mathASKS 152.4

FEATURING PROFESSOR SPIRO KARIGIANNIS

NOTRAYMO: ELI5, WHAT ARE DIFFERENTIAL GEOMETRY AND GEOMETRIC ANALYSIS?

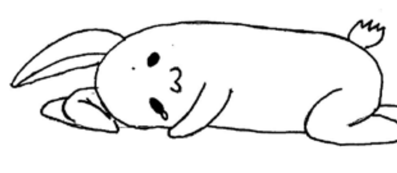
Since this is such a great question, I will give you a great answer.

The word "geometry" literally is Greek (I should know) for "measuring the earth". It is the oldest science and the oldest branch of mathematics. Broadly speaking, it is the study of the structure of a "space" equipped with a notion of "distance" between points. In classical (Euclidean) geometry, the space is just a 2-dimensional (or 3-dimensional) real vector space, equipped with a positive definite inner product (although this is certainly not how the ancient Greeks thought of it). This of course generalizes to any dimension, and while there are some interesting things that one can say about this situation, it is not that exciting, because the geometry basically "looks the same" at every point and in every direction. This is encoded in the fact that the inner product is "constant" in some sense. So if we want to study more interesting situations, we need to allow things to *change*—if the world didn't change, it would be very boring indeed. And of course, if things are changing in a sufficiently "smooth" way, we are doing *calculus*. Differential geometry is the study of "spaces" equipped with "smoothly varying" notions of distance and inner product. Strictly speaking, what I have just non-rigorously defined is *Riemannian* geometry, which is just a subset of differential geometry.

To get more precise, let's start with some basics. Before we can talk about "differentiability", we need to understand "continuity". The right setting to make sense of continuity is topology. A topological space is a space on which it makes sense to define continuous functions, or more generally continuous maps between topological spaces. Our good friend $\mathbb{R}^n$ with the standard metric space structure induced from the Euclidean inner product is just one example. All we need to understand "continuity" is some notion of "closeness", so no linear (vector space) structure is really needed. But if we want to be able to *differentiate* objects, we need to take a limit of a *difference quotient*. To make sense of this, it seems that we need a vector space structure. This is almost true, but we can get away with something that is *almost* a vector space, as I will now valiantly endeavour to explain. In kindergarten, we learn about the simplest spaces (vector spaces) and the simplest maps between them (linear maps). In elementary school, we decide to get a bit crazy, and consider *nonlinear* maps between linear spaces, as long as these are reasonably well-behaved. Here, "reasonable" means differentiable so that, near a given point, such maps are well-approximated by linear maps. Then in high school we go completely nuts, and consider nonlinear maps between *nonlinear spaces*. What could this even mean? Since we are doing calculus, the key is *linear approximation*. We want our "nonlinear spaces" to be be well-approximated by linear spaces, near a given point. This is the notion of a manifold. An n-manifold is a topological space that "looks like" $\mathbb{R}^n$ near each point. A smooth map between an n-manifold and and m-manifold "looks like" a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ near each point. Smooth manifolds are the most general spaces on which is makes sense to do calculus. This is the subject of PMATH 465, which you should all take. It's awesome.

But wait, there's more. Let M be an n-manifold. At each point $p \in M$, we have a "tangent space" T_pM, which is an n-dimensional real vector space approximating M in some sense. Abstractly, this is of course isomorphic to $\mathbb{R}^n$, but not canonically. (That is, there is no preferred basis.) Also, there is no preferred inner product on T_pM. A choice of "smoothly varying" inner product on all the tangent spaces of M is a *Riemannian metric* on M. It turns out such a metric always exists, but there are uncountably many such metrics, and no preferred choice on a random manifold. (If $M = \mathbb{R}^n$ then there is a preferred choice, the one from Ancient Greece, but there are uncountably many here too.) A natural question is, "what is the "best" Riemannian metric on a given manifold M?" The answer, of course, depends on what we mean by "best". This is where we start to get into *geometric analysis*. I'll explain that very soon, but bear with me a bit longer. (Hopefully you're all still reading this!) More generally than a Riemannian metric, we can consider "geometric structures" on any manifold M as follows. Whenever there's an algebraic structure that can exist on a vector space, we can try to "attach" such a structure to each tangent space T_pM of M in a "smoothly varying way". Depending on the structure, this may or may not always be possible. There may be "global topological obstructions." From inner products on vector spaces, we get Riemannian metrics, and this can always be done, which is not obvious. From orientations on vector spaces, we get manifold orientations. This *cannot* always be done (Google the Möbius strip or the Klein bottle, for example). And things get much more exotic than that, such as almost complex structures or G_2 structures, but I am rambling.

Now suppose you have a manifold M that admits a certain type of "geometric structure". If it does, it usually admits infinitely many. What is the best one? In most situations, the natural notion of "best" is characterized by that structure satisfying a natural (usually nonlinear) partial differential equation on the manifold. So even if M admits a certain type of geometric structure, it may not have a "best" one, because that geometric PDE may not have a solution. To be able to answer such questions, one uses the tools of functional analysis and partial differential equations in the setting of Riemannian geometry. So to do geometric analysis, you really need to know a bit about everything, and a lot about certain things, but that's why it's so interesting! I can go on and say much more, but you're probably already regretting asking me this question!



MOLASSES: AS A TOPOLOGIST, WHAT IS YOUR OPINION ON THE TOPOGRAPHY OF THE WATERLOO REGION?

Since I am not actually a topologist, perhaps I should not answer this question. But I will. If I *were* a topologist, the topography of Waterloo would be uninteresting, because topology is only concerned with structure up to continuous deformation, and the topography of Waterloo region (or any other region) is homeomorphic to a flat space. As a geometer, I care about lengths, distances, and curvatures. So the question is more meaningful. Sadly, Waterloo is (even geometrically) quite flat. It's not that interesting topographically. Except maybe for Elora Gorge. If you haven't been there, it's worth the trip.

BOLDBLAZER: WHAT DO YOU THINK OF THE TOPICS IN PMATH 340? WHAT ABOUT PMATH 333?

I've never taught PMATH 340, and I certainly never will, since I don't even know what quadratic reciprocity is. I can't really work well with numbers. Thankfully we now have machines that calculate the restaurant tip for me, because I can't do the arithmetic myself. I haven't yet taught PMATH 333, but I will actually teach it for the first time in Fall 2023. It's a course designed to get people ready for PMATH 351 if they did not take MATH 247 (which I've taught at least five or six times, and which I will also teach again in Fall 2023.) That material in 333, or 247, or 351, is certainly very cool, and you can't do geometry without it, but of course it's not as cool as geometry. Nothing is, except maybe German shepherds.

LABYRINTH: WHAT GOT YOU INTERESTED IN DIFFERENTIAL GEOMETRY AND GEOMETRIC ANALYSIS, AND WHAT'S YOUR FAVOURITE THING ABOUT YOUR RESEARCH?

This might be true of most mathematicians, but I am especially attracted to patterns. More specifically, the (mathematical) thing that really turns me on is when we find a structure that is very closely related to a previously well-understood structure, but also has some differences in subtle but important ways. For example (and this example is really fundamental), there are many similarities between real numbers and complex numbers. They are both fields, and are also real vector spaces equipped with natural norms which are compatible with the field multiplication. That is, $|ab| = |a|\,|b|$ for any a, b. But the real numbers are naturally ordered, while the complex numbers are not. There exists another such structure which is very similar, namely the quaternions, $\mathbb{H}$, which are a 4-dimensional real vector space equipped with a multiplication that makes them *almost* a field, they are just non-commutative. And their multiplication is compatible with the norm as for $\mathbb{R}$ or $\mathbb{C}$. In fact, there is exactly only one other such "real normed division algebra", called the *octonions* $\mathbb{O}$, which are not only non-commutative, but also *non-associative*. This makes them more complicated (but at the same time much more interesting) than $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The special structure of the octonions in 8-dimensions induces a special "cross product" operation on $\mathbb{R}^7$, thought of as the orthogonal complement of the identity element in $\mathbb{O}$. This is almost exactly the same as the cross product on $\mathbb{R}^3$ that we all learned about in first grade, except that the non-associativity introduces some complications. My research studies 7-dimensional and 8-dimensional manifolds that essentially have these special algebraic structures on each of their tangent spaces, in a smoothly varying way. These spaces are of potential application in theoretical physics, which is cool, but I would find them extremely interesting regardless. The amazing thing about geometric analysis, as I hinted at above in the first question, is that it mixes together algebra, topology, analysis, and geometry in a really beautiful way. In fact, the crowning achievement of 20th century mathematics is widely considered to be the Atiyah-Singer Index Theorem, which describes an incredible marriage between all four of these players (mathematical polygamy is fine and to be encouraged). In hindsight, the fact that I was interested in things which were "very similar, but only slightly different" was evident from my childhood. I remember being very young and being enthralled by a McDonald's marketing campaign that featured *two* of that creepy-looking blob guy Grimace, the traditional purple Grimace and a super-cool *green Grimace*. That blew my 5-year old mind. True story.

JEFF: WHAT IS YOUR FAVOURITE BATHROOM ON CAMPUS?

If only there were a decent bathroom on campus. I don't understand why the University administration is so cheap as to stock all the bathrooms with what is essentially *negative ply* toilet paper. Best to bring your own or go at home.

BOLDBLAZER: DO YOU HAVE A PREFERRED RESTAURANT AT THE UNIVERSITY PLAZA?

If I have to choose something in the Plaza, then I choose Harvey's just because I've been a Harvey's customer since I was a kid in Montréal. But the best restaurant in the Waterloo region is Urwa's, a Pakistani restaurant near the *other* Harvey's, at King and Weber. You should try their Lahori Chana. It is awesome. Just like PMATH 465, only spicier. (I am not getting kickbacks from Urwa's, but I would gladly accept them.)

AUTUMN: WHAT'S YOUR FAVOURITE SEASON?

Ironically, Autumn, my favourite season is Fall. My favourite time is when it starts to cool off and the leaves fall down. It puts me (perhaps weirdly) in the mindset of starting a new chapter of life. This made sense when I would start a new school year every September, but makes less sense since I stopped being a student. It also makes less sense because September is the new August. That is, the feeling I would get from the September weather when I was a kid in the 80's doesn't happen until October now. We've really messed up the climate on this planet. We may have to find a new one if we can't get our act together.

BOLDBLAZER: WHAT COLOUR CREWMATE WOULD YOU CHOOSE IN AMONG US?

I've never played this game, although I have watched my daughter play it. I can't say that it looks exciting. Certainly not anywhere near as cool as that monumental classic of 1983

Apple II games, “Canyon Climber” (Google it). Although I should add that I used to be *really good* at the triple jump on the Nintendo Entertainment System. There was a trick to that. But, since you asked me about my favourite colour (although you actually didn’t), my favourite colour is purple. Probably because I’m a big fan of the artist formerly known as his royal purpleness. May he rest in peace.

PSYCHGIRL: WHAT SUGGESTIONS DO YOU HAVE FOR MATH STUDENTS WHO ARE SOON TO GRADUATE? WHAT GENERAL ADVICE DO YOU HAVE FOR STUDENTS ABOUT LIFE OUTSIDE OF UNIVERSITY AND ACADEMIA?

The great thing now is that there are so many options to actually *do math* outside of academia. That didn’t used to be the case. Until about 2000, if you had a degree in math and didn’t stay in academia, you either ended up teaching math at the pre-university level (a very fine and noble profession, and we need more good people doing that!) or you went to Wall Street to work in financial consulting, with zero knowledge of what that means, and probably as a result helped cause the financial crisis of 2008 (I know several people who took this path). But now, there are so many jobs in private industry where you actually need to do non-trivial *math*. I had a postdoc here who went to San Francisco to work in the computer game industry, and he’s actually doing Riemannian geometry. It’s not just coding. Having given you this fantastic news, I do admit that I am not closely connected to these opportunities, I just know that they exist. So if this kind of thing appeals to you then I encourage you to seek out faculty members who may have such connections, to learn more.

If you want to continue in math, that’s great. Math is awesome. But, to quote the infamous Qui-Gon Jinn, it’s a hard life. Each year I learn as much new math as I did in the previous several years. Most of the math I know I learned after my PhD and after 5.5 years of postdoc (that is, in the 14 years since I have been at Waterloo). You never really stop learning, nor should you. In fact, this is probably good advice for anyone in any kind of profession: never stop learning. Since you asked for general advice, here are some pearls of wisdom I have attained through many trials and just as many errors:

0. Never stop learning.
1. You actually learn the most from the mistakes you make, not from things you do right the first time. Think about your past courses. You almost certainly understand something better if you *initially* didn’t understand it and had to work hard to get it. So don’t be afraid to make mistakes. That’s how we grow.
2. Related to (1) above, the best way to understand something is to try to *teach* it to someone else. See also the next question below.
3. Sleep is very important. All-nighters don’t work. Trust me, I tried. I learned the hard way. The body needs sleep. That being said, you will probably all have to learn this lesson on your own, if you haven’t already.
4. The mind also needs a break often. Trying to do math (or anything else) for several hours without stopping is not good for you. Take a step away. Go for a walk. Watch something stupid on TV. Read something light and fluffy. Just as you wouldn’t exercise your heart or your biceps without taking a break, the same is true for the brain.

JEFF: WHAT’S A GRAD-LEVEL COURSE YOU’D LIKE TO OFFER WHICH HASN’T BEEN OFFERED YET?

I have really had the great pleasure to teach *many* graduate-level special topics courses at Waterloo, probably averaging about one every 2.5 years or even slightly more. I almost always choose a topic which is something that I really don’t know that well but would like to know much better. That’s why I taught courses on the “Atiyah-Singer Index Theorem” and on “Clifford algebras and spinors”. Another reason to teach a topics course is to organize the material better in my head, for an eventual book. I taught the first three iterations of PMATH 868: Connections and Riemannian Geometry, and have produced about ⅔ of an eventual book. Hopefully it will be done in the next two years. Probably the next topics course I teach (maybe in 2024–2025, because I am on sabbatical in early 2024) will be on harmonic maps, as I have lately become very interested in these objects in my own research but don’t know enough about them. Other topics I am interested in teaching one day are: Einstein metrics, geometric flows, and symmetric spaces. Again, all things I wish I knew better, and if I did, I would produce better and more interesting research. So I will teach these at some point in the coming decade (see items 0 and 2 in the question above).

GEOMETER: FAVOURITE GEOMETRY RESULT?

I’ve already mentioned the Atiyah-Singer Index Theorem, which is truly incredible. But, to choose something more specific to geometric analysis, I would have to say the Calabi-Yau Theorem. This was really the spark that ignited the fire which was to become geometric analysis. My former PhD supervisor (and Fields Medalist) Shing-Tung Yau proved this theorem in the mid-1970's, when it was known as the “Calabi conjecture”. The simplest version of this theorem says that if M is a compact Kähler manifold, then it admits a unique Ricci-flat Kähler metric in each Kähler cohomology class if and only if its first Chern class vanishes. That’s quite a mouthful, I know. I encourage you to read his popular science book “The Shape of Inner Space” which attempts to explain this to a general audience. It’s really quite well-written. He’s not paying me to say that, honest.

Green Grimace blew my 5-year old mind.

PROF. SPIRO KARIGIANNIS