Introduction to Quantum Information Processing

QIC 710 / CS 768 / PH 767 / CO 681 / AM 871

Lecture 12 (2017)

Jon Yard

QNC 3126

jyard@uwaterloo.ca

http://math.uwaterloo.ca/~jyard/qic710

Thursday class in **QNC 1501**

Sept. 19	QNC 1501	Sept. 21	MC 4058
Sept. 26	QNC 1501	Sept. 28	QNC 0101
Oct. 3	OPT 309	Oct. 5	QNC 1501
Oct. 10	QNC 0101	Oct. 12	QNC 0101
Oct. 17	QNC 0101	Oct. 19	QNC 0101
Oct. 24	QNC 0101	Oct. 26	**QNC 1501**
Oct. 31	QNC 0101	Nov. 2	QNC 0101
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Nov. 14	QNC 0101	Nov. 16	QNC 0101
Nov. 21	QNC 0101	Nov. 23	QNC 0101
Nov. 28	QNC 0101	Nov. 30	QNC 0101

Preliminary remarks about quantum communication

Quantum information can apparently be used to substantially reduce ***computation*** costs for a number of interesting problems

How does quantum information affect the ***communication costs*** of information processing tasks?

We explore this issue ...

Entanglement and signaling

Recall that Entangled states, such as $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$,

can be used to perform some intriguing feats, such as ***teleportation*** and ***superdense coding***

—but they ***cannot*** be used to "signal instantaneously"

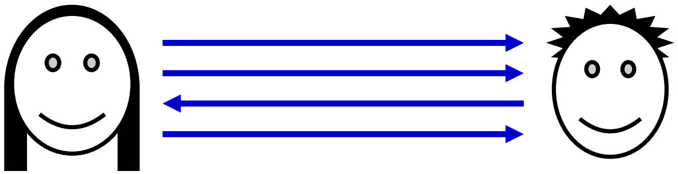
Any operation performed on one system has no affect on the state of the other system (its reduced density matrix)

Basic communication scenario

Goal: convey n bits from Alice to Bob

Basic communication scenario

Bit communication:

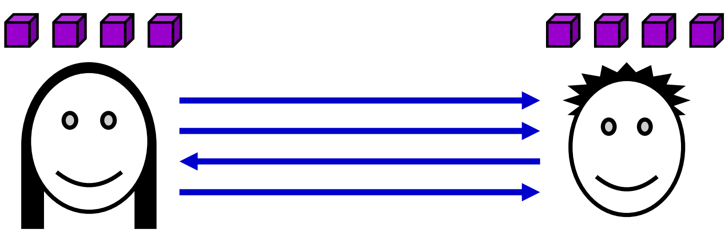


Cost: n

Qubit communication:

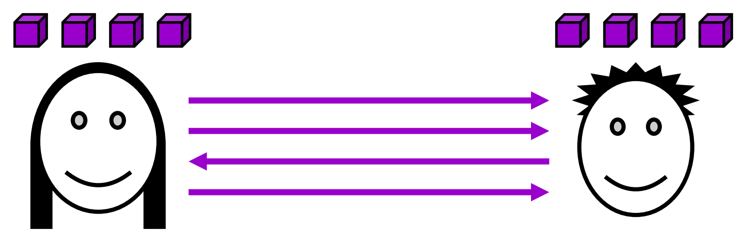
Cost: n [Holevo’s Theorem, 1973]

Bit communication & prior entanglement:



Cost: n (can be deduced)

Qubit communication & prior entanglement:



Cost: $\frac{n}{2}$ superdense coding

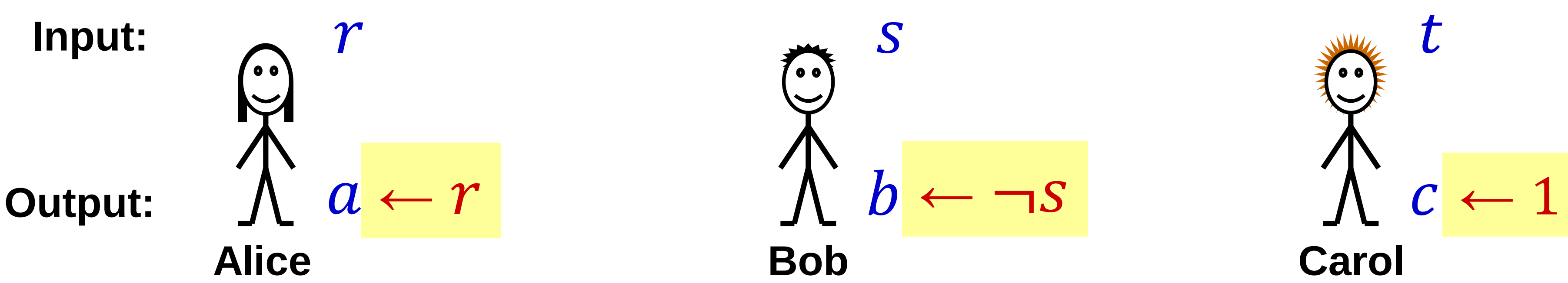
[Bennett & Wiesner, 1992]

The GHZ “paradox”

(Greenberger-Horne-Zeilinger)

GHZ scenario

[Greenberger, Horne, Zeilinger, 1980]



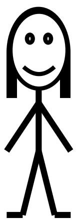
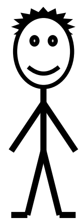
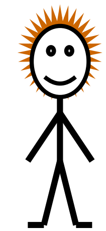
Rules of the game:

1. It is promised that $r + s + t = 0$
2. No communication after inputs received
3. They ***win*** if $r \vee s \vee t = a + b + c$

rst	$a + b + c$	abc
000	0	011
011	1	001
101	1	111
110	1	101

No perfect strategy for GHZ

Input: r s t

Output: a b c

rst	$a+b+c$
000	0
011	1
101	1
110	1

General deterministic strategy:

$a_0, a_1, b_0, b_1, c_0, c_1$

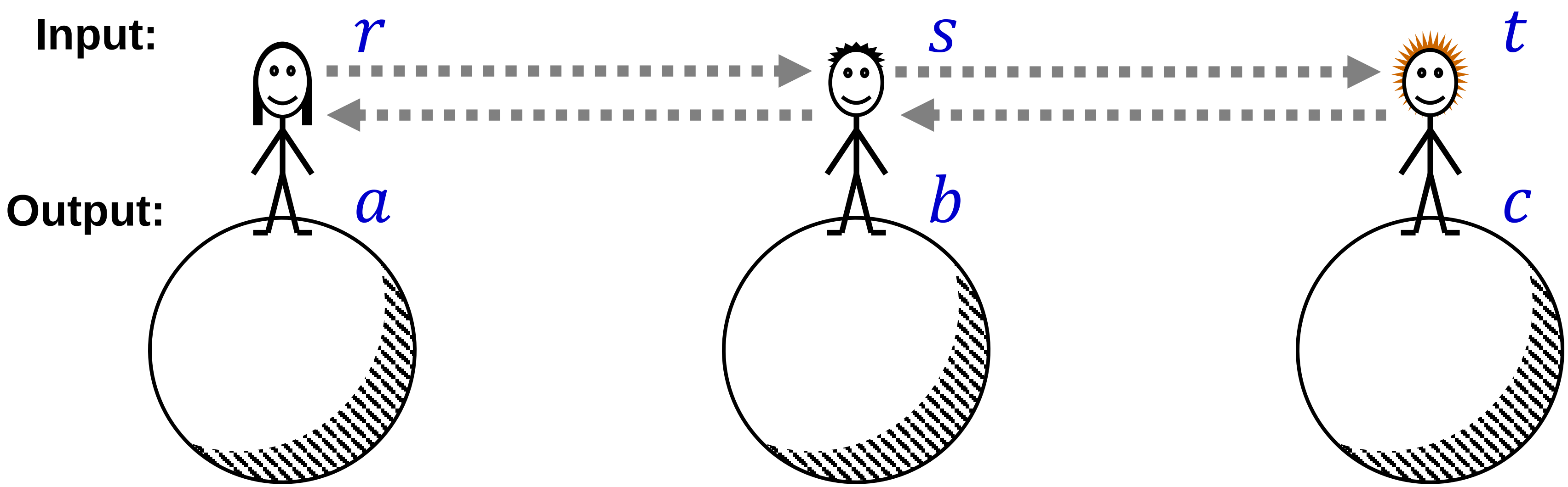
Winning conditions:

$$\begin{cases} a_0 + b_0 + c_0 = 0 \\ a_0 + b_1 + c_1 = 1 \\ a_1 + b_0 + c_1 = 1 \\ a_1 + b_1 + c_0 = 1 \end{cases}$$

Has no solution (why?), thus no perfect strategy exists.

$$2(a_0 + a_1 + b_0 + b_1 + c_0 + c_1) = 0 \neq 1 = 1 + 1 + 1$$

GHZ: preventing communication

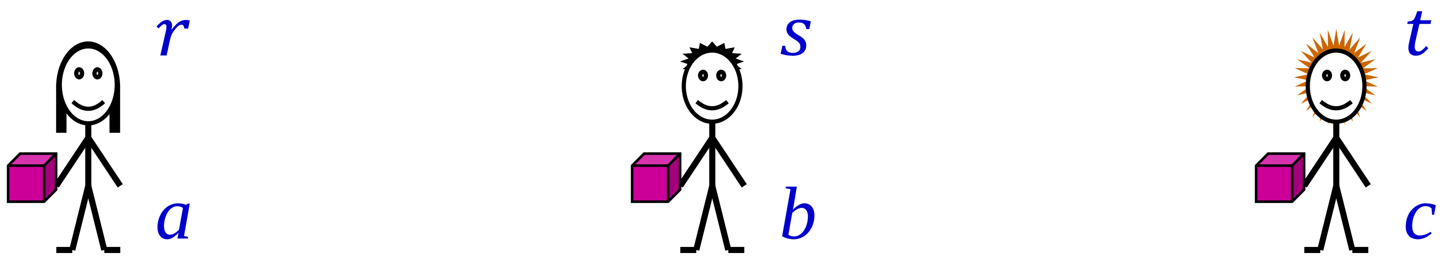


Input and output events can be ***space-like*** separated:
so signals at the speed of light are not fast enough for cheating

What if Alice, Bob, and Carol ***still*** keep on winning?

"GHZ Paradox" explained

Prior entanglement: $|\psi\rangle = |000\rangle - |011\rangle - |101\rangle - |110\rangle$



<u>Alice's strategy:</u>

1. if $r = 1$ then apply H to qubit (else I)
2. measure qubit and set a to result

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

<u>Bob's & Carol's strategies:</u> similar

Case 1 ($rst = 000$)**:** state is measured directly …

Case 2 ($rst = 011$)**:** new state $|001\rangle + |010\rangle - |100\rangle + |111\rangle$

Cases 3 & 4 ($rst = 101$ & 110)**:** similar by symmetry

GHZ: conclusions

- For the GHZ game, any ***classical*** team succeeds with probability at most ¾
- Allowing the players to communicate would enable them to succeed with probability 1
- Entanglement cannot be used to communicate
- Nevertheless, allowing the players to have entanglement enables them to succeed with probability 1 (but not by using entanglement to communicate)
- Thus, entanglement is a useful resource for the task of ***winning the GHZ game***

The Bell inequality and its violation

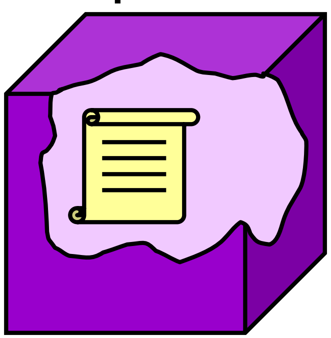
– Physicist's perspective

Bell's Inequality and its violation

Part I: physicist's view:

Can a quantum state have ***pre-determined*** outcomes for each possible measurement that can be applied to it?

qubit:



where the "manuscript" is something like this:

if {|0⟩,|1⟩} measurement then output **0**

if {|+⟩,|−⟩} measurement then output **1**

if **...** (etc)

called ***hidden variables***

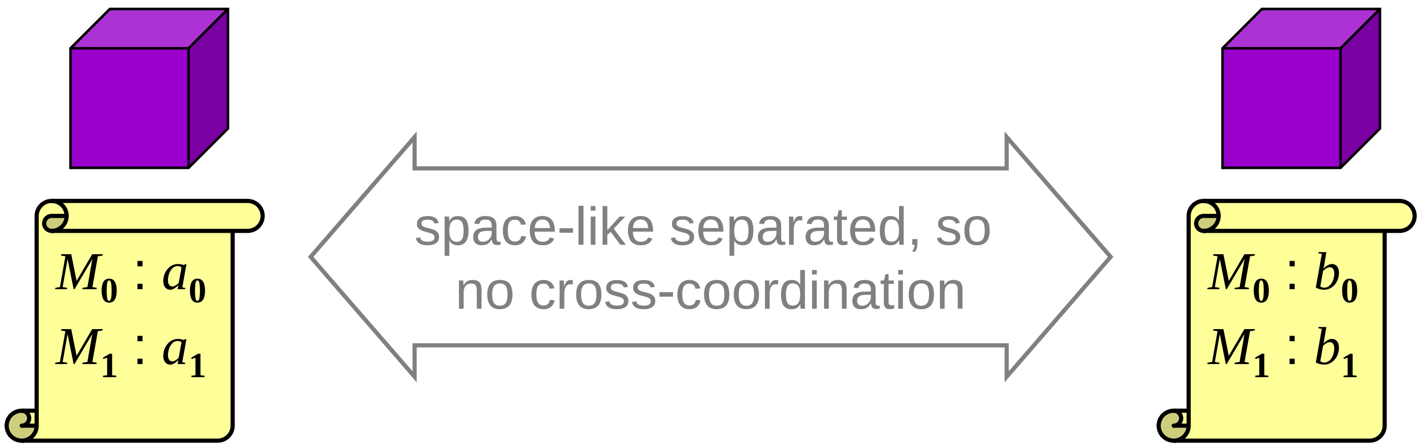
table could be implicitly given by some formula

[Bell, 1964]

[Clauser, Horne, Shimony, Holt, 1969]

Bell Inequality

Imagine a two-qubit system, where one of two measurements, called M_0 and M_1, will be applied to each qubit:



Define:

$$A_0 = (-1)^{a_0}$$
$$A_1 = (-1)^{a_1}$$
$$B_0 = (-1)^{b_0}$$
$$B_1 = (-1)^{b_1}$$

Claim: $A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1 \leq 2$

Proof: $A_0(B_0 + B_1) + A_1(B_0 - B_1) \leq 2$

one is ± 2 and the other is 0

Bell Inequality

$A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2$ is called a ***Bell Inequality****

Question: could one, in principle, design an experiment to check if this Bell Inequality holds for a particular system?

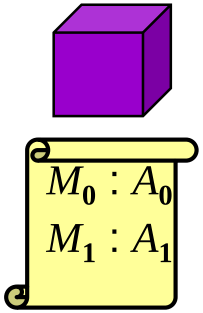
Answer 1: ***no, not directly***, because A_0, A_1, B_0, B_1 cannot all be measured (only ***one*** $A_s B_t$ term can be measured)

Answer 2: ***yes, indirectly***, by making many runs of this experiment: pick a random $st \in \{00, 01, 10, 11\}$ and then measure with M_s and M_t to get the value of $A_s B_t$

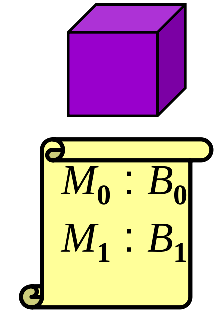
The ***average*** of $A_0 B_0$, $A_0 B_1$, $A_1 B_0$, $-A_1 B_1$ should be ≤ ½

* also called CHSH Inequality

Recap of Bell Inequality



Assume local hidden variables framework is correct



Consider the following experiment:

1. pick a random $st \in \{00, 01, 10, 11\}$ (uniform distribution)
2. perform M_s measurement on 1st qubit (outcome $A_s \in \{+1, -1\}$)
3. perform M_t measurement on 2nd qubit (outcome $B_t \in \{+1, -1\}$)
4. output the value of $(-1)^{s \cdot t} A_s B_t$

In any run of this experiment, the output is an element of $\{+1, -1\}$ (according to probabilities that depend on what A_0, A_1, B_0, B_1 are)

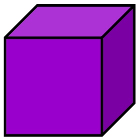
How large can the ***expected*** value of the outcome be?

$$\tfrac{1}{4}(A_0 B_0) + \tfrac{1}{4}(A_0 B_1) + \tfrac{1}{4}(A_1 B_0) + \tfrac{1}{4}(-A_1 B_1)$$

$$= \tfrac{1}{4}(A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1) \leq \tfrac{1}{4} 2 = \tfrac{1}{2}$$

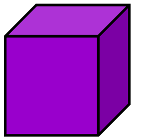
Violating the Bell Inequality

Assume the quantum mechanical framework is correct



Two-qubit system in state

$|\phi\rangle = |00\rangle - |11\rangle$



$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

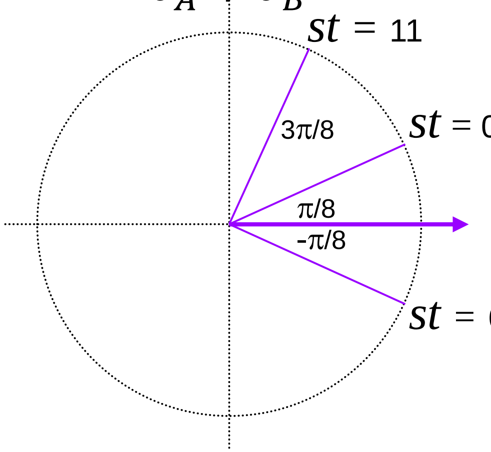
It can be shown that applying rotations $R_{\theta_A} \otimes R_{\theta_B}$ yields:

$$\cos(\theta_A + \theta_B)\underbrace{(|00\rangle - |11\rangle)}_{AB=+1} + \sin(\theta_A + \theta_B)\underbrace{(|01\rangle + |10\rangle)}_{AB=-1}$$

Define

M_0: rotate by $-\pi/16$ then measure

M_1: rotate by $+3\pi/16$ then measure



Then $A_0 B_0$, $A_0 B_1$, $A_1 B_0$, $-A_1 B_1$ all have expected value

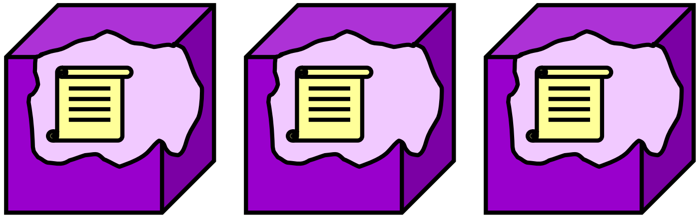
$$\cos^2\left(\frac{\pi}{8}\right) - \sin^2\left(\frac{\pi}{8}\right) = \left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right) - \left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right) = \frac{\sqrt{2}}{2},$$

which ***contradicts*** the upper bound of $1/2$.

Therefore, QM framework implies LHV framework is false

Bell Inequality violation: summary

Assuming that quantum systems are governed by ***local hidden variables*** leads to the Bell inequality



$$A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2$$

But this is ***violated*** in the case of Bell states (by a factor of $\sqrt{2}$)

Therefore, no such hidden variables exist

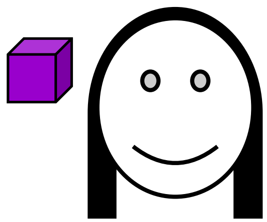
This is, in principle, experimentally verifiable, and experiments along these lines have actually been conducted

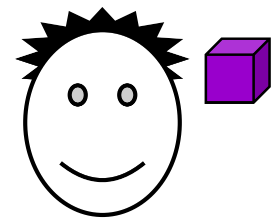
The Bell inequality and its violation – Computer Scientist’s perspective

Bell's Inequality and its violation

Part II: computer scientist's view:

input: s t





output: a b

Rules: 1. No communication after inputs received
2. They ***win*** if $a + b = s \wedge t$

st	$a + b$
00	0
01	0
10	0
11	1

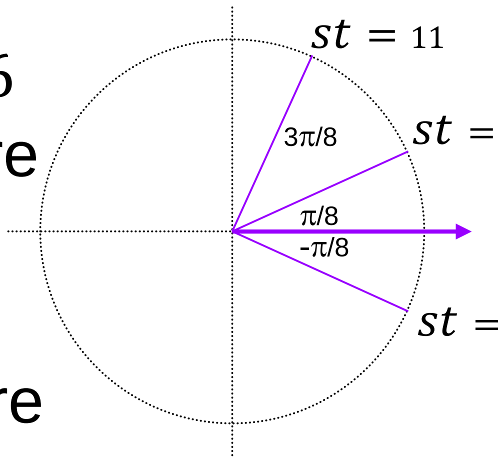
With classical resources, $\Pr[a + b = s \wedge t] \leq 0.75$

But, with prior entanglement state $|00\rangle - |11\rangle$,

$$\Pr[a + b = s \wedge t] = \cos^2(\pi/8) = \frac{1}{2} + \frac{\sqrt{2}}{4} = 0.853 \ldots$$

The quantum strategy

- Alice and Bob start with entanglement $|\phi\rangle = |00\rangle - |11\rangle$
- **Alice:** if s = 0 then rotate by $\theta_A = -\pi/16$ else rotate by $\theta_A = +3\pi/16$ and measure
- **Bob:** if t = 0 then rotate by $\theta_B = -\pi/16$ else rotate by $\theta_B = +3\pi/16$ and measure



$$\cos(\theta_A + \theta_B)(|00\rangle - |11\rangle) + \sin(\theta_A + \theta_B)(|01\rangle + |10\rangle)$$

$$a + b = 0 \qquad\qquad a + b = 1$$

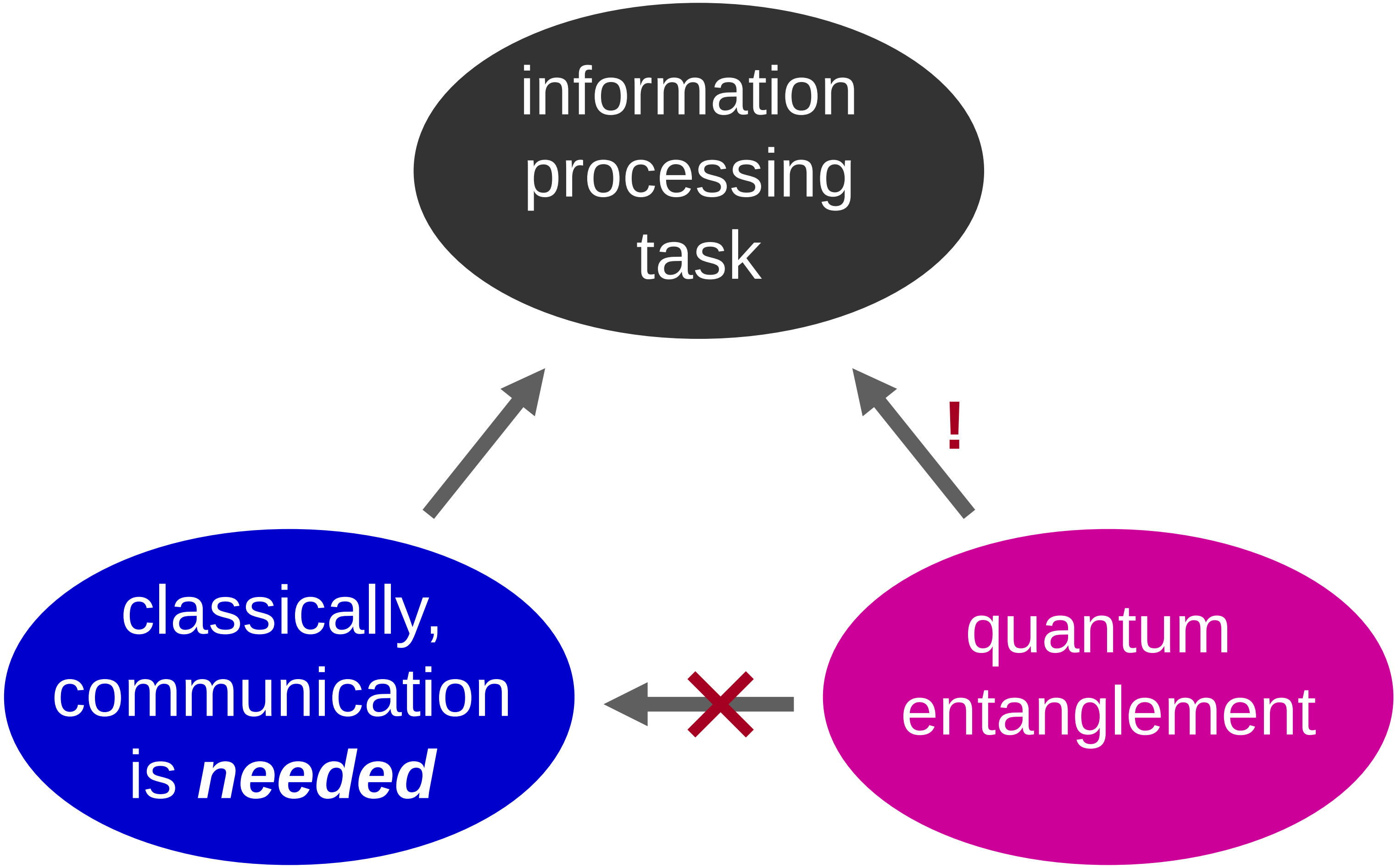
Success probability:

$\Pr[s \wedge t = a + b]$

$$= \Pr[s \wedge t = 0] \cos^2(\pm\pi/8) + \Pr[s \wedge t = 1] \sin^2(3\pi/8)$$

$$= \frac{3}{4}\left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right) + \frac{1}{4}\left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right) = \frac{1}{2} + \frac{\sqrt{2}}{4} = 0.853\ldots$$

Nonlocality in operational terms



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