in U_i using edges in $G[U_i]$, we may assume without loss of generality, that M misses all these vertices. The graph $M \Delta N$ is a disjoint union of even cycles and even paths since $|M| = |N| = \nu(G)$. Consider the k disjoint paths starting at the vertices $v_1, \dots, v_k$ in $M \Delta N$. We observe that at least one of the M edges in each of these paths should belong to F, otherwise we can obtain a maximum matching in H that exposes the starting vertex v_i, thus contradicting v_i being an essential vertex in H. Hence $|F| \geq k$. The result follows by Proposition 1. □

4 The M-Stabilizer problem

In this section, we address the M-stabilizer problem. We prove the hardness result and give a 2-approximate algorithm. We also exhibit an example to highlight that the size of the M-stabilizer highly depends on the choice of the maximum matching M.

Proposition 2 *The M-stabilizer problem is NP-hard, and no polynomial-time $(2-\varepsilon)$-approximation algorithm exists for any $\varepsilon > 0$ assuming the Unique Games Conjecture [10].*

Proof Let $G = (V, E)$ be a vertex cover instance. Without loss of generality, we may assume that G has no isolated vertices. The approach is to extend the graph by connecting every vertex of V by a two-edge-path to a super-source v_0. We observe that every edge in G induces a cycle of length 5 including v_0 in the new graph. We choose the maximum matching M such that each of the 5-cycle forms an M-flower with the exposed vertex v_0. Two of these M-flowers are edge-disjoint if and only if the two corresponding edges in G are vertex-disjoint. We show that in order to ensure that there are no M-flowers, the stabilizer should remove as many edges in the new graph as the number of vertices needed to cover all edges in G.

Formally, we construct the new graph $G' = (V', E')$ as follows:

$$V' := \{v_0\} \cup \{v', v'' \colon v \in V\} \quad \text{and}$$

$$E' = \{v_0 v', v'' v' \colon v \in V\} \cup \{u'' v'' \colon uv \in E\}.$$

We set the matching $M := \{v'' v' \colon v \in V\}$. Since G has no isolated vertices, we observe that G' is factor-critical and M is a maximum matching exposing v_0. See figure 1 for the graph G' and the matching M.

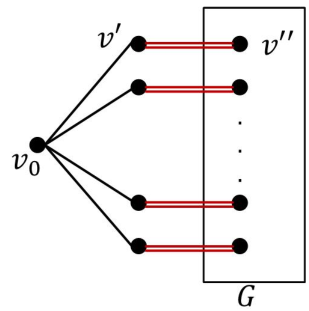


Fig. 1 Minimum M-stabilizer instance G' constructed from vertex cover instance G. The double edges are present in M.

Every edge $uv \in E$ in G corresponds to an M-flower $\{v_0 v', v' v'', u'' v'', u' u'', v_0 u'\}$ in G'. We further observe that by the choice of M, all M-flowers in G' are necessarily of this kind.

We will show that (a) any vertex cover in G induces an M-stabilizer in G' of the same size, and (b) any M-stabilizer F in G' induces a vertex cover in G of size at most $|F|$. This implies that the reduction is approximation preserving and the inapproximability results for the vertex cover problem [23, 24] carry over to the problem of finding a minimum M-stabilizer.

For (a), let W be a vertex cover in G. Consider the set $F = \{v_0 v' \colon v \in W\}$. We observe that $G \setminus F$ has no M-flowers since any M-flower corresponding to an edge $uv \in E$ in G contains at least one edge from F. This is because W is a vertex cover. Thus, $G \setminus F$ is stable by Theorem 6 and hence F is a stabilizer.

For (b), suppose that F is an M-stabilizer that contains an edge $e \in E'$ that is of the type $u''v''$ for some edge $uv \in E$. Then, the set of edges $F' = (F \setminus \{e\}) \cup \{v_0 u'\}$ is still an M-stabilizer since e intersects only with the M-flower corresponding to the edge $uv \in E$. Repeating this for every edge $u''v''$ in F, we obtain an M-stabilizer F' such that $|F'| \leq |F|$ and F' consists of edges only of the type $v_0 v'$ for some vertices $v \in V$. This allows us to construct a vertex cover from such an M-stabilizer F' as follows: For every edge $v_0 v'$ in the M-stabilizer F', we take the corresponding vertex v into the vertex cover. By construction, the resulting set of vertices is a vertex cover, since an uncovered edge in G would induce an M-flower in $G' \setminus F'$ thus contradicting the stability of $G' \setminus F'$ (Theorem 6). We further note that the cardinality of the resulting vertex cover is the same as that of $|F'| \leq |F|$.

This concludes the proof of inapproximability. □

Next, we obtain a 2-approximation algorithm for finding a minimum M-stabilizer for a given maximum matching M in graph G by showing that a suitable linear program has a half-integral optimum solution. We first prove that the formulation has an integral optimum solution for bipartite graphs. Then we construct a suitable new bipartite graph G' from our original instance G whose LP solution allows us to derive a M-stabilizer for G that is at most twice as large as the minimum M-stabilizer.

Proposition 3 *The M-stabilizer problem admits an efficient 2-approximation algorithm.*

Proof Let $V(M) \subseteq V$ denote the set of vertices that are incident to an edge in the given matching M. We introduce a variable x_v for every vertex $v \in V(M)$. We consider the following covering linear program:

$$\begin{aligned} \min \quad & \sum_{e \in E \setminus M} z_e && (\bar{P}) \\ \text{s.t.} \quad & y_u + y_v = 1 \quad \forall uv \in M \\ & y_u + y_v + z_e \geq 1 \quad \forall e = uv \in E \setminus M \text{ and } u, v \in V(M) \\ & y_v + z_e \geq 1 \quad \forall e = uv \in E \setminus M \text{ and } v \in V(M), u \notin V(M) \\ & y, z \geq 0 \end{aligned}$$

The first set of constraints enforces that $|M| = \sum_{v \in V(M)} y_v$. The subsequent two sets of constraints imply that every edge not in M is covered by the corresponding z-variable or the y-variables of its end points. We note that at least one of the end points u, v of any edge $uv \in E$ is in $V(M)$ since M is a maximal matching. Consequently, every edge in G has exactly one covering constraint in $(\bar{P})$.

We observe that if a feasible solution (y, z) of $(\bar{P})$ satisfies $z \in \{0,1\}^{E \setminus M}$, then $F := \{e \in E \colon z_e = 1\}$ is an M-stabilizer. This is because we have a fractional vertex cover y in $G \setminus F$ of the same size as the maximum matching in $G \setminus F$.

In the other direction, suppose we have an M-stabilizer F, then, we can construct a solution (y, z) of $(\bar{P})$ satisfying $z \in \{0,1\}^{E \setminus M}$ as follows: Take $z_e = 1$ if $e \in F$ and $z_e = 0$ for every $e \in E \setminus (M \cup F)$. Take y to be a minimum fractional vertex cover in $G \setminus F$. Since $G \setminus F$ is stable, we have that y and M form a primal-dual optimal pair to the fractional vertex cover and fractional matching linear programs in $G \setminus F$. Hence, by complementary slackness $y_v = 0$ for every vertex v that is exposed by M and hence $\text{Support}(y) \subseteq V(M)$. Thus, (y, z) is a feasible solution to $(\bar{P})$.

Claim For a bipartite graph $G = (V, E)$ and a maximal matching M in G, the linear program $(\bar{P})$ has an integral optimum solution (y^*, z^*).

Proof Let A denote the coefficient matrix of the constraints in $(\bar{P})$ for G. Then the matrix A has the form

$$A = \begin{bmatrix} A' & I \end{bmatrix}$$

where A' is a $|E| \times |V(M)|$ sub matrix of the edge-vertex incidence matrix A_G of G and I is a $|E| \times |E \setminus M|$ sub matrix of the $|E| \times |E|$ identity matrix $I_{|E|}$ after removing the columns corresponding to the edges in M. Now, we observe that the matrix $\begin{bmatrix} A_G & I_{|E|} \end{bmatrix}$ is totally unimodular since G is bipartite and thus A_G is totally unimodular. Since A is a (column-indexed) sub matrix of $\begin{bmatrix} A_G & I_{|E|} \end{bmatrix}$, we conclude that A is totally unimodular as well. This implies that there is an integral optimum solution (y^*, z^*) of $(\bar{P})$. □

We will use the above claim to find an M-stabilizer in G that is at most twice as large as the optimum by constructing a bipartite graph as follows. Let $G' = (V', E')$ denote the new bipartite graph with $V' := V_1 \cup V_2$ for $V_i := \{v_i \colon v \in V\}$ and $E' := \{u_1v_2, u_2v_1 \colon uv \in E\}$.

We set $M' := \{u_1v_2, u_2v_1 \colon uv \in M\}$. We note that M' is a maximal matching in G', but may not necessarily be maximum (M-flowers in G correspond to M'-augmenting paths in G'). Let $(\bar{P}'), (\bar{P})$ denote the corresponding linear programs for G' and G, respectively.

We first show that the minimum M-stabilizer F in G induces a solution (y', z') for $(\bar{P}')$ with cost $2|F|$ such that z' is integral. To see this, let y denote the fractional vertex cover of $G \setminus F$ with size $\sum_{v \in V(M)} y_v = |M|$. Such a fractional vertex cover exists because $G \setminus F$ is stable. Since $G \setminus F$ is stable, we have that y and M form a primal-dual optimal pair to the fractional vertex cover and fractional matching linear programs in $G \setminus F$. Hence, by complementary slackness $y_v = 0$ for every vertex v that is exposed by M and hence $\text{Support}(y) \subseteq V(M)$. We set $y'_{u_i} := y_u$ for all $u \in V, i = 1, 2$, and $z'_{u_1v_2} = z'_{u_2v_1} = 1$ for all $uv \in F$. Now, (y', z') is a feasible solution of $(\bar{P}')$ of cost $2|F|$ with integral z'.

Next, we show that the optimum integral solution of $(\bar{P}')$ can be used to find a half-integral solution of $(\bar{P})$. Let (y', z') be the optimal integral solution of $(\bar{P}')$. Then $y_u := (1/2)(y'_{u_1} + y'_{u_2})$ and $z_{uv} = \max\{z'_{u_1v_2}, z'_{u_2v_1}\}$ defines a feasible solution for $(\bar{P})$: For $uv \in M$, we get $y_u + y_v = (y'_{u_1} + y'_{v_2} + y'_{u_2} + y'_{v_1})/2 = 1$ and for $uv \in E \setminus M$ with $u, v \in V(M)$, we get

$$\begin{aligned} y_u + y_v &= \frac{y'_{u_1} + y'_{v_2} + y'_{u_2} + y'_{v_1}}{2} \geq \frac{1 - z'_{u_1 v_2} + 1 - z'_{u_2 v_1}}{2} \\ &= 1 - \frac{z'_{u_1 v_2} + z'_{u_2 v_1}}{2} \geq 1 - \max\{z'_{u_1 v_2}, z'_{u_2 v_1}\} = 1 - z_{uv}. \end{aligned}$$

The case $uv \in E \setminus M$ with $u \in V(M)$ and $v \notin V(M)$ follows in an analogous manner. As the cost of (y', z') is at most $2|F|$, the cost of the solution (y, z) of $(\bar{P})$ that we constructed is also bounded by $2|F|$. However, z is integral and thus defines an M-stabilizer in G of size at most twice the size of the minimum M-stabilizer. □

Propositions 2 and 3 jointly imply Theorem 2.

4.1 Ratio between M-stabilizer and minimum stabilizer

In this section we illustrate that the size of the M-stabilizer could vary by a large factor depending on the choice of the maximum matching M.

Proposition 4 *There exist a planar factor-critical graph $G = (V, E)$ and two different maximum matchings M, M' in G such that the sizes of the minimum M-stabilizer and the minimum M'-stabilizer differ by a factor of $\Omega(|V|)$.*

Proof Let G' be the graph as shown in Figure 2. Its vertex and edge sets are given by

$$V = \{a_i, b_i, c_i, d_i : i \in [t]\} \cup \{u_1, u_2, v_1, v_2, r\}$$

$$E = \{u_1 v_1, u_2 v_2, u_1 u_2, v_1 v_2\} \cup \{u_2 a_i, v_2 c_i, a_i b_i, c_i d_i, b_i r, d_i r : i \in [t]\}.$$

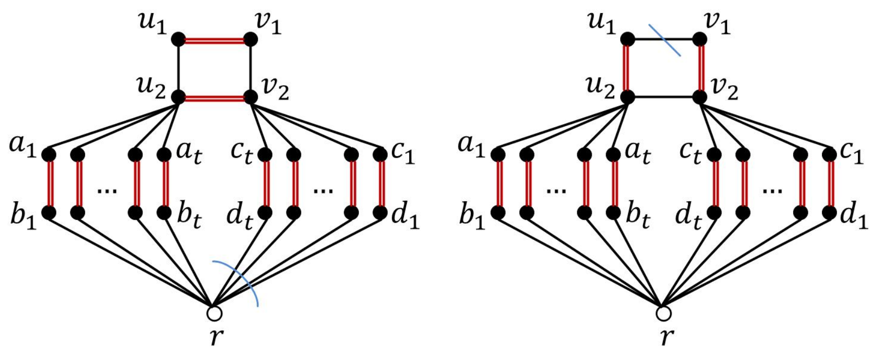
By figure 2 the graph is planar. Factor-criticality of the graph is also straightforward to verify. Now, we take $M = \{a_i b_i, c_i d_i : i \in [t]\} \cup \{u_1 v_1, u_2 v_2\}$ and $M' = (M \setminus \{u_1 v_1, u_2 v_2\}) \cup \{u_1 u_2, v_1 v_2\}$.

The minimum M'-stabilizer is $\{u_1 v_1\}$ as deleting this edge ensures that there are no M'-flowers in the resulting graph. In contrast, any M-stabilizer necessarily has to delete $t = \Omega(|V|)$ edges to ensure that the resulting graph has no M-flowers. This is because there exist t M-flowers that are disjoint in their non-M-edges.

□

5 Finding small stabilizers

In this section, we return to the problem of finding small stabilizers. The following two sections present algorithms for the problem in sparse and regular graphs, respectively.



(a) M-stabilizer of size t (b) M'-stabilizer of size 1

Fig. 2 Instance showing that the size of a minimum M-stabilizer is far from the size of a minimum stabilizer.

5.1 An $O(\omega)$-approximation algorithm for sparse graphs

Before proving Theorem 3, we state and prove the following lemma that is the main ingredient of our algorithm.

Lemma 2 *Let G be a graph with $\nu_f(G) > \nu(G)$. There exists an efficient algorithm to find a set of edges L with $|L| = O(\omega)$, such that*

(i) $\nu(G \setminus L) = \nu(G)$,
(ii) $\nu_f(G \setminus L) \leq \nu_f(G) - \frac{1}{2}$.

In other words, Lemma 2 shows that we can find a small subset of edges to remove from G without decreasing the size of the maximum matching but reducing the size of the minimum fractional vertex cover.

Proof (Proof of Lemma 2) Consider $\hat{x}$ and M as in Theorem 9 for the graph G. By duality theory, there exists a fractional vertex cover y with $\mathbf{1}^T y = \mathbf{1}^T \hat{x}$ satisfying complementary slackness conditions with $\hat{x}$. Moreover, we can always find such a vector y with half integral components (e.g., see [9]). We will give an efficient algorithm to find a vertex u with the following properties:

(a) $y_u = \frac{1}{2}$,
(b) $L_u := \{uw : y_w = \frac{1}{2}\}$ satisfies $\nu(G \setminus L_u) = \nu(G)$ and $|L_u| \leq 4\omega$.

First, let us argue that (a) and (b) together imply the result. Assume we can find such a vertex u. The only non-trivial conclusion that needs to be verified is that $\nu_f(G \setminus L_u) \leq \nu_f(G) - 1/2$. Consider the vector y' defined as $y'_v = y_v$ for all $v \neq u$ and $y'_u = 0$ otherwise. Note that u cannot be adjacent to vertices w with $y_w = 0$ since y is a feasible fractional vertex cover in G. Furthermore, $y_w = 1$ whenever $uw \in E \setminus L_u$. This implies that y' is a feasible fractional vertex cover in $G \setminus L_u$, and its objective value of $y(V) - 1/2$ is an upper-bound on the value of a fractional matching in $G \setminus L_u$.

Now let us prove that a vertex u satisfying (a) and (b) can be found efficiently. Consider an arbitrary odd cycle in $\hat{x}$, e.g., C_1. Since $\hat{x}_e > 0$ for every edge $e = uv$ in C_1, it follows that the vertex cover constraint is tight (i.e., $y_u + y_v = 1$ holds) for all edges in C_1, and therefore $y_v = \frac{1}{2}$ for all vertices in C_1. By choice of M we know that it induces a maximum matching in C_1, and M exposes exactly one vertex of C_1.

Set $H := C_1$, and *mark* all vertices in C_1. Since C_1 is an odd cycle, and since M induces a matching in C_1 that exposes a single vertex, removing from G any subset of edges incident to a (marked) vertex in H does not decrease the size of a maximum integral matching in the resulting graph. Repeat the following process, which will maintain a collection of four invariants for the graph H: (i) Every vertex in H has y-value $\frac{1}{2}$, (ii) removing any subset of edges incident to one marked vertex of H does not decrease the size of a maximum matching, (iii) from any marked vertex, there is an even-length M-alternating path to a vertex in C_1, (iv) at least half of the vertices of H are marked. All properties clearly hold initially when H consists of C_1 only.

1. If there is a marked vertex in H with $|L_u| \leq 4\omega$, then u satisfies properties (a) and (b). STOP.
2. Otherwise, consider an arbitrary marked vertex u in H that is adjacent to a vertex $w \notin H$ with $y_w = \frac{1}{2}$. Such a w must be matched in M as otherwise, we could obtain an M-augmenting path in G by concatenating wu, the even-length M-alternating path from u to C_1 guaranteed by property (iii) and an appropriate even-length M-alternating path along C_1 to the M-exposed vertex on C_1.
3. Let z be the vertex matched to w by M. By complementary slackness, $y_z = \frac{1}{2}$. Add w and z to H and mark z. Go to 1.

It is straightforward to verify that properties (i)–(iv) continue to hold throughout the execution of the above process. Thus, it only remains to show that we can always find a vertex w as specified in Step 2 above; i.e., if all marked vertices u have $|L_u| > 4\omega$, then there exists a marked vertex in H that is adjacent to a vertex $w \notin H$ with $y_w = 1/2$. Suppose not. Consider the subgraph $G[H]$ induced by the vertices in H. This subgraph has the property that the degree of every marked vertex u in $G[H]$ is at least $|L_u| > 4\omega$. However, by (iv), the number of marked vertices is more than half the total number of vertices in $G[H]$. This contradicts the ω-sparsity of $V(H)$ in G. Finally, it is easy to see that the above process runs in polynomial time. □

With this Lemma at hand, we are now ready to prove our main theorem. We will use the following algorithm:

Algorithm 1.
INITIALIZE $G' = G$.
FOR $i = 1, \ldots, 2(\nu_f(G) - \nu(G))$:

1. Let L be the set of edges returned by the algorithm in Lemma 2 when its input is the current graph G'.

2. Set $G' \leftarrow G' \setminus L$.
3. If G' is stable, STOP.

Proof (Proof of Theorem 3)
Let G be an unstable graph. We use Algorithm 1. We will now prove that (a) whenever the above algorithm stops, the current graph G' is stable, and (b) the total number of edges removed during the complete execution of the algorithm is $O(\omega) \cdot |F^*|$, where F^* is a minimum stabilizer. Clearly (a) + (b) implies the result.

First, let us argue about stability. If the algorithm stops in step (iii) for some iteration $i < 2(\nu_f(G) - \nu(G))$, this is clear. So we may assume that the algorithm stops after performing all $2(\nu_f(G) - \nu(G))$ iterations. The graph G' output at this point has $\nu_f(G') \leq \nu_f(G) - \frac{1}{2}(2(\nu_f(G) - \nu(G))) = \nu(G) = \nu(G')$. This is because, by Lemma 2, in each iteration the size of a minimum fractional vertex cover decreases by at least $\frac{1}{2}$ while the size of the maximum matching is maintained. Hence, by definition of stability, G' is stable.

By Lemma 2, in each iteration we remove $O(\omega)$ edges and the total number of iterations is at most $2(\nu_f(G) - \nu(G))$. The bound on the approximation factor follows from Theorem 11. The running time bound also follows since the number of applications of the algorithm in Lemma 2 is at most $2(\nu_f(G) - \nu(G)) \leq |F^*| \leq |E|$ times. □

We end the section with an observation about our algorithm that will be useful for our approximation results on regular graphs.

Proposition 5 *Let $\Delta(G)$ be the maximum degree of a vertex in G. Then, the stabilizer output by Algorithm 1 has size at most $2(\nu_f(G) - \nu(G))\Delta(G)$.*

Proof In each iteration of the algorithm, we remove a subset of edges incident to some vertex. Therefore we remove at most $\Delta(G)$ edges in each iteration. Further, the number of iterations is at most $2(\nu_f(G) - \nu(G))$. □

5.2 A 2-approximation algorithm for regular graphs

In this section, we give a 2-approximation algorithm for solving the stabilizer problem in regular graphs.

Proof (Proof of Theorem 4) We use Algorithm 1. Consider a d-regular graph G, i.e., a graph where every vertex has degree d. Let $k := 2(\nu_f(G) - \nu(G))$. By Proposition 5, the size of F output by the algorithm is at most kd. We complete the proof by showing that every stabilizer in G is of size at least $kd/2$.

Consider the Edmonds-Gallai decomposition of G, namely $B(G)$, $C(G)$, $D(G)$. Let S denote the isolated vertices in $G[B]$. Consider a maximum matching M in G that also matches the maximum possible number of vertices in S. By Proposition 1, the number of non-trivial factor-critical components in $G[B(G)]$ with at least one vertex exposed by M is equal to k.

Let S_u denote the vertices in S that are exposed by M. We first observe that the size $\nu(G)$ of the maximum matching in G is $(|V|-k-|S_u|)/2$. Consider the following primal and dual linear programs.

$$\begin{aligned} \min \sum_{e\in E} z_e & \qquad (\mathcal{P}) \\ y_u + y_v + z_{uv} \geq 1 \ \forall\ uv \in E & \\ \sum_{u\in V} y_u = \nu(G) & \\ y, z \geq 0 & \end{aligned}$$

$$\begin{aligned} \max \sum_{e\in E} \alpha_e - \gamma\nu(G) & \qquad (\mathcal{D}) \\ \alpha(\delta(u)) \leq \gamma \ \forall\ u \in V & \\ 0 \leq \alpha \leq 1 & \end{aligned}$$

By setting z to be the indicator vector of the minimum stabilizer, we can obtain y such that (y, z) is a feasible solution to the primal program. This is because, if z is the indicator vector of a stabilizer in G, then by definition there exists a fractional vertex cover y in $G \setminus \text{Support}(z)$ with size equal to $\nu(G \setminus \text{Support}(z))$. We also know by Theorem 1 that for every minimum stabilizer F, $\nu(G \setminus F) = \nu(G)$.

Thus, the primal program is a relaxation of the stabilizer problem. Consequently, the objective value of any feasible solution to the dual program is a lower bound on the size of a minimum stabilizer. We will provide a dual feasible solution with objective value at least $kd/2$.

Consider the dual solution $(\gamma = d,\ \alpha_e = 1\ \forall\ e \in E)$. Since the graph is d-regular we have that $\alpha(\delta(u)) = d$. Thus, all dual constraints are satisfied and hence, it is a dual feasible solution. The objective value is

$$\sum_{e\in E} \alpha_e - \gamma\nu(G) = \frac{d|V|}{2} - d\left(\frac{|V| - k - |S_u|}{2}\right) = d\left(\frac{k + |S_u|}{2}\right) \geq \frac{kd}{2}.$$

$\square$

5.3 Integrality Gap

In this section, we show an integrality gap example for a natural integer programming formulation of the stabilizer problem.

If we consider the linear program $(\mathcal{P})$ and add integrality constraints on the z variables, we obtain an integer program (IP) and it follows by our result that the integrality gap of the resulting IP is at most 2 for d-regular graphs. Könemann et al. [12] proved a $\Theta(n)$-bound on the integrality gap of the IP for general graphs. However, the resulting IP is *not* a formulation for our minimum stabilizer problem, since the integral optimum solution of the IP could be $\Omega(n)$ away from the size of a minimum stabilizer for arbitrary graphs (not necessarily regular). In order to obtain a formulation for our stabilizer problem, we could introduce additional variables x and impose the existence of a matching in $G \setminus \text{Support}(z)$ of size $\nu(G)$:

$$\begin{aligned}
&\min \sum_{e \in E} z_e \\
&y_u + y_v + z_{uv} \geq 1 \ \forall\ uv \in E, \\
&\sum_{u \in V} y_u = \nu(G), \\
&x(\delta(v)) \leq 1 \ \forall\ v \in V,\ \sum_{e \in E} x_e = \nu(G),\ x(E[S]) \leq \frac{|S|-1}{2} \ \forall\ S \subseteq V,\ |S| \text{ odd}, \\
&x_e + z_e \leq 1 \ \forall\ e \in E, \\
&x, y, z \geq 0,\ x, z \text{ integral.}
\end{aligned}$$

However, we can show a lower bound of $\Omega(n)$ on the integrality gap of the above formulation for factor critical graphs.

Proposition 6 *There exists a graph $G = (V, E)$ such that the integrality gap of the above formulation for G is $\Omega(|V|)$.*

Proof The gap instance $G = (V, E)$ is constructed as follows: Let $V = \{v_i : i \in [2r+1]\} \cup \{a_i, b_i : i \in [4r]\}$. The edges are $E = \{v_i v_j : i, j \in [2r+1]\} \cup \{a_i b_i : i \in [4r]\} \cup \{a_i v_j, b_i v_j : i, \in [4r], j \in [r]\}$.

For notational convenience, let $Q = \{v_i : i \in [2r+1]\}$ and $S = \{v_i : i \in [r]\}$. The instance consists of an odd clique on $2r+1$ vertices in Q. The gap instance and a feasible solution are shown in figures 3 and 4. The number of vertices in the graph is $|V| = 10r + 1$.

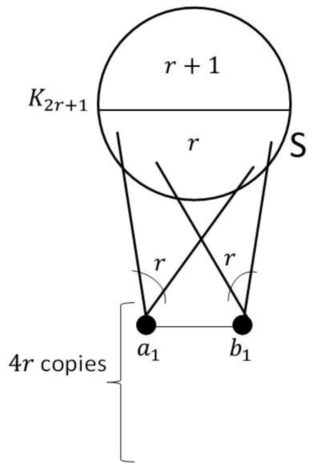


Fig. 3 Integrality gap instance G

Claim The graph G is factor critical.

Proof By Lemma 1, it is sufficient to give an ear construction of the graph using only odd ears. Here is an ear construction of G: first construct K_{2r+1} using odd ears. Repeat for $i = 1, \ldots, 4r$: add the odd ear (ua_i, a_ib_i, b_iu) where $u \in S$. Next add edges a_iv followed by b_iv for all vertices $v \in S$ and all $i \in [4r]$. □

Next we construct a feasible solution (x, y, z) to the linear program for this instance (see Figure 4). Fix a vertex $v \in S$ in the clique and let M be a perfect matching in the clique that exposes v. Set

$$x_e = \begin{cases} 1 & \text{if } e \in M \\ \frac{7r-1}{7r} & \text{if } e = a_ib_i \ i \in [4r] \\ \frac{1}{7r} & \text{if } e = a_iv,\ i \in [4r] \\ 0 & \text{otherwise,} \end{cases}$$

$$y_u = \begin{cases} \frac{1}{2} & \text{if } u \in Q \setminus S \\ \frac{7r+1}{14r} & \text{if } u \in S \\ \frac{7r-1}{14r} & \text{if } u = a_i \text{ or } b_i,\ i \in [4r] \\ 0 & \text{otherwise,} \end{cases}$$

$$z_e = \begin{cases} \frac{1}{7r} & \text{if } e = a_ib_i,\ i \in [4r] \\ 0 & \text{otherwise.} \end{cases}$$

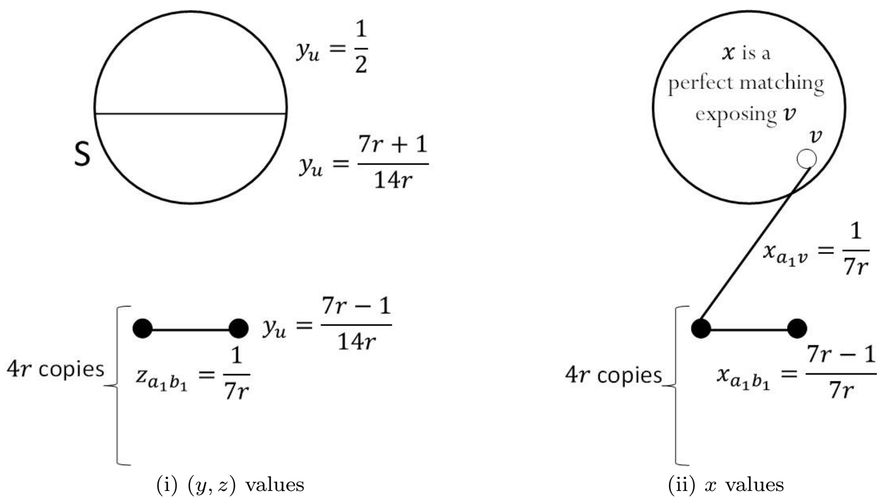


Fig. 4 Feasible solution for the integrality gap instance

Claim The solution (x, y, z) is feasible and has objective value $4/7$.

Proof We show feasibility of the solution by verifying that x satisfies all odd-set constraints. The rest of the constraints can be verified easily. In order to show that x satisfies all matching constraints, we will express it as a convex combination of $4r+1$ integral matchings. Take $M_0 = M \cup \{a_i b_i : i \in [4r]\}$. Now, for each $i = 1, \dots, 4r$, take $M_i = M \cup \{a_j b_j : j \in [4r], j \neq i\} \cup \{a_i v\}$. It is immediately seen that

$$x = \frac{3}{7}\chi_{M_0} + \sum_{i=1}^{4r} \frac{1}{7r}\chi_{M_i}$$

where χ_M denotes the indicator vector of M. The objective value of the linear program is easy to verify. □

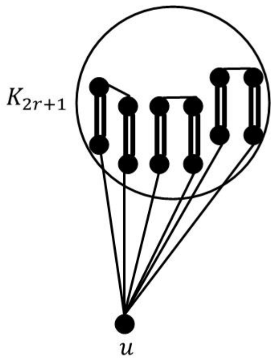
In order to exhibit the integrality gap, it remains to show that the minimum stabilizer in this graph is of size $\Omega(r)$. The following claim completes the proof. □

Claim The minimum stabilizer for G is of size $\Omega(r)$.

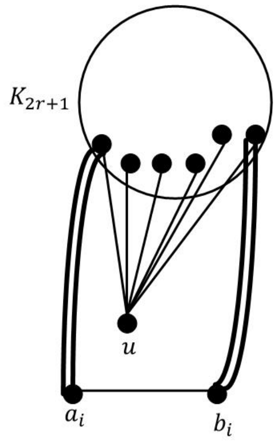
Proof By Theorem 1, we know that there exists a maximum matching M^* in G and a minimum stabilizer F that is disjoint from M^*. Let $H = G \setminus F$. Since the graph G is factor-critical, we know that the matching M^* exposes exactly one vertex u. We have two cases.

1. Suppose this vertex u is in the odd clique. We note that at most r of the vertices in Q have neighbors outside Q and therefore can be matched outside. Hence, at least r of the vertices in Q are matched to vertices in Q. Therefore, we have at least $r/2$ edge-disjoint triangles through u containing exactly one edge from M^*. Since H is stable, H cannot contain any of these triangles. Hence $|F| \geq r/2$.
2. Suppose this vertex $u \in \{a_1, \dots, a_{4r}, b_1, \dots, b_{4r}\}$. Consider the vertices in S. All these are neighbors of u in G. If t among these vertices in S are matched inside the odd clique by M^*, then we can pair up these matching edges and find $t/2$ disjoint 5-cycles through u in G each containing exactly two edges from M^* (See figure (a)). Since H is stable, H cannot contain any of these 5-cycles. Thus, the stabilizer has to remove at least $t/2$ edges.
   (a) If $t \geq r/2$, then the stabilizer has to remove at least $t/2 \geq r/4$ edges.
   (b) If $t < r/2$, then M^* matches $r - t \geq r/2$ vertices to vertices outside outside the clique. If one of the $r-t$ vertices is matched to some a_i, then the vertex b_i is either matched to a vertex inside the clique or $b_i = u$ (see figure (b)). Thus, we can once again identify $(r-t-1)/2 \geq (r-2)/4$ disjoint 5-cycles through u in G each containing exactly two edges from M^*. Since H is stable, H cannot contain any of these 5-cycles. Thus, the stabilizer has to remove at least $(r-2)/4 \geq r/8$ edges.

Thus, we have a lower bound of $\Omega(r) = \Omega(|V|)$ on the size of the minimum stabilizer. □



(a) Disjoint 5-cycles using matching edges *inside* the clique



(b) Disjoint 5-cycles using matching edges *outside* the clique

6 Hardness of the Stabilizer problem

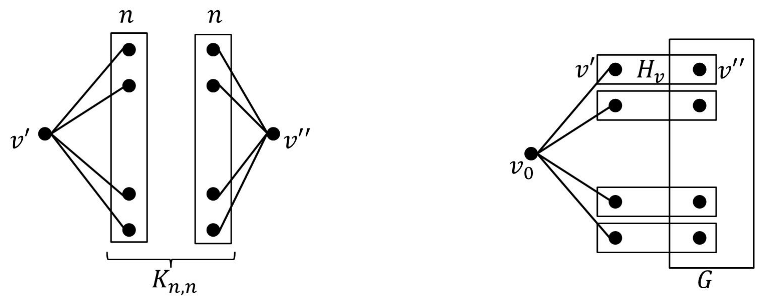
In this section, we show that the stabilizer problem is at least as hard as the vertex cover problem. The construction is very similar to the one used in the proof of Proposition 2, except that we introduce a gadget graph H instead of a two-edge-path in order to enforce that a minimum stabilizer selects edges incident to the super-source.

Proof (of Theorem 5) As in the proof of Proposition 2, we give a reduction from the vertex cover problem. Let $G = (V, E)$ be a vertex cover instance. We may assume that G has no isolated vertices. We construct a new graph G' as in the proof of Proposition 2, but where the edges of the type $v'v''$ are replaced by a gadget graph H. The gadget graph H connects v' and v'', each to one of the parts of a $K_{n,n}$, the complete bipartite graph on n vertices, where $n = |V|$ is the number of vertices in the vertex cover instance. See Figure 5(a) for an illustration of the gadget graph H and Figure 5(b) for the instance G' constructed from the vertex cover instance G. In the rest of the proof, for every vertex $v \in V$, we will refer to H_v as the gadget graph H between the vertices v' and v'' in G'.

We observe that the gadget graph H_v is precisely $K_{n,n} \setminus \{v'v''\}$ and hence bipartite. Further, it is straightforward to verify that the instance G' is factor-critical.

Claim Let W be a vertex cover in G. Then, $F = \{v_0v' : v \in W\}$ is a stabilizer in G' and moreover, $|F| = |W|$.

Proof By Theorem 6, it is sufficient to show that F is an N-stabilizer for any chosen maximum matching N in G'. We choose N to be the maximum matching that leaves v_0 exposed and has a perfect matching in each gadget graph H_v. Suppose for contradiction that there is an N-flower in $G' \setminus F$. Since



(a) Gadget graph H_v (b) Stabilizer instance G'

Fig. 5 Minimum stabilizer instance constructed from vertex cover instance G

v_0 is the only exposed vertex, every N-flower has to contain it. By the choice of N and the construction of the gadget graph H, this N-flower has to contain an edge of the type $u''v''$ for an edge $uv \in E$ and also the edges v_0u' and v_0v'. This is a contradiction to W being a vertex cover. $\square$

By the above claim, we can also conclude that a minimum stabilizer in G' is of size strictly smaller than n.

Claim Given any stabilizer $\bar{F}$ in G', there exists another stabilizer F' in G' that only consists of edges of the type v_0v' for some vertices $v \in V$ and $|F'| \leq |\bar{F}|$.

Proof We may assume that $|\bar{F}| < n$ for otherwise, we can take $F' = \{v_0v : v \in V\}$ to obtain the conclusion. The algorithm used in the proof of Theorem 1 yields a stabilizer F and a maximum matching M in G' with $F \cap M = \emptyset$ and $F \subseteq \bar{F}$. To prove the claim, we will show that we can replace every edge in F of a different type by an edge of the type v_0v' for some vertex $v \in V$. Since G' is factor-critical, every maximum matching in G' can contain at most one edge of the type $u''v''$ for an edge $uv \in E$; otherwise the matching would expose more than one vertex, a contradiction to G' being factor-critical. Thus we are left with only two kinds of maximum matchings in G':

Case 1: M does not contain any edge of the type $u''v''$ for an edge $uv \in E$.
If M leaves v_0 exposed, we observe that every M-flower in G' corresponds to an odd cycle of the form $v_0, v', \ldots, v'', u'', \ldots, u', v_0$ (with some path of odd length through the gadgets H_u, H_v to connect v' to v'' and u'' to u', respectively). This implies that an edge $u''v'' \in F$ can be replaced by v_0u' without violating the stabilizing property, since every flower containing $u''v''$ must also contain v_0u'. Similarly, an edge in F that belongs to a gadget H_v for some $v \in V$ can also be replaced by v_0v', since every flower containing this edge must also contain v_0v'.

If $v_0u' \in M$, then the exposed vertex is either u'' or adjacent to u' in H_u. Then there exists an even M-alternating path P from the exposed vertex to v_0 of length two or four. We change M along P to a new maximum matching

$M' := M \Delta P$. Now, if P contains an edge $f \in F$, we can exchange f with v_0u'. The resulting set $F' := (F \setminus \{f\}) \cup \{v_0u'\}$ is an M'-stabilizer. To see this, observe first that every M'-flower in G' is an odd cycle of the form $v_0, p', \ldots, p'', w'', \ldots, w', v_0$ (with some path of odd length through the gadgets H_p, H_w to connect p' to p'' and w'' to w', respectively). Suppose there was an M'-flower Z in $G' \setminus F'$. Z cannot contain u' or u'' since $v_0u' \in F'$. Hence Z corresponds to the blossom of at least n different M-flowers that are disjoint on the edges neither in M nor in Z. This contradicts either the fact that $G' \setminus F$ is stable or $|F| < n$.

Case 2: M contains an edge $u''v''$.
Because M is a maximum matching, M has to contain by construction either v_0u' or v_0v'. Suppose w.l.o.g. $v_0u' \in M$. We have two possibilities:

Case 2(a): The exposed vertex is v'. We observe that if the edge v_0v' is not contained in the stabilizer F, then the number of M-flowers with disjoint non-M-edges is at least n due to the n edge-disjoint paths through H_u and H_v. Hence, by characterization (iii) of Theorem 6, F is of size at least n, a contradiction. We further observe that for any $p, w \notin \{u, v\}$, the edge $p''w''$ in F can be replaced by v_0p' as in previous case 1, since they only belong to M-blossoms with base v_0 (reached via an even M-alternating path from v' through v'', u'' and u'). A similar argument applies to the edges of F within a gadget H_w, since every flower has to contain the edge v_0w' as well. Finally, suppose F contains edges e from the gadgets H_v or H_u. Then, either all these edges can be removed from F still ensuring the stabilizer property or $|F| \geq n$. This is because, if removing all these edges from F leads to M-flowers in the resulting graph, then there are in fact more than n M-flowers with disjoint non-M-edges (due to the n edge-disjoint paths from v' to v'' through the gadget H_v).

Case 2(b): The exposed vertex is a vertex t adjacent to v'' in the gadget H_v. As in the previous case, a stabilizer cannot contain edges from the gadgets H_v or H_u, since otherwise the stabilizer would have size at least n. We further observe that for any $p, w \notin \{u, v\}$, the edge $p''w''$ in F can be replaced by v_0p' as before, since they only belong to M-blossoms with base v_0 (reached via an even M-alternating path from t through v'', u'' and u').

This proves our claim. □

We set $W := \{v \colon v_0v' \in F'\}$ for the stabilizer F' with the property mentioned in the claim. We now claim that W is a vertex cover in G. For contradiction, suppose that an edge uv is not covered by W. This implies that neither v_0v' nor v_0u' is in F'. Then there exists a cycle $v_0, v', \ldots, v'', u'', \ldots, u', v_0$ (with some path of length three through the gadgets H_u, H_v) in $G' \setminus F'$. We observe that the matching N defined in the beginning of the proof is a maximum matching in $G' \setminus F'$ and thus this cycle forms an N-flower in $G' \setminus F'$ contradicting that F' is a stabilizer in G'.

Hence, we have shown that any vertex cover in G induces a stabilizer in G' of the same cardinality, and moreover, any stabilizer F in G' induces a vertex cover in G of size at most $|F|$. Thus, we have an approximation preserving reduction from the vertex cover problem. □

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