Introduction to Quantum Information Processing

QIC 710 / CS 768 / PH 767 / CO 681 / AM 871

Lecture 6 (2017)

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Homework on Crowdmark

- We will be using Crowdmard to grade homeworks.
- Upload .pdf once then drag each problem solution to the box for that problem.
- New page per solution (but okay this time if not, just scan relevant pages twice).
- Will accept paper for this assignment, but understand that we will need to scan it in.
- For this reason, okay to turn in on **Friday**.

Tuesday class in **OPT 309**

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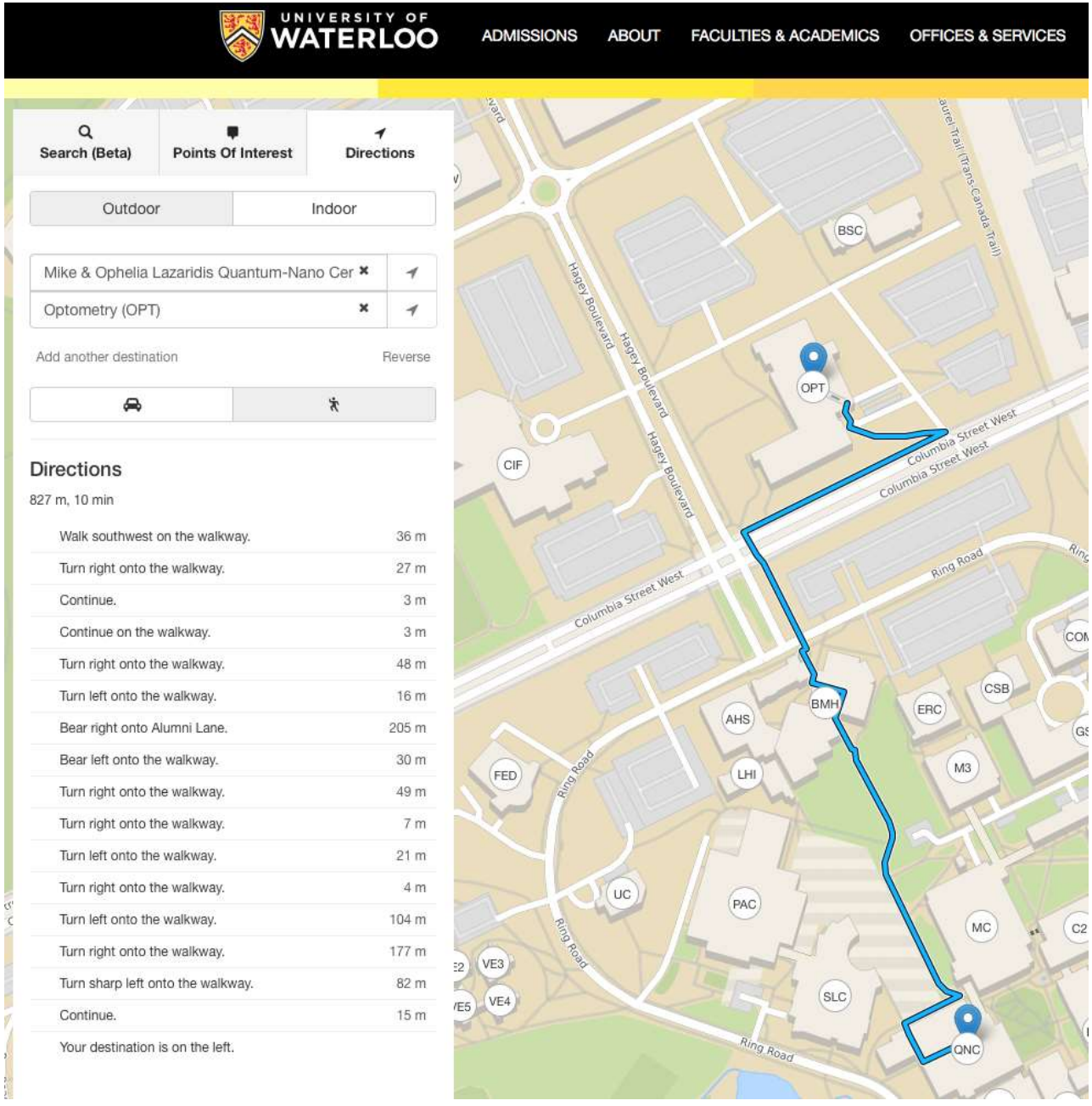
Nov. 21	QNC 0101	Nov. 23	QNC 0101
Nov. 28	QNC 0101	Nov. 30	QNC 0101

OPT 309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Ring Road
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BMH
ERC
CSB
AHS
LHI
FED
M3
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PAC
MC
SLC
VE3
VE4
QNC

Discrete log problem

Discrete logarithm problem (DLP)

Input: p (prime), g (generator of $\mathbb{Z}_p^\times$), $a \in \mathbb{Z}_p^\times$
Output: $r \in \mathbb{Z}_{p-1}$ such that $g^r \equiv a \bmod p$

Example: $p = 7$, $\mathbb{Z}_7^\times = \{1, 2, 3, 4, 5, 6\} = \{3^0, 3^2, 3^1, 3^4, 3^5, 3^3\}$ (hence 3 is a generator of $\mathbb{Z}_7^\times$)

For $a = 6$, since $3^3 = 6$, the output should be $r = 3$

Note: No efficient classical algorithm for ***DLP*** is known, and cryptosystems exist (El Gamal encryption, Diffie-Hellman key exchange, Digital Signature Algorithm) whose security is based on the computational difficulty of DLP.

Efficient quantum algorithm for DLP?
(**Hint:** it can be made to look like Simon's problem!)

DLP similar to Simon's problem

Clever idea (of Shor): define $f: \mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1} \to \mathbb{Z}_p^\times$ as $f(x_1, x_2) = g^{x_1} a^{-x_2} \bmod p$. Can be efficiently computed *classically* (e.g. in $O(n^3)$-time by repeated squaring).

When is $f(x_1, x_2) = f(y_1, y_2)$?

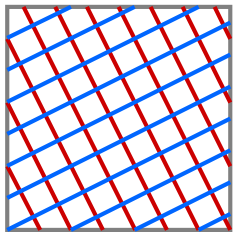
We know $a = g^r$ for ***some*** r, so $f(x_1, x_2) = g^{x_1 - r x_2} \bmod p$

Thus, $f(x_1, x_2) = f(y_1, y_2)$ iff $x_1 - r x_2 \equiv y_1 - r y_2 \bmod p - 1$

iff $(x_1, x_2) \cdot (1, -r) \equiv (y_1, y_2) \cdot (1, -r) \bmod p - 1$

iff $((x_1, x_2) - (y_1, y_2)) \cdot (1, -r) \equiv 0 \bmod p - 1$

iff $(x_1, x_2) - (y_1, y_2) \equiv k(r, 1) \bmod p - 1$



cf. Simon's property: $f(x) = f(y)$ iff $x - y \equiv r \bmod 2$.

Simon's problem modulo m

The function arising in DLP can be abstracted to the following:

Given: $f: \mathbb{Z}_m \times \mathbb{Z}_m \to T$ with the property that

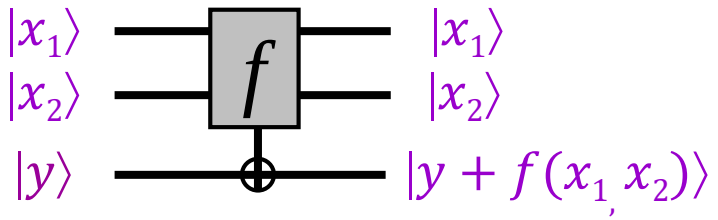
$f(x_1, x_2) = f(y_1, y_2)$ iff $(x_1, x_2) - (y_1, y_2) \equiv k(r_1, r_2) \bmod m$,

where (r_1, r_2) is the hidden data.

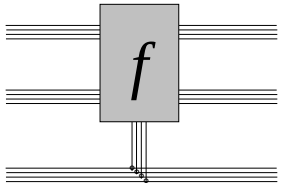
Goal: determine (r_1, r_2)

Note: in DLP case, $(r_1, r_2) = (r, 1)$

The reversible query box for f is



where each "wire" denotes many qubit wires, to represent elements of $\mathbb{Z}_m$ like:



Not a "black" box, because we can simulate it by 1-qubit and 2-qubit gates (and this can be done efficiently)…

Digression:
on simulating black boxes

How *not* to simulate a black box

Given an efficiently (classically) computable function, over some finite domain, such as $f(x) = g^{x_1} a^{-x_2} \bmod p$, need to simulate f-queries over that domain.

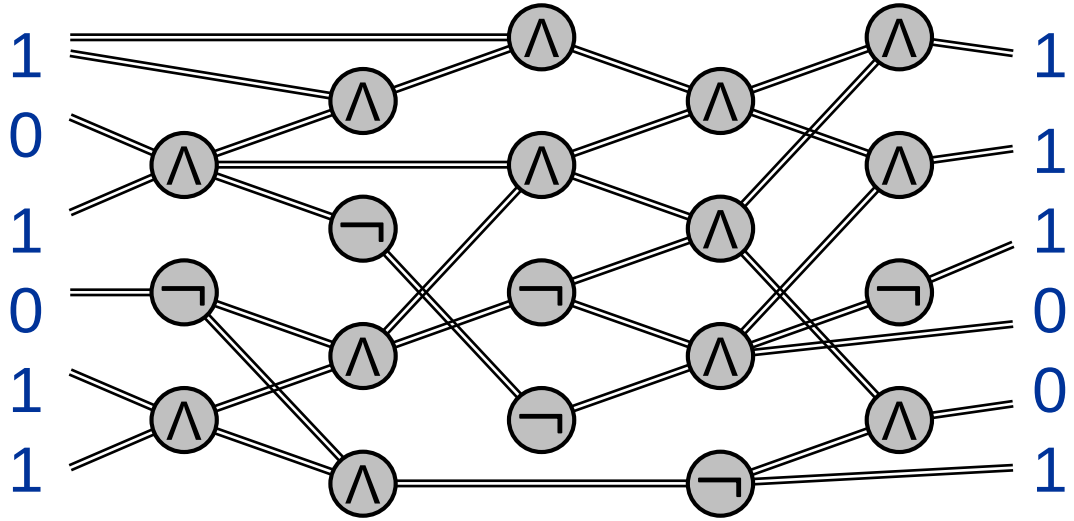
Easy to compute mapping $|x\rangle|y\rangle|0\cdots 0\rangle \mapsto |x\rangle|y + f(x)\rangle|g(x)\rangle$, where the third register is "work space" with accumulated "garbage" (e.g., two such bits arise when a Toffoli gate is used to simulate an AND gate inside a classical circuit).

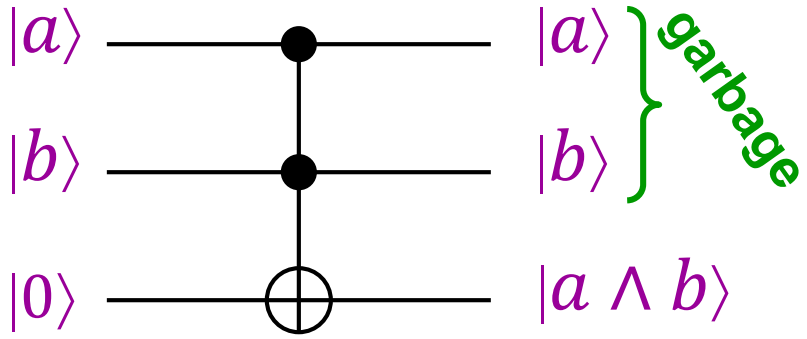
This works fine ***as long as f is not queried in superposition***

If f is queried in superposition then the resulting state can be $\sum_x \alpha_x |x\rangle|y + f(x)\rangle|g(x)\rangle$.
Can't we just discard the third register?

No ... there could be entanglement ...

Intermediate AND gates





How *to* simulate a black box

Simulate the mapping $|x\rangle|y\rangle|0\cdots 0\rangle \mapsto |x\rangle|y+f(x)\rangle|0\cdots 0\rangle$,
(i.e., clean up the "garbage")

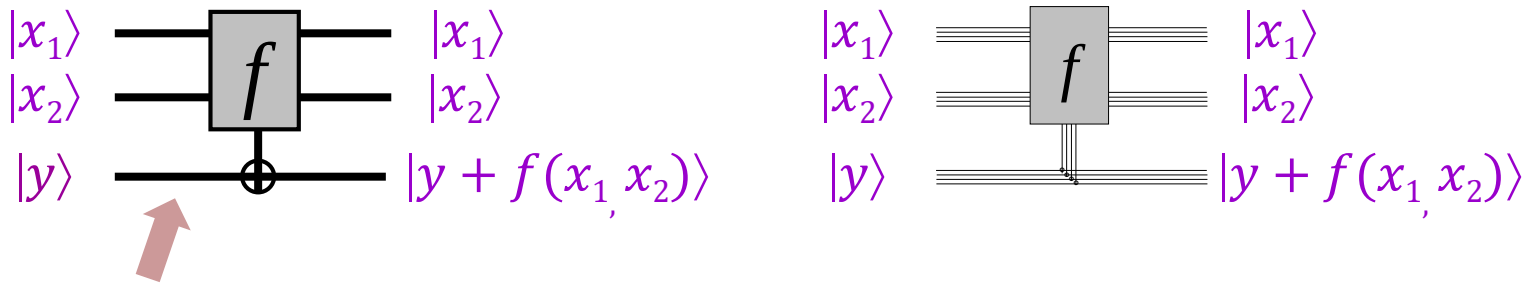
To do this, use an additional register, and:

1. compute $|x\rangle|y\rangle|0\cdots 0\rangle|0\cdots 0\rangle \mapsto |x\rangle|y\rangle|f(x)\rangle|g(x)\rangle$
   (ignoring the 2nd register in this step)
2. compute $|x\rangle|y\rangle|f(x)\rangle|g(x)\rangle \mapsto |x\rangle|y+f(x)\rangle|f(x)\rangle|g(x)\rangle$
   (using CNOT gates between the 2nd and 3rd registers)
3. compute $|x\rangle|y+f(x)\rangle|f(x)\rangle|g(x)\rangle \mapsto |x\rangle|y+f(x)\rangle|0\cdots 0\rangle|0\cdots 0\rangle$
   (by reversing, i.e. "*uncomputing*", the procedure in step 1)

Total cost about twice the classical cost of computing f,
plus n auxiliary CNOT gates.

Simon's problem modulo m

So now we have an efficient way of implementing the reversible black box for f



Reminder: each "thick wire" denotes several qubits, to represent an element of $\mathbb{Z}_m$ (e.g. $\{0,1,2,3,4,5,6\} = \{000,001,010,011,100,101,110\}$)

OK, so what about a quantum algorithm for this problem?

To get one, we go beyond the Hadamard transform, which has been our main tool so far, to…

Quantum Fourier transform (QFT)

Quantum Fourier transform

$$F_m = \frac{1}{\sqrt{m}} \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{m-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(m-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(m-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{m-1} & \omega^{2(m-1)} & \omega^{3(m-1)} & \cdots & \omega^{(m-1)^2} \end{pmatrix}$$

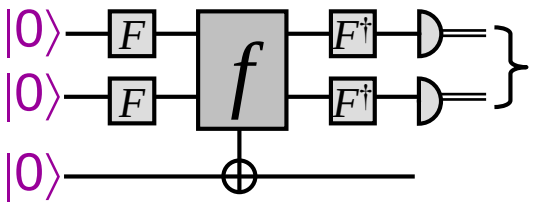
where $\omega = e^{2\pi i/m}$ (for n qubits, $m = 2^n$).

This is unitary and generalizes the Hadmard transform $F_2 = H$.

The quantum Fourier transform is an important component of several interesting quantum algorithms …

Quantum algorithm for Simon mod m

$f(x_1, x_2) = f(y_1, y_2)$ iff $(x_1, x_2) - (y_1, y_2) \equiv k(r, 1) \bmod m$



Turns out that the result is a random (s_1, s_2) such that $(s_1, s_2) \cdot (r, 1) \equiv 0 \bmod m$.
(generalizes idea from Simon's algorithm)

If $\gcd(s_1, m) = 1$, then s_1 has an inverse mod m, and r can be computed as $r \equiv -s_2/s_1 \bmod m$.
(The details of computing $s_1^{-1} \bmod m$ follow from the extended Euclidean algorithm).

Moreover, the probability that $\gcd(s_1, m) = 1$ occurs is not too small (if it fails the algorithm can be run again).

Quantum algorithm for Simon mod m

Steps that have been shown to be efficiently implementable (i.e., in terms of a number of 1- and 2-qubit/bit gates that scales polynomially with respect to the number of bits of m):

- Implementation of reversible gate for f
- The classical post-processing at the end

What's missing?

- Implementation of the QFT f modulo m ($= p - 1$ for DLP)
- Proof that outcome is random s.t. $(s_1, s_2) \cdot (r, 1) \equiv 0 \bmod m$

Next time, we'll show how to implement the QFT for $m = 2^n$.

Shor did this too, and showed that if the modulus is within a factor of 2 from $p - 1$, by using careful error-analysis, this was good enough, though the calculations and analysis become more complicated (we'll omit the details of this).

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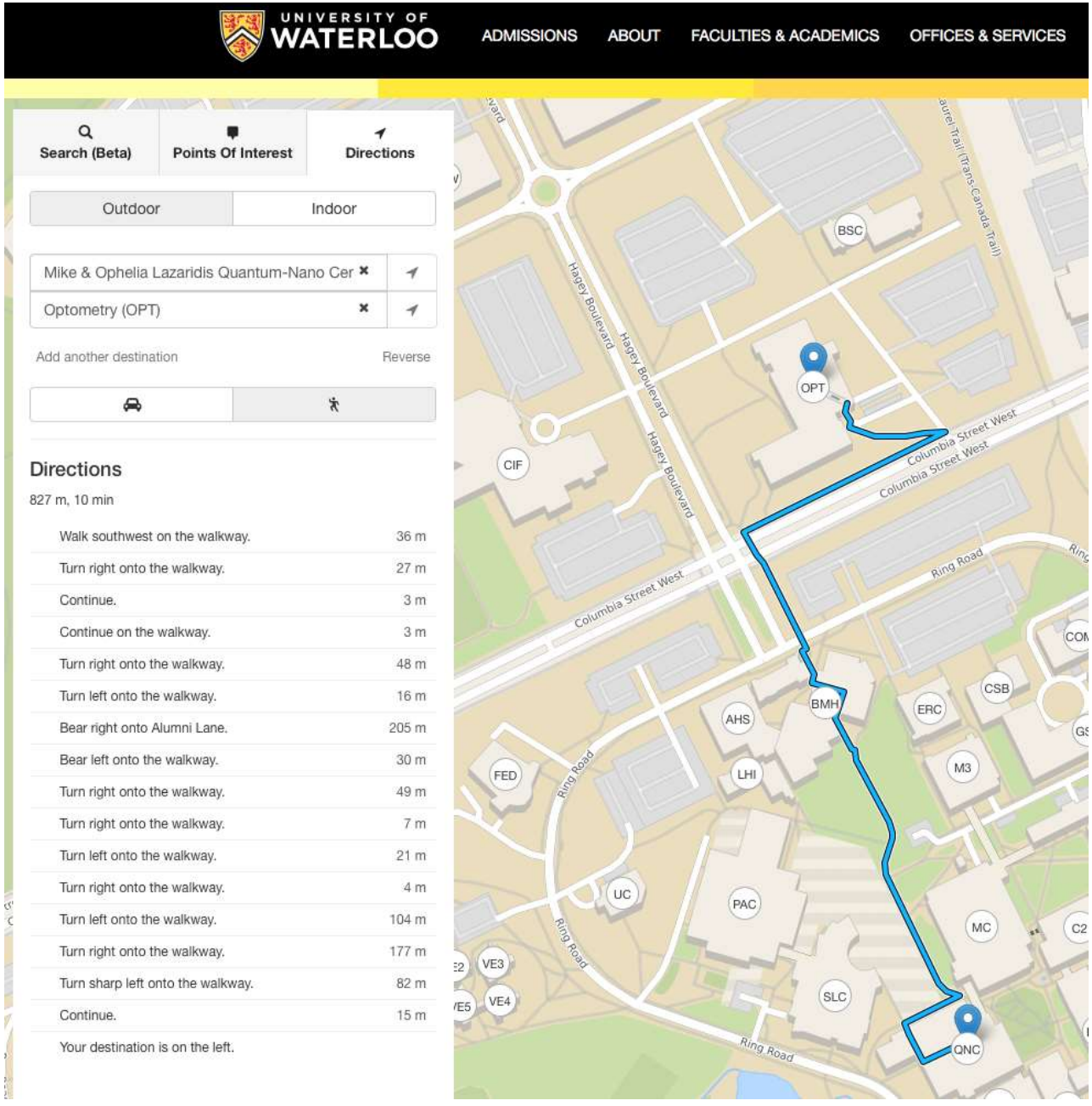
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