Introduction to Quantum Information Processing

QIC 710 / CS 768 / PH 767 / CO 681 / AM 871

Lecture 13 (2017)

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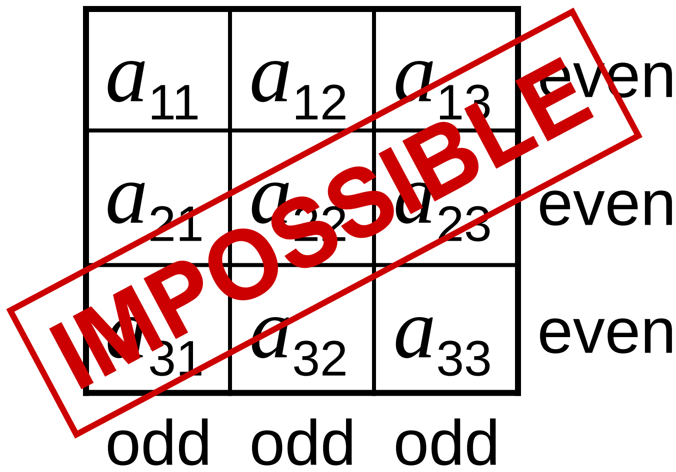
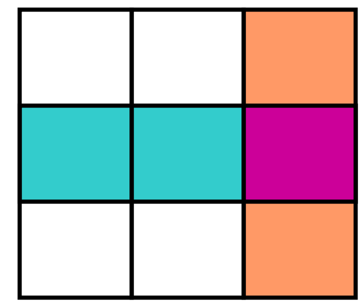
The magic square game

Magic square game

Problem: fill in the matrix with bits such that each row has even parity and each column has odd parity





Game: ask Alice to fill in one row and Bob to fill in one column

They ***win*** iff parities are correct and bits agree at intersection

Success probabilities: $8/9$ classical and 1 quantum

[Mermin, 1990] (For more details, Google “Pseudo-telepathy)

Distance measures for quantum states

Distance measures

Some simple (and often useful) measures:

- **Euclidean distance:** $\| \, |\phi\rangle - |\psi\rangle \, \|_2$
- **Fidelity:** $|\langle\phi|\psi\rangle|$

Small Euclidean distance implies "closeness" but large Euclidean distance need not imply "far away"

(for example, $|\psi\rangle$ vs $-|\psi\rangle$)

Not so clear how to extend these for mixed states …

… though fidelity does generalize, to $F(\rho,\sigma) = \mathrm{Tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}$

Trace norm – preliminaries (1)

For a normal matrix M and a function $f: \mathbb{C} \to \mathbb{C}$, we define the matrix $f(M)$ as follows:

Write $M = U^\dagger D U$, where D is diagonal (we can do this because normal matrices are unitarily diagonalizable).

Now, define $f(M) = U^\dagger f(D) U$, where

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_d \end{pmatrix} \qquad f(D) = \begin{pmatrix} f(\lambda_1) & 0 & \cdots & 0 \\ 0 & f(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(\lambda_d) \end{pmatrix}$$

Trace norm – preliminaries (2)

For a normal matrix $M = U^\dagger D U$, define $|M|$ in terms of replacing D with

$$|D| = \begin{pmatrix} |\lambda_1| & 0 & \cdots & 0 \\ 0 & |\lambda_2| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |\lambda_d| \end{pmatrix}$$

More generally, can define $|M| = \sqrt{M^\dagger M}$ for ***all*** matrices M (not necessarily normal ones), since $M^\dagger M$ is positive semidefinite.

Trace norm/distance – definition

The ***trace norm*** of M is $\|M\|_{\mathrm{tr}} = \|M\|_1 = \mathrm{Tr}|M| = \mathrm{Tr}\sqrt{M^\dagger M}$.

Intuitively, it's the 1-norm of the eigenvalues (or, in the non-normal case, the *singular values*) of M

The ***trace distance*** between ρ and σ is defined as $\|\rho - \sigma\|_1$.

Why is this a meaningful distance measure between quantum states?

Theorem: for any two quantum states ρ and σ, the ***optimal*** measurement procedure for distinguishing between them succeeds with probability $\frac{1}{2} + \frac{1}{4}\|\rho - \sigma\|_1$

Distinguishing between two arbitrary quantum states

Holevo-Helstrom Theorem (1)

Theorem: for any two quantum states ρ and σ, the optimal measurement procedure for distinguishing between them succeeds with probability $\frac{1}{2}+\frac{1}{4}\|\rho-\sigma\|_1$ (equal prior probs.)

Proof* (the attainability part):

Since $\rho-\sigma$ is Hermitian, its eigenvalues are real

Let Π_+ be the projector onto the positive eigenspaces

Let Π_- be the projector onto the non-positive eigenspaces

Take the POVM measurement specified by Π_+ and Π_- with the associations $+\equiv\rho$ and $-\equiv\sigma$

* The other direction of the theorem (optimality) is omitted here

Holevo-Helstrom Theorem (2)

Claim: this succeeds with probability $\frac{1}{2} + \frac{1}{4}\|\rho - \sigma\|_1$

Proof of Claim:

A key observation is $\mathrm{Tr}(\Pi_+ - \Pi_-)(\rho - \sigma) = \|\rho - \sigma\|_1$

The success probability is $p_s = \frac{1}{2}\mathrm{Tr}(\Pi_+\rho) + \frac{1}{2}\mathrm{Tr}(\Pi_-\sigma)$.

& the failure probability is $p_f = \frac{1}{2}\mathrm{Tr}(\Pi_+\sigma) + \frac{1}{2}\mathrm{Tr}(\Pi_-\rho)$.

Therefore, $p_s - p_f = \frac{1}{2}\mathrm{Tr}(\Pi_+ - \Pi_-)(\rho - \sigma) = \frac{1}{2}\|\rho - \sigma\|_1$.

From this, the result follows ∎

Purifications & Ulhmann's Theorem

Any density matrix ρ can be obtained by tracing out part of some larger ***pure*** state:

$$\rho = \sum_{j=1}^{d} \lambda_j |\phi_j\rangle\langle\phi_j| = \mathrm{Tr}_2 \left(\sum_{j=1}^{d} \sqrt{\lambda_j}\, |\phi_j\rangle|j\rangle \right) \left(\sum_{j=1}^{d} \sqrt{\lambda_j}\, \langle\phi_j|\langle j| \right)$$

a purification of ρ

Ulhmann's Theorem*: The ***fidelity*** between ρ and σ is the maximum of $|\langle\phi|\psi\rangle|$ taken over all purifications $|\psi\rangle$ and $|\phi\rangle$

* See [Nielsen & Chuang, pp. 410-411] for a proof of this using the singular-value decomposition.

Recall our previous definition of fidelity as

$$F(\rho,\sigma) = \mathrm{Tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}} = \left\|\rho^{1/2}\sigma^{1/2}\right\|_1.$$

Relationships between fidelity and trace distance

$$1 - F(\rho, \sigma) \leq \frac{1}{2} \|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho, \sigma)^2}$$

See [Nielsen & Chuang, pp. 415-416] for more details