Introduction to Quantum Information Processing

QIC 710 / CS 768 / PH 767 / CO 681 / AM 871

Lecture 2 (2017)

Jon Yard
QNC 3126
jyard@uwaterloo.ca

Superdense coding

How much classical information in n qubits?

2^n–1 complex numbers apparently needed to describe an arbitrary n-qubit pure quantum state:

$$\alpha_{000}|000\rangle + \alpha_{001}|001\rangle + \alpha_{010}|010\rangle + \cdots + \alpha_{111}|111\rangle$$

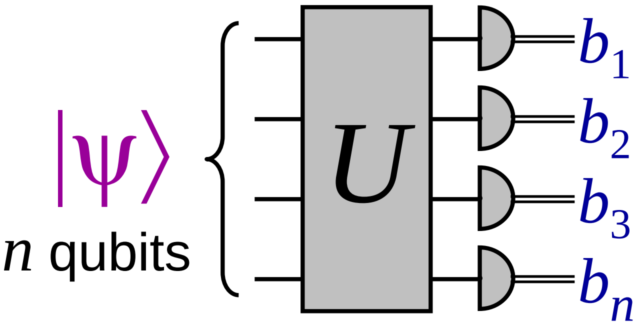
Does this mean that an exponential amount of classical information is somehow “stored” in n qubits?

Not in an operational sense ...

For example, Holevo’s Theorem (from 1973) implies: one cannot convey more than n classical bits of information in n qubits

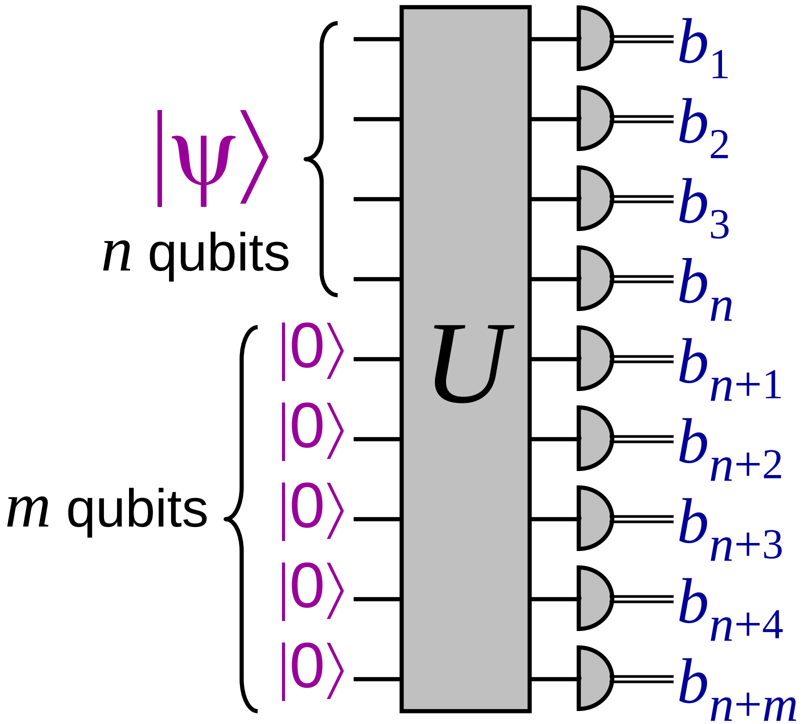
Holevo's Theorem

Easy case:



$b_1, b_2, \ldots, b_n$ certainly cannot convey more than n bits!

Hard case (the general case)**:**

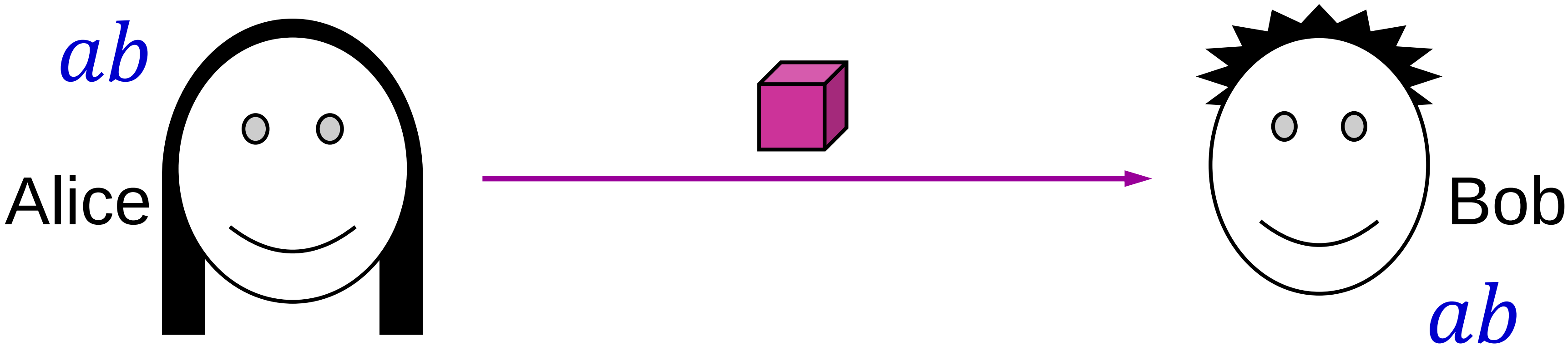


Proof is beyond the scope of this course

(but see CS 766 / QIC 820)

Superdense coding (prelude)

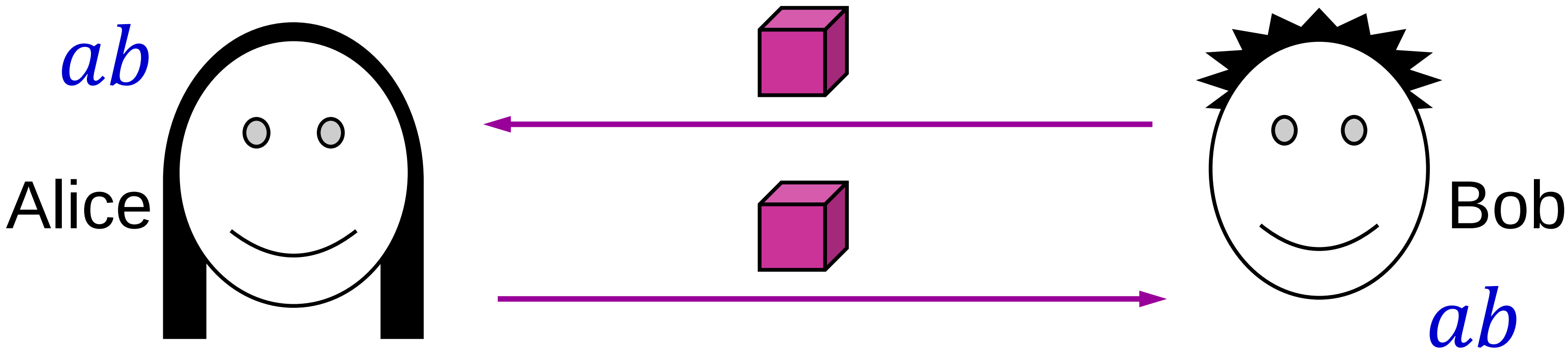
Suppose that Alice wants to convey ***two*** classical bits to Bob sending just ***one*** qubit



By Holevo's Theorem, this is ***impossible***

Superdense coding

In ***superdense coding***, Bob is allowed to send a qubit to Alice first



How can this help?

How superdense coding works

1. Bob creates the state $|00\rangle + |11\rangle$ and sends the ***first*** qubit to Alice

2. Alice: If $a = 1$, apply X to her qubit.
   Then if $b = 1$, apply Z to her qubit.
   Then send the qubit back to Bob.

Recall that

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

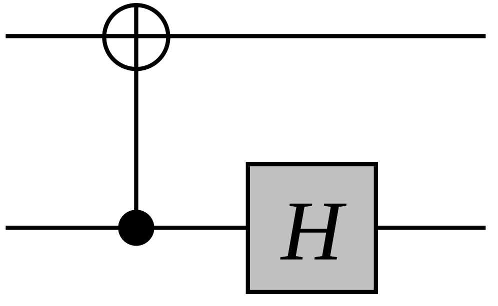
ab	$(Z^b X^a \otimes I)(|00\rangle + |11\rangle)$
00	$|00\rangle + |11\rangle = |\Psi_+\rangle$
01	$|00\rangle - |11\rangle = |\Psi_-\rangle$
10	$|01\rangle + |10\rangle = |\Phi_+\rangle$
11	$|01\rangle - |10\rangle = |\Phi_-\rangle$

} Bell basis

3. Bob measures the two qubits in the ***Bell basis***

Measurement in the Bell basis

Specifically, Bob applies



input	output
$\lvert 00\rangle + \lvert 11\rangle$	$\lvert 00\rangle$
$\lvert 00\rangle - \lvert 11\rangle$	$\lvert 01\rangle$
$\lvert 01\rangle + \lvert 10\rangle$	$\lvert 10\rangle$
$\lvert 01\rangle - \lvert 10\rangle$	$-\lvert 11\rangle$

to his two qubits ...

and then measures them, yielding ab

This concludes superdense coding

Teleportation

Recap

- ***n*-qubit quantum state:** 2^n-dimensional unit vector

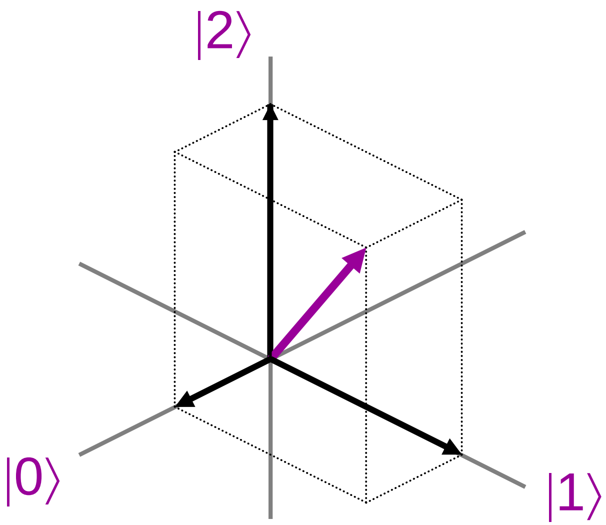
- **Unitary op:** $2^n \times 2^n$ linear operation U such that $U^\dagger U = I$
  (where $U^\dagger$ denotes the conjugate transpose of U)

  $U|0000\rangle$ = the 1st column of U
  $U|0001\rangle$ = the 2nd column of U
  : : : : : :
  $U|1111\rangle$ = the (2^n)th column of U

  the columns of U are orthonormal

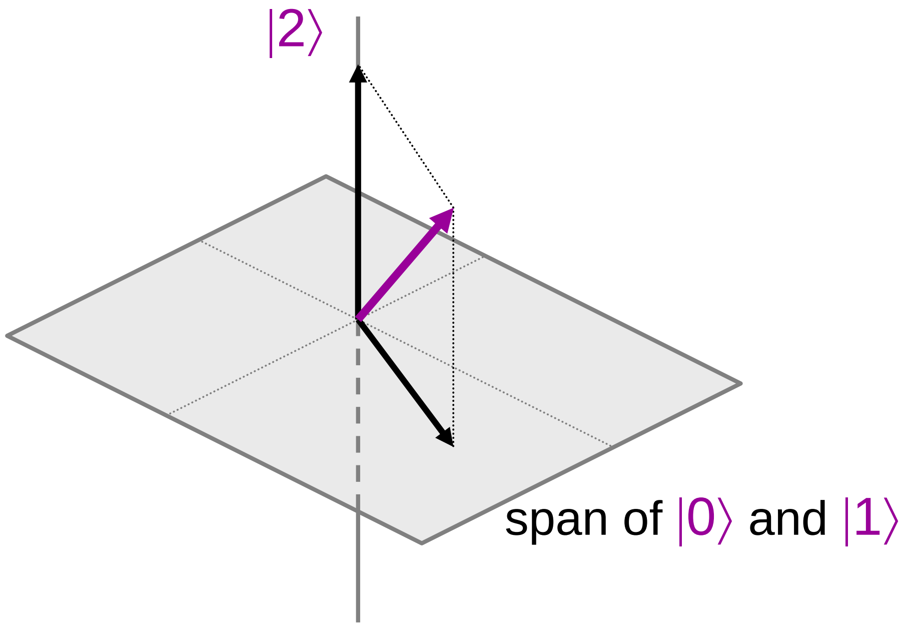
Incomplete measurements (I)

Measurements up until now are with respect to orthogonal one-dimensional subspaces:



(qutrit)

The orthogonal subspaces can have other dimensions:



Incomplete measurements (II)

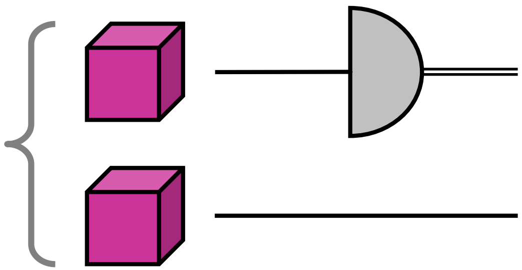
Such a measurement on $\alpha_0|0\rangle + \alpha_1|1\rangle + \alpha_2|2\rangle$

(renormalized)

results in $\begin{cases} \alpha_0|0\rangle + \alpha_1|1\rangle & \text{with prob } |\alpha_0|^2 + |\alpha_1|^2 \\ |2\rangle & \text{with prob } |\alpha_2|^2 \end{cases}$

Measuring the first qubit of a two-qubit system

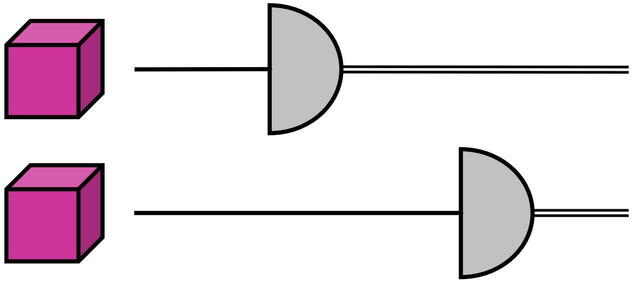
$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|01\rangle + \alpha_{11}|11\rangle$



Defined as the incomplete measurement with respect to the two dimensional subspaces:

- span of $|00\rangle$ & $|01\rangle$ (all states with first qubit 0), and
- span of $|10\rangle$ & $|11\rangle$ (all states with first qubit 1)

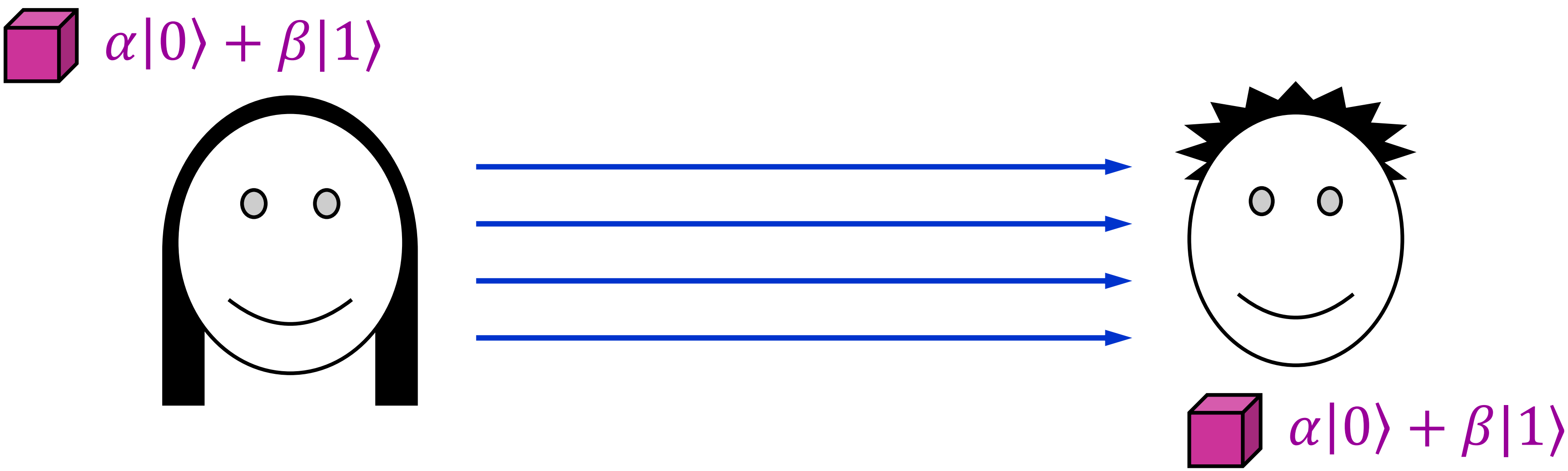
$$\text{Result is } \begin{cases} 0,\ \alpha_{00}|00\rangle + \alpha_{01}|01\rangle \text{ with prob } |\alpha_{00}|^2 + |\alpha_{01}|^2 \\ 1,\ \alpha_{10}|10\rangle + \alpha_{11}|11\rangle \text{ with prob } |\alpha_{10}|^2 + |\alpha_{11}|^2 \end{cases}$$



Easy exercise: show that measuring the first qubit and *then* measuring the second qubit gives the same result as measuring both qubits at once

Teleportation (prelude)

Suppose Alice wishes to convey a qubit to Bob by sending just classical bits



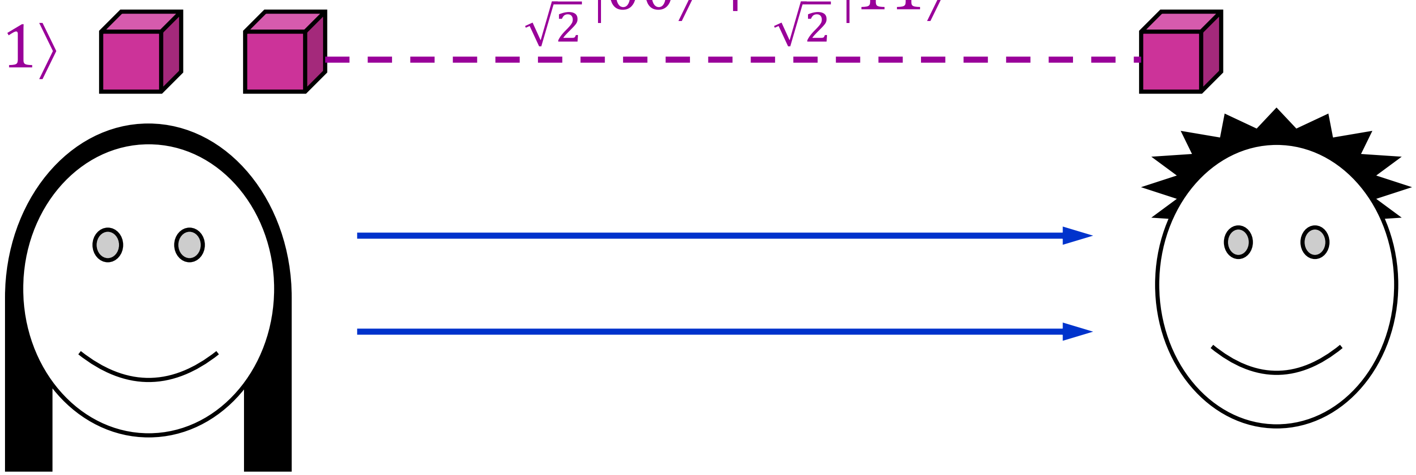
If Alice ***knows*** α and β, she can send approximations of them —but this still requires infinitely many bits for perfect precision

Moreover, if Alice does ***not*** know α or β, she can at best acquire ***one bit*** about them by a measurement

Teleportation scenario

In teleportation, Alice and Bob also start with a Bell state

$$\alpha|0\rangle + \beta|1\rangle \qquad \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$



and Alice can send two classical bits to Bob

Note that the initial state of the three qubit system is

$$(\alpha|0\rangle + \beta|1\rangle)\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) = \frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\alpha}{\sqrt{2}}|011\rangle + \frac{\beta}{\sqrt{2}}|100\rangle + \frac{\beta}{\sqrt{2}}|111\rangle$$

How teleportation works

Initial state

$$(\alpha|0\rangle + \beta|1\rangle)\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right)$$

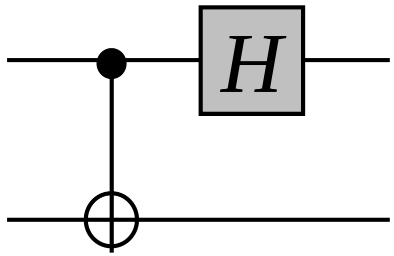
$$= \frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\alpha}{\sqrt{2}}|011\rangle + \frac{\beta}{\sqrt{2}}|100\rangle + \frac{\beta}{\sqrt{2}}|111\rangle$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right)(\alpha|0\rangle + \beta|1\rangle)$$
$$+ \frac{1}{2}\left(\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle\right)(\alpha|1\rangle + \beta|0\rangle)$$
$$+ \frac{1}{2}\left(\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle\right)(\alpha|0\rangle - \beta|1\rangle)$$
$$+ \frac{1}{2}\left(\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle\right)(\alpha|1\rangle - \beta|0\rangle)$$

Protocol: Alice measures her two qubits ***in the Bell basis*** and sends the result to Bob (who then “corrects” his state)

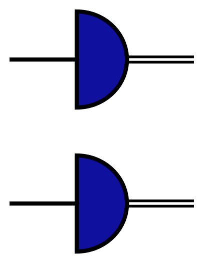
Measuring the Bell basis

Alice applies



to her two qubits, yielding:

$$\begin{cases} \frac{1}{2}|00\rangle(\alpha|0\rangle + \beta|1\rangle) \\ +\frac{1}{2}|01\rangle(\alpha|1\rangle + \beta|0\rangle) \\ +\frac{1}{2}|10\rangle(\alpha|0\rangle - \beta|1\rangle) \\ +\frac{1}{2}|11\rangle(\alpha|1\rangle - \beta|0\rangle) \end{cases}$$



$$\begin{cases} (00,\ \alpha|0\rangle + \beta|1\rangle) & \text{with prob. } 1/4 \\ (01,\ \alpha|1\rangle + \beta|0\rangle) & \text{with prob. } 1/4 \\ (10,\ \alpha|0\rangle - \beta|1\rangle) & \text{with prob. } 1/4 \\ (11,\ \alpha|1\rangle - \beta|0\rangle) & \text{with prob. } 1/4 \end{cases}$$

Then Alice sends her two classical bits to Bob, who then adjusts his qubit to be $\alpha|0\rangle + \beta|1\rangle$ depending on the values of the bits.

Bob's adjustment procedure

Bob receives two classical bits a, b from Alice, and:

if $b = 1$ he applies X to qubit

if $a = 1$ he applies Z to qubit

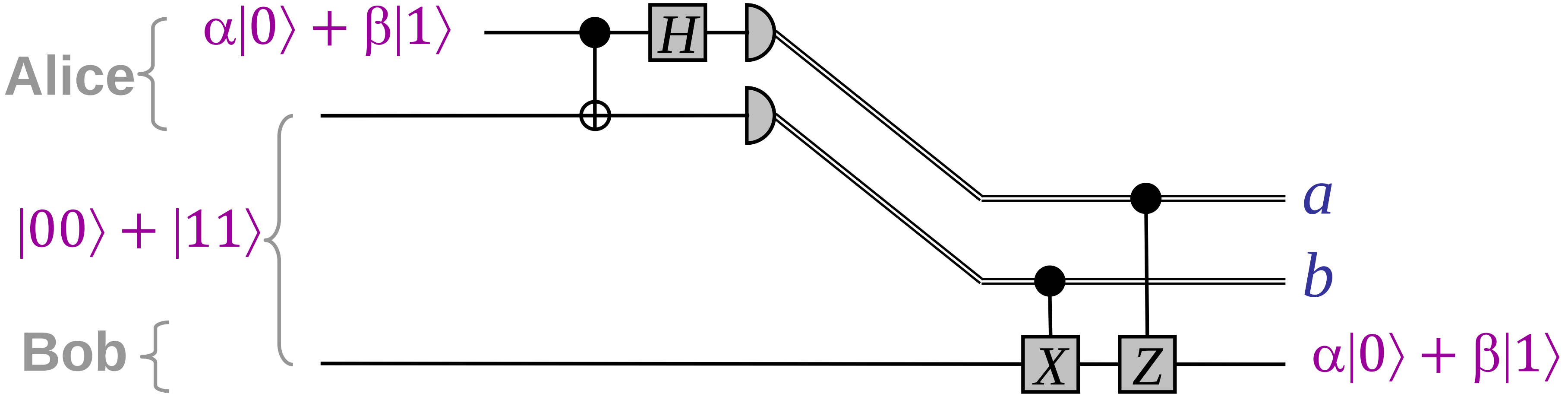
$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

yielding:

$$\begin{cases} 00, & \alpha|0\rangle + \beta|1\rangle \\ 01, & X(\alpha|1\rangle + \beta|0\rangle) = \alpha|0\rangle + \beta|1\rangle \\ 10, & Z(\alpha|0\rangle - \beta|1\rangle) = \alpha|0\rangle + \beta|1\rangle \\ 11, & ZX(\alpha|1\rangle - \beta|0\rangle) = \alpha|0\rangle + \beta|1\rangle \end{cases}$$

Note that Bob acquires the correct state in each case

Summary of teleportation



Quantum circuit exercise: try to work through the details of the analysis of this teleportation protocol