Introduction to Quantum Information Processing

QIC 710 / CS 768 / PH 767 / CO 681 / AM 871

Lecture 10 (2017)

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Distinguishing mixed states

Distinguishing mixed states (1)

Whats the best distinguishing strategy for distinguishing between these two mixed states?

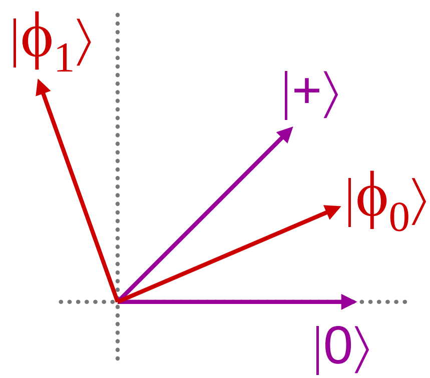
$\begin{cases} |0\rangle & \text{with prob. } ½ \\ |0\rangle + |1\rangle & \text{with prob. } ½ \end{cases}$

$\begin{cases} |0\rangle & \text{with prob. } ½ \\ |1\rangle & \text{with prob. } ½ \end{cases}$

$$\rho_1 = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}$$

$$\rho_2 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

ρ_1 also arises from this orthogonal mixture:



… as does ρ_2 from:

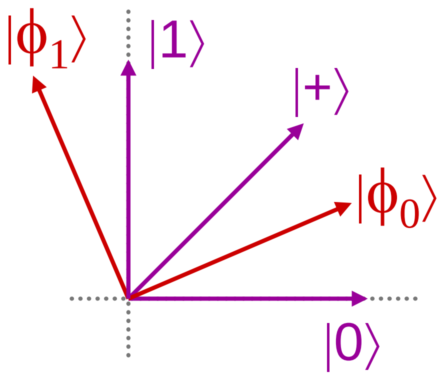
$$\begin{cases} |\phi_0\rangle = \frac{|0\rangle + |+\rangle}{\sqrt{2}} & \text{with prob. } \cos^2(\pi/8) \\ |\phi_1\rangle = \frac{|1\rangle + |-\rangle}{\sqrt{2}} & \text{with prob. } \sin^2(\pi/8) \end{cases}$$

$$\begin{cases} |\phi_0\rangle & \text{with prob. } ½ \\ |\phi_1\rangle & \text{with prob. } ½ \end{cases}$$

Distinguishing mixed states (2)

We managed to find an orthonormal basis $|\phi_0\rangle, |\phi_1\rangle$ in which both density matrices are diagonal:

$$\rho_1' = \begin{pmatrix} \cos^2(\pi/8) & 0 \\ 0 & \sin^2(\pi/8) \end{pmatrix} \qquad \rho_2' = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$



Rotating $|\phi_0\rangle, |\phi_1\rangle$ to $|0\rangle$, $|1\rangle$ the scenario can now be examined using classical probability theory:

Distinguish between two ***classical*** coins, whose probabilities of "heads" are $\cos^2(\pi/8) \simeq .853$ and ½ respectively

Question: what do we do if we aren't so lucky to get two density matrices that are simultaneously diagonalizable?

Quantum Operations (quantum channels)

Quantum operations (1)

Also known as:
"quantum channels"
"completely positive trace preserving (CPTP) maps",
"general quantum operations"

Let $A_1, A_2, \ldots, A_m$ be matrices satisfying $\sum_{j=1}^{m} A_j^\dagger A_j = I$

Then the mapping $\rho \mapsto \sum_{j=1}^{m} A_j \rho A_j^\dagger$ is a quantum operation.

Note: $A_1, A_2, \ldots, A_m$ do not have to be square matrices.

Example 1 (unitary): applying U to ρ yields $U \rho U^\dagger$.

Quantum operations (2)

Example 2 (decoherence): let $A_0 = |0\rangle\langle 0|$ and $A_1 = |1\rangle\langle 1|$

This quantum op maps ρ to $|0\rangle\langle 0|\rho|0\rangle\langle 0| \;+\; |1\rangle\langle 1|\rho|1\rangle\langle 1|$

For $|\psi\rangle = \alpha|0\rangle \;+ \beta|1\rangle$

$$\begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} \mapsto \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}$$

Corresponds to measuring ρ "without looking at the outcome"

After looking at the outcome, ρ becomes $\begin{cases} |0\rangle\langle 0| & \text{with prob. } |\alpha|^2 \\ |1\rangle\langle 1| & \text{with prob. } |\beta|^2 \end{cases}$

Quantum operations (3)

Example 3

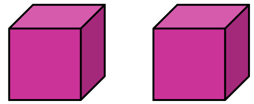
Let $A_0 = I \otimes \langle 0| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $A_1 = I \otimes \langle 1| = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

- Any state of the form $\rho \otimes \sigma$ (product state) becomes ρ
- Entangled state $\frac{|00\rangle + |11\rangle}{\sqrt{2}} \frac{\langle 00| + \langle 11|}{\sqrt{2}}$ becomes $\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$

  It's the same density matrix as for $\big((½, |0\rangle), (½, |1\rangle)\big)$
- Corresponds to "discarding the second register"

The operation is called the ***partial trace*** $\mathrm{Tr}_2\, \rho$

More about the partial trace

Two quantum registers  in states σ and μ (resp.) are ***independent*** when the combined system is in state $\rho = \sigma \otimes \mu$

If the 2nd register is discarded, state of the 1st register remains σ

In general, the state of a two-register system may not be of the form $\sigma \otimes \mu$ (it may contain ***entanglement*** or ***correlations***)

The ***partial trace*** Tr_2 gives the effective state of the first register

For d-dimensional registers, Tr_2 is defined with respect to the operators $A_k = I \otimes \langle\phi_k|$, where $|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_d\rangle$ can be any orthonormal basis

The **partial trace** $\mathrm{Tr}_2\, \rho$, can also be characterized as the unique linear operator satisfying the identity $\mathrm{Tr}_2(\sigma \otimes \mu) = \sigma$.

Partial trace continued

For 2-qubit systems, the partial trace is explicitly

$$\mathrm{Tr}_2\begin{bmatrix} \rho_{00,00} & \rho_{00,01} & \rho_{00,10} & \rho_{00,11} \\ \rho_{01,00} & \rho_{01,01} & \rho_{01,10} & \rho_{01,11} \\ \rho_{10,00} & \rho_{10,01} & \rho_{10,10} & \rho_{10,11} \\ \rho_{11,00} & \rho_{11,01} & \rho_{11,10} & \rho_{11,11} \end{bmatrix} = \begin{bmatrix} \rho_{00,00}+\rho_{01,01} & \rho_{00,10}+\rho_{01,11} \\ \rho_{10,00}+\rho_{11,01} & \rho_{10,10}+\rho_{11,11} \end{bmatrix}$$

and

$$\mathrm{Tr}_1\begin{bmatrix} \rho_{00,00} & \rho_{00,01} & \rho_{00,10} & \rho_{00,11} \\ \rho_{01,00} & \rho_{01,01} & \rho_{01,10} & \rho_{01,11} \\ \rho_{10,00} & \rho_{10,01} & \rho_{10,10} & \rho_{10,11} \\ \rho_{11,00} & \rho_{11,01} & \rho_{11,10} & \rho_{11,11} \end{bmatrix} = \begin{bmatrix} \rho_{00,00}+\rho_{10,10} & \rho_{00,01}+\rho_{10,11} \\ \rho_{01,00}+\rho_{11,10} & \rho_{01,01}+\rho_{11,11} \end{bmatrix}$$

Quantum operations (4)

Example 4 (adding an extra qubit):

Just one operator $A_0 = I \otimes |0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

States of the form ρ become $\rho \otimes |0\rangle\langle 0|$

More generally, to add a register in state $|\phi\rangle$, use the operator $A_0 = I \otimes |\phi\rangle$.

Quantum operations (5)

Let $\mathcal{A}(\rho) = \sum_{j=1}^{m} A_j \rho A_j^\dagger$ be a quantum operation.

Then $(\mathcal{A} \otimes \mathrm{id})(\rho) = \sum_{j=1}^{m} (A_j \otimes I) \rho (A_j^\dagger \otimes I)$

is a quantum operation.

Why? Physical interpretation?

Properties of quantum operations

Let $\mathcal{A}(\rho) = \sum_{j=1}^{m} A_j \rho A_j^\dagger$ be a quantum operation.

What properties does it satisfy?

Linear: $\mathcal{A}(X+Y) = \mathcal{A}(X) + \mathcal{A}(Y)$

Trace-preserving: $\mathrm{Tr}\,\mathcal{A}(X) = \mathrm{Tr}\,X$

Hermitian preserving: $X^\dagger = X \Rightarrow \mathcal{A}(X)^\dagger = \mathcal{A}(X)$

Positivity preserving: X p.s.d $\Rightarrow$ $\mathcal{A}(X)$ p.s.d.

Completely positive: Y p.s.d $\Rightarrow$ $(\mathcal{A} \otimes \mathrm{id})(Y)$ p.s.d.

This is a full characterization: Quantum operations are **CPTP** (Completely Positive and Trace-Preserving) linear maps

Quantum channels

Quantum operations also model noisy quantum processes, where they are often referred to as **quantum channels.**

Qubit flip: $\rho \mapsto (1-p)\rho + pX\rho X$

Phase flip: $\rho \mapsto (1-p)\rho + pZ\rho Z$

Completely depolarizing: $\rho \mapsto \frac{1}{4}(\rho + X\rho X + Y\rho Y + \ Z\rho Z) = \frac{1}{2}I$

Depolarizing: $\rho \mapsto (1-p)\rho + \frac{p}{2}I$

(exercise: show that this is a quantum operation)