The Atiyah-Hitchin-Singer Theorem and an 8-dimensional generalization

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Abstract

The Atiyah-Hitchin-Singer theorem states that the twistor almost complex structure on a certain S^2 bundle over an oriented Riemannian 4-manifold (M, g) is integrable if and only if the Weyl curvature tensor of g is self-dual.

These ideas were developed by Roger Penrose connecting 4-dimensional Riemannian geometry with complex geometry.

We present a new approach to the Atiyah-Hitchin-Singer theorem using horizontal lifts and their respective flows, cross products and the quaternions to show that the Nijenhuis tensor vanishes if and only if the Weyl curvature tensor of g is anti-self-dual. An eight dimensional generalization is presented when the manifold is $\mathbb{R}^8$.

Contents

1 Introduction

It is be possible that certain types of PDEs have underlying geometric structure. Knowing the type of geometric structure can reduce the complexity of certain PDEs.

Penrose used twistors to describe solutions of these types of PDEs [5]. He first showed how the wave equation on complexified Minkowski space was related to a contour integral in complex geometry. A twistor space is a space Z with a projection to a space M such that its fibres are complex manifolds. Penrose's twistor space is a complex 4 dimensional space Z which has a double fibration to the complex Grassmanian and to $\mathbb{CP}^3$. Based on this construction, Richard Ward [14] showed that instantons on complexified Minkowski space correspond to holomorphic vector bundles on $\mathbb{CP}^3$. Other correspondences came about such as instantons on the 4-sphere correspond to holomorphic bundles on complex projective 3-space [3] where instantons are special kinds of solutions to the Yang-Mills equations in 4d. They are connections whose curvature is self-dual or anti-self-dual. Atiyah, Hitchin, and Singer further developed [2] the ideas of Penrose in the setting of 4-manifolds and produced the paper "Self duality in four dimensional Riemannian geometry". The Atiyah-Hitchin-Singer theorem states that the twistor almost complex structure on a certain S^2 bundle over an oriented Riemannian 4-manifold (M, g) is integrable if and only if the Weyl curvature tensor of g is self-dual. The Newlander-Nirenberg Theorem states that the vanishing of the Nijenhuis tensor corresponds to the integrability of an almost complex structure. In this thesis we show the Nijenhuis tensor vanishes hence the almost complex structure is integrable if and only if the anti-self-dual part of the Weyl curvature vanishes.

In section 2 we introduce the vertical bundle, horizontal bundle, and horizontal lifts of a vector fields. We then describe the flow of the horizontal lift which has the property that it preserves the cross product and is an isometry in the vertical direction. The Lie bracket of two horizontal lifts involves the curvature which will be needed in calculating the Nijenhuis tensor. We then show how horizontal lifts act on the tautological 2-form.

In section 3 we set up some linear algebra for our later calculations. We introduce orthogonal complex structures and orientations. We show how 2-forms can be related to endomorphisms, and define an almost complex structure on the 2-sphere.

In section 4, the results of the previous sections are applied in the case of the vector bundle $E = \Lambda^2_-(T^*M)$. We consider the $\sqrt{2}$-sphere bundle $Z \subset E$ and define an almost complex structure on Z compatible with the horizontal and vertical splitting. The almost complex structure on HZ uses the tautological 2-form, while the almost complex structure on VZ is related to the quaternions and the cross product.

In section 5 we calculate the Nijenhuis tensor of the almost complex structure defined in section 4. We show that the Nijenhuis tensor of a horizontal lift and of a vertical vector field vanish and the Nijenhuis tensor of two horizontal lifts reduces to curvature. The Nijenhuis tensor of two vertical vector fields vanishes and its calculation depends on the cross product on the fibre of E. Through the use of quaternions in particular their relation to the cross product together with the Kulkarni-Nomizu product, we arrive at a condition for integrability only involving the anti-self-dual part of the Weyl curvature. We then show that this condition is satisfied if and only if the anti-self-dual part of the Weyl curvature vanishes

In section 6 we show how self-dual instantons are related to holomorphic vector bundles through the use of the techniques we built up in the previous sections.

In section 7 we apply the techniques of the previous sections to an 8-dimensional generalization where the base is $\mathbb{R}^8$ with standard Spin(7) structure and we calculate the Nijenhuis tensor.

Some further references for further reading related to this topic are [12], [11], [6], [14], [7], [10] and [1].

2 Vector bundles

Let E be a rank k vector bundle over a smooth manifold M. Let $\pi : E \to M$ be the projection map. Let $U \subseteq M$ be a local chart on M. For $p \in M$, let $E_p = \pi^{-1}(p)$ be the fibre of E over p and $v \in E_p$.

Recall that a section $\sigma \in \Gamma(E)$ is a smooth map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathbb{I}_M$. Suppose that E can be trivialized over U and let $\phi_U : E_U \to U \times \mathbb{R}^k$ be a trivialization map. The standard basis in $\mathbb{R}^k$ induces via ϕ_U a local frame $\{\sigma_i\}_{i=1}^k$ for E_U. The local coordinates induced on E_U by the local frame and the local coordinates on U are $(x^1, \dots, x^n, y^1, \dots, y^k)$. In particular, a local section $\sigma \in \Gamma(E_U)$ can be given by $\sigma = \sum_{i=1}^k y^i \sigma_i$. The induced coordinates on E are given by the map

$$(y^i \sigma_i)_p \mapsto (p, y^1, y^2, y^3, \dots, y^k) \tag{1}$$

where $p = (x^1, \dots, x^n)$. These adapted local coordinates for E are the only ones we will use.

Recall the pullback bundle of E along π is denoted as $\pi^* E$, where $\pi^* E = \{(u, v) \in E \times E : \pi(u) = \pi(v)\}$. For $v \in E_p$ we have $(\pi^* E)_v = E_p$.

Definition 2.1. Let E^* be the dual bundle of E. A fibre metric on E is a section

$g_E \in \Gamma(E^* \otimes E^*)$ so that $g_E(\cdot,\cdot)$ is symmetric and positive definite inner product for each $p \in M$.

2.1 The vertical subbundle VE

The pushforward by π is denoted by $\pi_* : TE \to TM$ and we define the vertical subbundle of TE as $\ker(\pi_*) = VE$. We will show that a local frame for the vertical space is given by $\left\{\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \ldots, \frac{\partial}{\partial y^k}\right\}$. We denote $V_\upsilon E := (VE)_\upsilon$ for $\upsilon \in E$.

Lemma 2.2. *(Vector Bundle Construction Lemma [9] page 108)*
Let M be a smooth manifold and suppose we are given:

- *for each $p \in M$, a real vector space E_p of some fixed dimension k*
- *for each $\alpha \in A$ a bijective map $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose restriction to E_p is a linear isomorphism from E_p to $\{p\} \times \mathbb{R}^k$*
- *for each $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$, a smooth map $\rho_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ such that the composite map $\Phi_\alpha \circ \Phi_\beta^{-1}$ from $(U_\alpha \cap U_\beta) \times \mathbb{R}^k$ to itself has the form $\Phi_\alpha \circ \Phi_\beta^{-1} = (p, \rho_{\alpha\beta}(p)v)$*

Then E has a unique smooth manifold structure making it into a smooth vector bundle of rank k over M, with π as projection and the maps Φ_α as smooth local trivializations.

Theorem 2.3. *Let $(\upsilon, \zeta) \in (\pi^*E)_\upsilon$. The vertical subbundle VE is isomorphic to the pullback bundle π^*E via the map*

$$E_{\pi(\upsilon)} \ni \zeta \xrightarrow{l_\upsilon} \left.\frac{d}{dt}\right|_{t=0} (\upsilon + t\zeta) \in V_\upsilon E,$$

where $\left.\frac{d}{dt}\right|_{t=0} (\upsilon + t\zeta) \in T_\upsilon(E_{\pi(\upsilon)}) \cong E_{\pi(\upsilon)}$.

Proof. Let $\upsilon \in E_{\pi(\upsilon)}$ and define a map $l_\upsilon : (\pi^*E)_\upsilon \to V_\upsilon E$ by

$$l_\upsilon : (\upsilon, \zeta) \mapsto \left.\frac{d}{dt}\right|_{t=0} (\upsilon + t\zeta) \in V_\upsilon E. \tag{2}$$

The map l_υ is one-to-one and onto and linear since $\frac{d}{dt}$ is linear hence it is an isomorphism of vector spaces. Then the inverse $l_\upsilon^{-1} : V_\upsilon E \to (\pi^* E)_\upsilon$ is given by

$$\frac{d}{dt}\bigg|_{t=0}(\upsilon + t\zeta) \mapsto (\upsilon, \zeta). \tag{3}$$

We will show $\pi^* E \cong VE$ using the vector bundle construction Lemma. Given a frame $\{\sigma_i\}$ for E over U, we obtain a local frame $\{\pi^*\sigma_i\}$ for the pullback bundle $\pi^* E$ over E_U. Let $(\upsilon, \kappa), (\upsilon, \kappa') \in (\pi^* E)_\upsilon$. We will show that the transition functions on VE come from the transition functions on $\pi^* E$ and by Lemma 2.2, VE will be a vector bundle. As $\pi^* E$ is a vector bundle $\pi^* E = \bigsqcup_\upsilon (\pi^* E)_\upsilon$. Let $\{U_\alpha\} \subset M$ be a trivializing open cover for E. Then on the intersection $U_\alpha \cap U_\beta$, there exists $\rho_{\alpha\beta}$ on $\pi^*(E_{U_\alpha \cap U_\beta})$ such that $(\upsilon, \kappa) = (\upsilon, \rho_{\alpha\beta}\kappa')$. Since l is linear and $\rho_{\alpha\beta}$ is linear we get

$$\begin{aligned} l_\upsilon(\upsilon, \kappa) &= l_\upsilon(\rho_{\alpha\beta}(p)\upsilon, \rho_{\alpha\beta}(p)\kappa') \\ &= \frac{d}{dt}\Big|_{t=0}(\rho_{\alpha\beta}(p)\upsilon + t\rho_{\alpha\beta}(p)\kappa') \\ &= \rho_{\alpha\beta}(p)\frac{d}{dt}\Big|_{t=0}(\upsilon + t\kappa') \\ &= (\rho_{\alpha\beta}(p))l_\upsilon(\upsilon, \kappa')). \end{aligned} \tag{4}$$

Hence the transition functions for $(\pi^* E)_\upsilon$ are the same as the transition functions on $V_\upsilon E$ and the map l_υ varies smoothly over the fibres and similarly for the map l_υ^{-1}. Then by Lemma 2.2 we get $VE = \bigsqcup_\upsilon V_\upsilon E$, where VE is constructed from the transition functions on $\pi^* E$, and we get vector bundle isomorphisms $l : \pi^* E \to VE$ and $l^{-1} : VE \to \pi^* E$. $\square$

Corollary 2.4. *Under the isomorphism between $\pi^* E$ and VE, and for local frames $\{\pi^*\sigma_i\}$, $\{\frac{\partial}{\partial y^i}\}$ of $\pi^* E$ and VE respectively, the map l sends $\{\pi^*\sigma_i\} \to \{\frac{\partial}{\partial y^i}\}$.*

Proof. Recall that for $U \subset M$, U a trivializing open set for E

$$\sigma_i(p) = \phi_U^{-1}(p, (0, 0, \ldots, 1, 0 \ldots, 0))$$

where the 1 is in the ith position. Let $\gamma(t) = \upsilon + t(0, 0, \ldots, 1, 0 \ldots, 0)$ where the 1 is in the ith position. For $p \in M$, f a function on E_p, then $\gamma_* \frac{\partial}{\partial t}$ at $t = 0$ applied to f is

$$\left(\gamma_* \frac{\partial}{\partial t}\right) f = \frac{d}{dt}\bigg|_{t=0} f(\upsilon + t(0, 0, \ldots, 1, 0 \ldots, 0)) = \frac{\partial}{\partial y^i}\bigg|_{\upsilon} f. \tag{5}$$

Then $l_\upsilon((\pi^*\sigma_i)_\upsilon)f = \gamma_* \frac{\partial}{\partial t} f = \frac{\partial}{\partial y^i}\big|_\upsilon f$. □

Definition 2.5. The fundamental vector field $\xi \in \Gamma(VE)$ is given by

$$\xi_\upsilon = \frac{d}{dt}\bigg|_{t=0} (\upsilon + t\upsilon).$$

Theorem 2.6. *For $\upsilon \in E$, $\xi_\upsilon \in V_\upsilon E$ is given in local coordinates by $\xi_\upsilon = y_i \frac{\partial}{\partial y_i}\big|_\upsilon$.*

Proof. Let $\upsilon \in E$ and $\xi_\upsilon \in V_\upsilon E$. Then $(\upsilon, \upsilon) = y^i(\pi^*\sigma_i)_\upsilon$ and $\xi_\upsilon = l_\upsilon(\upsilon, \upsilon) = \frac{d}{dt}\big|_{t=0}(\upsilon + t\upsilon) \in V_\upsilon E$. By Corollary 2.4 we get

$$\begin{aligned} l_\upsilon(\upsilon, \upsilon) &= l_\upsilon(y^i(\pi^*\sigma_i)_\upsilon) \\ &= y^i l_\upsilon((\pi^*\sigma_i)_\upsilon) \\ &= y^i \frac{\partial}{\partial yi}\bigg|_\upsilon . \quad \square \end{aligned} \tag{6}$$

Given that we have $\pi : E \to M$, we have the induced map $\pi_* : TE \to TM$. The bundle TE will have a local frame $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^k}\}$.

For the local coordinates on E given by (1), let π^i be defined by

$$\pi^i(x^1, \ldots, x^n, y^1, \ldots, y^k) = x^i. \tag{7}$$

The matrix of π_* is given by

$$\begin{bmatrix} \frac{\partial \pi^1}{\partial x^1} & \cdots & \frac{\partial \pi^1}{\partial x^n} & \frac{\partial \pi^1}{\partial y^1} & \cdots & \frac{\partial \pi^1}{\partial y^k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \pi^n}{\partial x^1} & \cdots & \frac{\partial \pi^n}{\partial x^n} & \frac{\partial \pi^n}{\partial y^1} & \cdots & \frac{\partial \pi^n}{\partial y^k} \end{bmatrix} = \begin{pmatrix} I_{n\times n} & 0_{n\times k} \end{pmatrix}$$

and as a consequence we see that for $\upsilon \in E_p$, $c^i \frac{\partial}{\partial x^i}\big|_\upsilon + b^i \frac{\partial}{\partial y_i}\big|_\upsilon \in T_\upsilon E$ then

$$\begin{aligned} \pi_*(c^i \frac{\partial}{\partial x^i}\big|_\upsilon + b^i \frac{\partial}{\partial y^i}\big|_\upsilon) &= \pi_*(c^i \frac{\partial}{\partial x^i}\big|_\upsilon) + \pi_*(b^i \frac{\partial}{\partial y^i}\big|_\upsilon) \\ &= c^i \frac{\partial}{\partial x^i}\big|_p \end{aligned} \tag{8}$$

and hence $\ker \pi_*$ has the basis $\left\{ \frac{\partial}{\partial y^i}\big|_\upsilon \right\}$.

Theorem 2.7. *Let $A \in \Gamma(TM)$. Suppose $A_1 \in \Gamma(TE)$ and assume that $\pi_* A_1 = A$. Let $\phi_t^A : M \to M$ and $\phi_t^{A_1} : E \to E$ be the flows of A and A_1 respectively. Then the following diagram commutes for all t.*

$$\begin{array}{ccc} E & \xrightarrow{\phi_t^{A_1}} & E \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\ M & \xrightarrow[\phi_t^{A}]{} & M \end{array}$$

Hence the flow $\phi_t^{A_1}$ is a family of fibre preserving diffeomorphisms.

Proof. See [9] Lemma 18.4 page 468. □

2.2 The horizontal lift

Let $\gamma : [0,1] \to U$ be an arbitrary curve in a coordinate chart $U \subset M$, given in local coordinates by $\gamma(t) = (x^1(t), x^2(t), \ldots, x^n(t))$ with $\gamma(0) = p$ and $\gamma'(0) = X_p \in T_p M$.

A lift of γ to E is denoted $\widehat{\gamma} \in \Gamma(\gamma^* E)$. It is given in the local coordinates defined by a local frame $\{\sigma_i\}$ of E in (1) by

$$\begin{aligned}\widehat{\gamma}(t) &= (x^1(t), x^2(t), \ldots, x^n(t), y^1(t), y^2(t), \ldots, y^k(t)) \\ &= (\gamma(t), q(t)).\end{aligned} \tag{9}$$

where $\gamma(t) = (x^1(t), x^2(t), \ldots, x^n(t))$ and $q(t) = (y^1(t), y^2(t), \ldots, y^k(t))$. In other words, $\widehat{\gamma} = y^i \sigma_i$. In this chart the derivative of $\widehat{\gamma}$ is given by

$$\widehat{\gamma}' = (\gamma'(t), q'(t)).$$

More explicitly

$$q'(t) = \frac{dy^a}{dt} \frac{\partial}{\partial y^a}\bigg|_{\widehat{\gamma}}$$

and

$$\gamma'(t) = \frac{dx^i}{dt} \frac{\partial}{\partial x^i}\bigg|_{\gamma} = X^i(t) \frac{\partial}{\partial x^i}\bigg|_{\gamma}.$$

Then we have

$$\widehat{\gamma}'(t) = X^i(t) \frac{\partial}{\partial x^i}\bigg|_{\widehat{\gamma}} + \frac{dy^a}{dt} \frac{\partial}{\partial y^a}\bigg|_{\widehat{\gamma}}. \tag{10}$$

Let ∇ be a connection on E. For $\sigma \in \Gamma(E)$ and $h \in C^\infty(M)$ then $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ and $\nabla(\sigma h) = dh \otimes \sigma + h \nabla \sigma$. Denote $\nabla_i := \nabla_{\partial_{x_i}}$.

Definition 2.8. Given a local frame $\{\sigma_a\}$ on E, and a connection ∇ on E, the connection coefficients Γ^b_{ai} are defined by $\nabla_i \sigma_a := \Gamma^b_{ai} \sigma_b$.

Definition 2.9. Let $\gamma : [0, 1] \to M$ be a smooth curve, and ∇ a connection on E. A lifting $\widehat{\gamma} : [0, 1] \to E$ of γ to E is horizontal with respect to ∇ if $(\gamma^* \nabla)\widehat{\gamma} = 0$, where $\gamma^* \nabla$ is the pullback connection on $\gamma^* E$.

We will solve the equation

$$(\gamma^* \nabla)\widehat{\gamma} = (\nabla_{\gamma'(t)} \gamma^*(y^a \sigma_a)) = 0. \tag{11}$$

We use $\{\gamma^*\sigma_1, \gamma^*\sigma_2, \ldots, \gamma^*\sigma_k\}$ as the induced frame on γ^*E from the local frame $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ on E. We drop the γ^* for convenience.

In local coordinates $(\gamma^*\nabla)\widehat{\gamma} = 0$ is equivalent to

$$\begin{aligned}\nabla_{\gamma'}\widehat{\gamma} &= \frac{dy^a}{dt}\sigma_a + y^a\nabla_{\gamma'}\sigma_a \\ &= \frac{dy^a}{dt}\sigma_a + y^a X^i\Gamma^b_{ai}\sigma_b \\ &= (\frac{dy_b}{dt} + y^a X^i\Gamma^b_{ai})\sigma_b \\ &= 0 \end{aligned} \tag{12}$$

which gives the system of equations for (y^a)'s

$$\frac{dy^b}{dt} = -\Gamma^b_{ai}X^i y^a. \tag{13}$$

We substitute equation (13) into equation (10) yielding

$$\widehat{\gamma}'(t) = X^i\frac{\partial}{\partial x^i}\Big|_{q(t)} - \Gamma^b_{ai}X^i y^a\frac{\partial}{\partial y^b}\Big|_{q(t)}. \tag{14}$$

Notice that $\widehat{\gamma}'(t)$ given in (14) only depends on ∇ and $\gamma'(t)$. This section of $\gamma^*(TE)$ is known as a horizontal lift of $X \in \Gamma(TM)$ along γ. Soon we will see that this will lead to a splitting of TE into horizontal and vertical subbundles where the horizontal lift is a section of the horizontal bundle.

Definition 2.10. Let $\gamma : [0,1] \to M$ be a curve in M. A horizontal lift of γ at $\upsilon \in E_p$ is $\gamma^h_\upsilon(t) : [0,1] \to E$, such that $\gamma^h_\upsilon(0) = \upsilon$, $\pi \circ \gamma^h_\upsilon = \gamma$ and γ^h_υ is horizontal.

Theorem 2.11. *For $\upsilon \in E_p$, in local coordinates $(x^1, \ldots, x^n)$ on M, let*

$$H_\upsilon E = \{X^h_\upsilon := X^i\frac{\partial}{\partial x^i} - \Gamma^b_{ai}y^a X^i\frac{\partial}{\partial y^b} \in T_\upsilon E \Big| X = X^i\frac{\partial}{\partial x^i} \in T_pM\}. \tag{15}$$

Then $H_\upsilon E$ is a subspace of $T_\upsilon E$ and

$$h_\upsilon : T_pM \to T_\upsilon E, \qquad X^i \frac{\partial}{\partial x^i} \mapsto X^i \frac{\partial}{\partial x^i} - \Gamma^b_{ai} y^a X^i \frac{\partial}{\partial y^b} \tag{16}$$

is a linear injection and $HE = \bigsqcup_{\upsilon \in E} H_\upsilon E$ is a vector bundle.

Proof. Let $X, Z \in T_pM$, and $X^h, Z^h \in T_\upsilon E$ be the respective horizontal lifts at υ. Denote $\Gamma^a_{bi} = \Gamma^a_{bi}(p)$ and $y^\alpha = y^\alpha(\upsilon)$. (We are dropping the p and υ for convenience). Write $X = X^i \frac{\partial}{\partial x^i}, Z = Z^i \frac{\partial}{\partial x^i}$. Let

$$\begin{aligned} X^h &= X^i \frac{\partial}{\partial x^i} - \Gamma^b_{ai} y^a X^i \frac{\partial}{\partial y^b}, \\ Z^h &= Z^i \frac{\partial}{\partial x^i} - \Gamma^b_{ai} y^a Z^i \frac{\partial}{\partial y^b}, \end{aligned} \tag{17}$$

and for $A, B \in \mathbb{R}$,

$$\begin{aligned} &AX^h + BZ^h \\ &= (AX^i + BZ^i) \frac{\partial}{\partial x^i} - \Gamma^b_{ai} (AX^i + BZ^i) y^a \frac{\partial}{\partial y^b} \end{aligned} \tag{18}$$

hence $AX^h + BZ^h$ is a horizontal lift of $AX + BX$ at υ.

We show that π_* is a left inverse of h:

$$\begin{aligned} &(\pi_* \circ h)(X) \\ &= (\pi_* \circ h)(X^i \frac{\partial}{\partial x^i}) \\ &= \pi_*(X^i \frac{\partial}{\partial x^i} - \Gamma^b_{ai} y^a X^i \frac{\partial}{\partial y^b}) \\ &= \pi_*(X^i \frac{\partial}{\partial x^i}) - \Gamma^b_{ai} y^a X^i \pi_*(\frac{\partial}{\partial y^b}) \\ &= X^i \frac{\partial}{\partial x^i} \\ &= X. \end{aligned} \tag{19}$$

Thus π_* is the inverse of h when restricted to the codomain $\mathrm{im}(h)$ and we showed $H_\upsilon E \cong T_pM$ is a linear isomorphism. Given a local frame for TM using the horizontal lift one can construct a local frame for HE.

The map h is linear and $HE = \bigsqcup_\upsilon H_\upsilon E \cong \bigsqcup_\upsilon (\pi^*TM)_\upsilon$, the transition functions for HE on $U_\alpha \cap U_\beta \subset M$ will be exactly the transition functions for TM on $U_\alpha \cap U_\beta \subset M$. Similar to Theorem 2.3, the bundle structure on HE is built from the linear injection and from the bundle structure on $\bigsqcup_\upsilon (\pi^*TM)_\upsilon = \pi^*TM$. □

We have now that $TE = \ker(\pi_*) \oplus \mathrm{im}(h) = VE \oplus HE$.

Definition 2.12. The horizontal lift of $X \in \Gamma(TM)$ is $X^h \in \Gamma(HE)$ such that at each point $p \in M, \upsilon \in E_p$, $X^h_\upsilon = h_\upsilon(X_p) \in H_\upsilon E \subset T_\upsilon E$.

We now calculate the integral curves of

$$X^h = X^i \frac{\partial}{\partial x^i} - X^i_p \Gamma^b_{ai} y^a \frac{\partial}{\partial y^b}. \tag{20}$$

We need to solve the equations

$$\frac{dx^i}{dt} = X^i, \qquad \frac{dy^a}{dt} = -\Gamma^b_{ai} X^i y^a. \tag{21}$$

with initial conditions $\gamma(0) = (x^1(0), \ldots, x^n(0)) = p, y(0) = (y^1(0), \ldots, y^k(0))$ such that $\upsilon = y^a(0)\sigma_a$. The solution is given by

$$\begin{aligned} x^i(t) &= \gamma^i(t) \\ y^a(t) &= e^{-\int_0^t \Gamma^b_{ai} X^i dt} y^a(0). \end{aligned} \tag{22}$$

where $\gamma(t)$ is the integral curve of X with initial conditions $\gamma(0) = p$. Then the flow for X^h is given by

$$\phi^{X^h}_t(\upsilon) := (\gamma(t), e^{-\int_0^t \Gamma^b_{ai} X^i dt} y^a(0)\sigma_b). \tag{23}$$

Let X^h be the horizontal lift of X, ϕ_t^X be the flow of X then $\phi_t^{X^h}(E_p) \subset E_{\phi_t^X(p)}$ and $\phi_t^{X^h}$ is a fibrewise morphism.

2.3 Properties of the flow $\phi_t^{X^h}$

We now define parallel transport of a point $\upsilon \in E$ along a curve $\gamma : [0,1] \to M$.

Definition 2.13. The parallel transport of $\upsilon \in E_p$ along γ starting at $\gamma(0) = p$ is the endpoint of the unique horizontal lift $\gamma_\upsilon^h(t)$ of $\gamma(t)$ such that $\gamma^h(0) = \upsilon$. For each $\upsilon \in E_p$ the integral curve of X^h associated with the flow $\phi_t^{X^h}$ is $\phi_t^{X^h}(\upsilon) := \gamma_\upsilon^h(t)$.

Let $g = \langle \cdot, \cdot \rangle_E$ be a metric on E and suppose ∇ preserves g. If $\{\sigma_a\}$ is a local frame for E, this means that

$$X\langle \sigma_a, \sigma_b \rangle_E = \langle \nabla_X \sigma_a, \sigma_b \rangle_E + \langle \sigma_a, \nabla_X \sigma_b \rangle_E. \tag{24}$$

Let $X \in \Gamma(TM)$ and X^h be its horizontal lift to TE. We claim that $\phi_t^{X^h}$ is an isometry in the vertical direction, that is $\phi_t^{X^h}$ preserves the metric g on E.

Theorem 2.14. *$\phi_t^{X^h} : E \to E$ is an isometry with respect to g the metric on E.*

Proof. Let $\{\sigma_a\}$ be a local orthonormal frame for E, in local coordinates $(x^1, \ldots, x^n)$, let $X \in \Gamma(TM)$, $X = X^i \frac{\partial}{\partial x^i}$. Then

$$\begin{aligned}
0 &= X\langle \sigma_a, \sigma_b \rangle_E \\
&= \langle \nabla_X \sigma_a, \sigma_b \rangle_E + \langle \sigma_a, \nabla_X \sigma_b \rangle_E \\
&= X^i(\langle \Gamma_{ai}^c \sigma_c, \sigma_b \rangle_E + \langle \sigma_a, \Gamma_{bi}^d \sigma_d \rangle_E) \\
&= X^i(\Gamma_{ai}^c \delta_{cb} + \Gamma_{bi}^d \delta_{ad}) \\
&= X^i(\Gamma_{ai}^b + \Gamma_{bi}^a).
\end{aligned} \tag{25}$$

Hence Γ_{ai}^b is antisymmetric in a and b thus the exponent in (23) is antisymmetric in a

and b. It follows that $\phi_t^{X^h}$ given by (23) is an orthogonal transformation in the second argument hence an isometry in the vertical direction. □

Next we show that $(\phi_t^{X^h})_*$ is an isometry when restricted to the subbundle VE.

Corollary 2.15. *The pushforward of* $\phi_t^{X^h}$, $(\phi_t^{X^h})_*$, *is an isometry with respect to the metric* $\langle\cdot,\cdot\rangle_{VE}$ *on* VE.

Proof. By (23) the pushforward of $\phi_t^{X^h}$, $(\phi_t^{X^h})_*$ is given in local coordinates $(\phi_t^{X^h})_*$ by

$$\begin{aligned}
(\phi_t^{X^h})_*(\frac{\partial}{\partial x^i}) &= \frac{\partial}{\partial x^i} + \frac{\partial y^b}{\partial x^i}\frac{\partial}{\partial y^b} \\
&= \frac{\partial}{\partial x^i} + \frac{\partial(e^{-\int_0^t \Gamma^b_{ai}X^i dt}y^a(0))}{\partial x^i}\frac{\partial}{\partial y^b} \\
(\phi_t^{X^h})_*(\frac{\partial}{\partial y^j}) &= \frac{\partial x^b}{\partial y^j}\frac{\partial}{\partial x^b} + \frac{\partial y^b}{\partial y^j}\frac{\partial}{\partial y^b} \\
&= (0)\frac{\partial}{\partial x^b} + e^{-\int_0^t \Gamma^b_{ai}X^i dt}\frac{\partial}{\partial y^b}
\end{aligned} \tag{26}$$

In matrix form

$$(\phi_t^{X^h})_* = \left(\begin{array}{c|c} 1 & 0 \\ \hline \frac{\partial(e^{-\int_0^t \Gamma^b_{ai}X^i dt}y^a(0))}{\partial x^i} & e^{-\int_0^t \Gamma^b_{ai}X^i dt} \end{array}\right).$$

The restriction of $(\phi_t^{X^h})_*$ to VE is $(\phi_t^{X^h})_*\big|_{VE} = e^{-\int_0^t \Gamma^b_{ai}X^i dt}$. By Theorem 2.14, $e^{-\int_0^t \Gamma^b_{ai}X^i dt}$ is an orthogonal transformation hence an isometry. Thus $(\phi_t^{X^h})_*\big|_{VE}$ is an isometry.

We make a note that the metric on E and the metric on VE are related by $\langle l(\alpha), l(\beta)\rangle_{VE} = \langle \alpha,\beta\rangle_E$ for $\alpha,\beta \in E_p$ where l is defined in Theorem 2.3. □

2.4 The Lie bracket of two horizontal lifts

For $s \in \Gamma(E)$, let $s^v := \pi^* s \in \Gamma(\pi^* E) \cong \Gamma(VE)$ be the section of VE using the identification of $\pi^* E$ and VE in Theorem 2.4, and is called the vertical lift.

For $f \in C^\infty(M)$, let $f^v := \pi^* f \in C^\infty(E)$.

Lemma 2.16. *Let X^h be the horizontal lift of $X \in \Gamma(TM)$, let $f \in C^\infty(M)$ then $(Xf)^v = X^h f^v$.*

Proof. Working out $X^h f^v$ yields

$$\begin{aligned} X^h f^v &= (\pi^* df)(X^h) \\ &= (df(\pi_* X^h)) \\ &= (df(X)) \circ \pi \\ &= (Xf) \circ \pi \\ &= \pi^*(Xf) \\ &= (Xf)^v. \end{aligned} \tag{27}$$

□

Theorem 2.17. *Let $s^v \in \Gamma(VE)$ be the vertical lift as above, let X^h be the horizontal lift of $X \in \Gamma(TM)$. Then*

$$(\nabla_X s)^v = [X^h, s^v].$$

Proof. Let $f \in C^\infty(M)$ and $s \in \Gamma(E)$. Then $s^v \in \Gamma(VE)$ is a vertical vector field on E. Note that $\pi^*(fs) = \pi^* f \pi^* s = f^v s^v$. Then $((Xf)s)^v = (X^h f^v)s^v$ and we have

$$\begin{aligned} (\nabla_X(fs))^v &= (f\nabla_X s + (Xf)s)^v \\ &= f^v(\nabla_X s)^v + (Xf)^v s^v \end{aligned} \tag{28}$$

and

$$(\nabla_{fX} s)^v = (f\nabla_X s)^v) = f^v(\nabla_X s)^v \tag{29}$$

Evaluating $[X^h, (fs)^v] = [X^h, f^v s^v]$ and $[(fX)^h, s^v]$ and noting that $s^v(f^v) = 0$, we have

$$
\begin{aligned}
[X^h, f^v s^v] &= f^v[X^h, s^v] + (X^h f^v)s^v \\
[(fX)^h, s^v] &= [f^v X^h, s^v] \\
&= f^v[X^h, s^v] - s^v(f^v)X^h \\
&= f^v[X^h, s^v]. \qquad (30)
\end{aligned}
$$

Equation (30) shows that $[X^h, s^v]$ has all the properties of a connection. In a local frame $\{\sigma_m\}$ for E, $k_m \in C^\infty(M)$, $s = \sum_m k_m\sigma_m$ then $s^v = \pi^*(\sum_m k_m\sigma_m) = \sum_m k_m^v\sigma_m^v$. Using Corollary 2.4 then $\sigma_m^v \in \Gamma(VE)$ coincides with $\frac{\partial}{\partial y^m}$. Let $X = \frac{\partial}{\partial x^i}$ so $X^h = \frac{\partial}{\partial x^i} - \Gamma_{ia}^b y^a \frac{\partial}{\partial y^b}$ then

$$
\begin{aligned}
\left[X^h, \frac{\partial}{\partial y^m}\right] &= \left[\frac{\partial}{\partial x^i} - \Gamma_{ia}^b y^a \frac{\partial}{\partial y^b}, \frac{\partial}{\partial y^m}\right] \\
&= \Gamma_{ia}^b \delta_m^a \frac{\partial}{\partial y^b} \\
&= \Gamma_{im}^b \frac{\partial}{\partial y^b}.
\end{aligned}
$$

It follows that $[X^h, \sigma_m^v] = (\nabla_{\frac{\partial}{\partial x^i}}\sigma_m)^v$ Using Corollary 2.4. (31)

By (30) $[X^h, s^v] = (\nabla_X s)^v$ for all $X \in \Gamma(TM)$ and $s \in \Gamma(E)$. □

Theorem 2.18. *For $X^h, Y^h \in \Gamma(HE)$ horizontal lifts of $X, Y \in \Gamma(TM)$ respectively, $\eta \in E$ then*

$$
[X^h, Y^h]_\eta - [X, Y]_\eta^h = -R(X, Y)\eta \qquad (32)
$$

where $R(X, Y) : \Gamma(E) \to \Gamma(E)$ is the curvature operator for ∇. The right hand side $R(X, Y)\eta$ is interpreted as an element in $T_\eta E$ via the identification $\pi^*E \cong VE$.

Proof. Let $(x^1, \dots, x^n, y^1, y^2, \dots, y^k)$ be adapted local coordinates on E from (1). Let

$X = \frac{\partial}{\partial x^i}$ and $Y = \frac{\partial}{\partial y^i}$. Let X^h and Y^h be horizontal lifts of X and Y. We compute

$$
\begin{aligned}
[X^h, Y^h] = [X^i(\frac{\partial}{\partial x^i} - \Gamma^b_{ia} y^a \frac{\partial}{\partial y^b}), Y^j(\frac{\partial}{\partial x^j} - \Gamma^e_{jd} y^d \frac{\partial}{\partial y^e})]\\
= [X^i \frac{\partial}{\partial x^i}, Y^j \frac{\partial}{\partial x^j}]\\
- [X^i \frac{\partial}{\partial x^i}, Y^j \Gamma^e_{jd} y^d \frac{\partial}{\partial y^e}]\\
- [X^i \Gamma^b_{ia} y^a \frac{\partial}{\partial y^b}, Y^j \frac{\partial}{\partial x^j}]\\
+ [X^i \Gamma^b_{ia} y^a \frac{\partial}{\partial y^b}, Y^j \Gamma^e_{jd} y^d \frac{\partial}{\partial y^e}].
\end{aligned} \tag{33}
$$

Expanding the Lie brackets

$$
\begin{aligned}
[X^h, Y^h] = (X^i(\frac{\partial Y^k}{\partial x^i}) - Y^j(\frac{\partial X^k}{\partial x^j}))\frac{\partial}{\partial x^k}\\
- (X^i \frac{\partial Y^j}{\partial x^i} \Gamma^e_{jd} y^d \frac{\partial}{\partial y^e} + X^i \frac{\partial \Gamma^e_{jd}}{\partial x^i} y^d Y^j \frac{\partial}{\partial y^e})\\
+ (Y^j \frac{\partial X^i}{\partial x^j} \Gamma^b_{ia} y^a \frac{\partial}{\partial y^b} + y^a X^i Y^j \frac{\partial \Gamma^b_{ia}}{\partial x^j} \frac{\partial}{\partial y^b})\\
+ X^i \Gamma^b_{ia} Y^j \Gamma^e_{jd} (y^a \delta^d_b \frac{\partial}{\partial y^e} - y^d \delta^a_e \frac{\partial}{\partial y^b}).
\end{aligned} \tag{34}
$$

After rearranging and collecting terms and relabelling indicies, we get

$$
\begin{aligned}
[X^h, Y^h] &= (X^i(\frac{\partial Y^k}{\partial x^i}) - Y^j(\frac{\partial X^k}{\partial x^j}))\frac{\partial}{\partial x^k} - (X^i \frac{\partial Y^j}{\partial x^i} \Gamma^e_{jd} y^d - Y^j \frac{\partial X^i}{\partial x^j} \Gamma^e_{id} y^d) \frac{\partial}{\partial y^e}\\
&\quad - X^i Y^j (\frac{\partial \Gamma^e_{jd}}{\partial x^i} y^d \frac{\partial}{\partial y^e} - y^a \frac{\partial \Gamma^b_{ia}}{\partial x^j} \frac{\partial}{\partial y^b} - \Gamma^d_{ia} \Gamma^e_{jd} y^a \frac{\partial}{\partial y^e} + \Gamma^b_{ia} \Gamma^a_{jd} y^d \frac{\partial}{\partial y^b})\\
&= (X^i(\frac{\partial Y^k}{\partial x^i}) - Y^j(\frac{\partial X^k}{\partial x^j}))\frac{\partial}{\partial x^k} - (X^i \frac{\partial Y^j}{\partial x^i} \Gamma^e_{jd} y^d - Y^j \frac{\partial X^i}{\partial x^j} \Gamma^e_{id} y^d) \frac{\partial}{\partial y^e}\\
&\quad - X^i Y^j (\frac{\partial \Gamma^e_{jd}}{\partial x^i} y^d \frac{\partial}{\partial y^e} - y^d \frac{\partial \Gamma^e_{id}}{\partial x^j} \frac{\partial}{\partial y^e} - \Gamma^a_{id} \Gamma^e_{ja} y^d \frac{\partial}{\partial y^e} + \Gamma^e_{ia} \Gamma^a_{jd} y^d \frac{\partial}{\partial y^e}).
\end{aligned} \tag{35}
$$

Note that $[X, Y]^h = (X^i(\frac{\partial Y^k}{\partial x^i}) - Y^j(\frac{\partial X^k}{\partial x^j}))\frac{\partial}{\partial x^k} - (X^i \frac{\partial Y^j}{\partial x^i} \Gamma^e_{jd} y^d - Y^j \frac{\partial X^i}{\partial x^j} \Gamma^e_{id} y^d)\frac{\partial}{\partial y^e}$.

Rearranging, we get

$$[X^h, Y^h] - [X, Y]^h = -X^i Y^j y^d R^e_{ijd} \frac{\partial}{\partial y^e}. \tag{36}$$

Evaluating the expression at $\eta = y^a \sigma_a$,

$$[X^h, Y^h]_\eta - [X, Y]^h_\eta = -X^i Y^j y^d R^e_{ijd} \frac{\partial}{\partial y^e} = -R(X, Y)\eta, \tag{37}$$

where the last step is the identification of VE with $\pi^* E$. □

2.5 Tautological 2-form

On $\Lambda^2(T^*M)$ there is a natural tautological 2-form Θ. Let $\alpha \in \Lambda^2(T^*M)$, $X, Y \in T_\alpha(\Lambda^2(T^*M))$, then $\pi_* X, \pi_* Y \in T_{\pi(\alpha)} M$. The value of Θ at α is given by

$$\Theta_\alpha(X, Y) = \alpha_{\pi(\alpha)}(\pi_* X, \pi_* Y). \tag{38}$$

Let $(x^1, \ldots, x^n)$ be local coordinates on M then $\{dx^i \wedge dx^j | i < j\}$ is a local frame for $\Lambda^2(T^*M)$. The corresponding induced local coordinates on $T^*(\Lambda^2(T^*M))$ are $x^1, .., x^n$ and $y_{ij},\ 1 \leq i, j \leq n$.
In this local frame

$$\Theta = \frac{1}{2} y_{ij} (dx^i \wedge dx^j). \tag{39}$$

2.6 Horizontal vector fields acting on tautological 2-forms

For a Riemannian manifold M, let ∇^{TM} be the Levi-Civita connection on TM. Let $\{X_i\}$ be a local frame for TM and $\{\alpha^i\}$ its dual local frame for T^*M. The connection coefficents with respect to ∇^{TM} are Γ^k_{ij}, i.e. $\nabla^{TM}_i X_j = \Gamma^l_{ij} X_l$. The induced connection ∇^{T^*M} on T^*M is then given by

$$\nabla_i^{T^*M}\alpha^k := -\Gamma_{ij}^k\alpha^j. \tag{40}$$

An induced connection $\nabla^{T^*M\otimes T^*M}$ on $T^*M\otimes T^*M$ is defined as follows, for $\alpha,\beta\in\Gamma(T^*M)$ and $i=1,\dots,n$

$$\nabla_i^{T^*M\otimes T^*M}(\alpha\otimes\beta) = \nabla_i^{T^*M}\alpha\otimes\beta + \alpha\otimes\nabla_i^{T^*M}\beta. \tag{41}$$

The local frame on $T^*M\otimes T^*M$ is $\{\alpha^i\otimes\alpha^j \mid i=1\dots n, j=1\dots n\}$. The connection coefficents are given by

$$\nabla_i^{T^*M\otimes T^*M}(\alpha^k\otimes\beta^j) = (-\Gamma_{im}^k\alpha^m\otimes\beta^j) + (\alpha^k\otimes -\Gamma_{im}^j\beta^m). \tag{42}$$

Definition 2.19. For $\{\alpha^i\wedge\alpha^j \mid i=1\leq i<j\leq n\}$ the induced frame on Λ^2T^*M from the local frame $\{\alpha^i\otimes\alpha^j \mid 1\leq i,j\leq n\}$ of $T^*M\otimes T^*M$, the induced connection $\nabla^{\Lambda^2T^*M}$ is defined by

$$\nabla_k^{\Lambda^2(T^*M)}(\alpha^i\wedge\beta^j) = ((-\Gamma_{km}^i)\alpha^m\wedge\beta^j) + (\alpha^i\wedge(-\Gamma_{km}^j\beta^m)). \tag{43}$$

Following the construction in determining (14), using (40) and Theorem 2.11, we apply the derivation of the horizontal lift to the vector bundle $\Lambda^2(T^*M)$.
In the induced local coordinates for Λ^2T^*M, given by $(x^1,\dots,x^n,y_{ij})$ for $i<j$, solving equation (12) gives

$$0 = \frac{dy_{ij}}{dt} - \frac{dx^c}{dt}\Gamma_{ic}^d y_{dj} - \frac{dx^c}{dt}\Gamma_{cj}^d y_{id}.$$

Hence the horizontal lift of $X\in\Gamma(TM)$ to $X^h\in\Gamma(\Lambda^2T^*M)$ is given in local coordinates by

$$X^h = (X^c \frac{\partial}{\partial x^c})^h = X^c (\frac{\partial}{\partial x^c} + (\Gamma^d_{ic} y_{dj} + \Gamma^d_{cj} y_{id}) \frac{\partial}{\partial y_{ij}}).$$

For $X^h, Y^h \in \Gamma(T(\Lambda^2(T^*M)))$ horizontal lifts of $X, Y \in \Gamma(TM)$, let $\omega \in E_p$, let local coordinates on $T^*(\Lambda^2(T^*M))$ be $(x^1, .., x^n, y_{12}, .., y_{ij})$, $i < j$. Then applying this to (38) we get

$$\Theta(X^h, Y^h) = \frac{1}{2} \left(y_{ij} (dx^i \wedge dx^j)(X, Y) \right) \circ \pi. \tag{44}$$

Evaluating (44) at ω then

$$\Theta_\omega(X^h_\omega, Y^h_\omega) = \omega_{\pi(\omega)}(X_{\pi(\omega)}, Y_{\pi(\omega)}). \tag{45}$$

We now apply the horizontal lift to the tautological 2-form (45) and work out $Z^h(\Theta(X^h, Y^h))$.

Theorem 2.20. *Let Z^h, X^h, Y^h be horizontal lifts of $Z, X, Y \in \Gamma(TM)$. Let Θ be the tautological 2-form (45). Then*

$$Z^h(\Theta(X^h, Y^h)) = \Theta((\nabla_Z X)^h, Y^h) + \Theta(X^h, (\nabla_Z Y)^h).$$

Proof. Let $X = X^d \frac{\partial}{\partial x^d}, Y = Y^m \frac{\partial}{\partial x^m}, Z = Z^i \frac{\partial}{\partial x^i}$

$$
\begin{aligned}
&Z^h(\Theta(X^h, Y^h)) \\
&= Z^i(\tfrac{\partial}{\partial x^i})^h \Theta(X^l(\tfrac{\partial}{\partial x^l})^h, Y^m(\tfrac{\partial}{\partial x^m})^h) \\
&= Z^i(\tfrac{\partial}{\partial x^i})^h (X^l Y^m \Theta((\tfrac{\partial}{\partial x^l})^h, (\tfrac{\partial}{\partial x^m})^h)) \\
&= Z^i Y^m (\tfrac{\partial X^l}{\partial x^i}) \Theta((\tfrac{\partial}{\partial x^l})^h, (\tfrac{\partial}{\partial x^m})^h)) + Z^i X^l (\tfrac{\partial Y^m}{\partial x^i}) \Theta((\tfrac{\partial}{\partial x^l})^h, (\tfrac{\partial}{\partial x^m})^h)) + Z^i X^l Y^m ((\tfrac{\partial}{\partial x^i})^h (y_{lm})) \\
&= \Theta(Z(X^l)(\tfrac{\partial}{\partial x^l})^h, Y^h) + \Theta(X^h, Z(Y^m)(\tfrac{\partial}{\partial x^m})^h) + Z^i X^l Y^m ((\tfrac{\partial}{\partial x^i})^h (y_{lm})) \\
&= \Theta(Z(X^l)(\tfrac{\partial}{\partial x^l})^h, Y^h) + \Theta(X^h, Z(Y^m)(\tfrac{\partial}{\partial x^m})^h) + Z^i X^l Y^m (\Gamma^j_{li} y_{jm} + \Gamma^j_{im} y_{lj}).
\end{aligned} \tag{46}
$$

We now look at the term $Z^i X^l Y^m (\Gamma^j_{li} y_{jm} + \Gamma^j_{im} y_{lj})$ which becomes

$$
\begin{aligned}
&Z^i X^l Y^m \Gamma^j_{li} y_{jm} + Z^i X^l Y^m \Gamma^j_{im} y_{lj} \\
&= Z^i X^l Y^m y_{jm} (dx^j (\nabla_{\frac{\partial}{\partial x^i}} \tfrac{\partial}{\partial x^l})) + Z^i X^l Y^m y_{lj} (dx^j (\nabla_{\frac{\partial}{\partial x^m}} \tfrac{\partial}{\partial x^i})) \\
&= Z^i X^l Y^m y_{jm} (dx^j (\nabla_{\frac{\partial}{\partial x^i}} \tfrac{\partial}{\partial x^l})) + Z^i X^l Y^m y_{lj} (dx^j (\nabla_{\frac{\partial}{\partial x^m}} \tfrac{\partial}{\partial x^i})) \\
&= Z^i X^l y_{jm} (dx^j (\pi_* (\nabla_{\frac{\partial}{\partial x^i}} \tfrac{\partial}{\partial x^l})^h)) \wedge dx^m (\pi_* Y^h) \\
&\qquad + Z^i Y^m y_{lj} dx^l (\pi_* X^h) \wedge (dx^j \pi_* ((\nabla_{\frac{\partial}{\partial x^m}} \tfrac{\partial}{\partial x^i})^h)) \\
&= \Theta(X^l (\nabla_Z \tfrac{\partial}{\partial x^l})^h, Y^h) + \Theta(X^h, Y^m (\nabla_Z \tfrac{\partial}{\partial x^m})^h).
\end{aligned} \tag{47}
$$

Combining the equations (46) and (47) yields

$$
\begin{aligned}
Z^h(\Theta(X^h, Y^h)) &= \Theta(Z(X^l)(\tfrac{\partial}{\partial x^l})^h + X^l (\nabla_Z \tfrac{\partial}{\partial x^l})^h, Y^h) \\
&\qquad + \Theta(X^h, Z(Y^m)(\tfrac{\partial}{\partial x^m})^h + Y^m (\nabla_Z \tfrac{\partial}{\partial x^m})^h) \\
&= \Theta((\nabla_Z X)^h, Y^h) + \Theta(X^h, (\nabla_Z Y)^h). \quad \square
\end{aligned} \tag{48}
$$

3 Linear algebra

3.1 Orthogonal complex structures and oriented manifolds

Let V be a $2n$ dimensional real vector space with metric g. Let $\{e_1, \ldots, e_{2n}\}$ be an orthonormal basis for V. We let μ be the volume form on V defined by $\mu = e_1 \wedge \ldots \wedge e_{2n}$. Using the metric g on V we can identify elements of V^* with V, that is $e^i = g(e_i, \cdot)$. The volume form on V is $e^1 \wedge \ldots \wedge e^{2n}$. Let $\{\tilde{e_1}, \ldots, \tilde{e_{2n}}\}$ be another orthonormal frame with volume form $\tilde{\mu} = \tilde{e_1} \wedge \ldots \wedge \tilde{e_{2n}}$. Let $A \in O(n)$ so that $A^i_j e_i = \tilde{e_j}$ then the two volume forms μ and $\tilde{\mu}$ are related by

$$\det(A)(e_1 \wedge \ldots \wedge e_{2n}) = \tilde{e_1} \wedge \ldots \wedge \tilde{e_{2n}}. \tag{49}$$

When $\det(A) = +1$, the two bases define the same orientation and when $\det(A) = -1$ the two bases define opposite orientation.

We now define an orthogonal almost complex structure.

Definition 3.1. Let V be a $2n$ dimensional real vector space with metric g. An orthogonal complex structure on V is a linear map $J : V \to V$ such that $J^2 = -\mathrm{id}_V$ and $g(JX, JX) = g(X, X)$ for all $X \in V$.

Given an orthogonal complex structure J there is an orthonormal basis such that J is given by

$$J : e_{2i-1} \to e_{2i}, e_{2i} \to -e_{2i-1}$$

for all $i = 1, 2, \ldots, n$. Then J induces a natural orientation on V given by

$$e_1 \wedge Je_1 \wedge \ldots \wedge e_{2n-1} \wedge Je_{2n-1} \tag{50}$$

We show this orientation is well defined on V.

Theorem 3.2. *Given an orthogonal complex structure J there is an orthonormal* *basis* $\{e_i\}$ *such that* J *is given by* $J : e_{2i-1} \mapsto e_{2i}, e_{2i} \mapsto -e_{2i-1}$. *Then the orientation on* V *given by*

$$e_1 \wedge Je_1 \wedge \ldots \wedge e_{2n-1} \wedge Je_{2n-1} \tag{51}$$

is well defined, i.e. independent of the choice of $\{e_i\}$.

Proof. Let $\{\tilde{e}_i\}$ be another such basis. We can write

$$\begin{aligned} \tilde{e}_k &= A^i_k e_i + B^i_k Je_i \\ J\tilde{e}_k &= A^i_k Je_i - B^i_k e_i, \end{aligned} \tag{52}$$

where A^i_k, B^i_k are real $n \times n$ matrices. Subsituting (52) into

$$\tilde{e}_1 \wedge J\tilde{e}_1 \wedge \ldots \wedge \tilde{e}_{2n-1} \wedge J\tilde{e}_{2n-1} \tag{53}$$

we get

$$\begin{aligned} &(A^1_k e_1 + JB^1_k e_1) \wedge (A^1_k Je_1 - B^1_k e_1) \wedge \ldots \\ &\quad \wedge (A^{2n-1}_k e_{2n-1} + JB^{2n-1}_k e_{2n-1}) \wedge (A^{2n-1}_k Je_{2n-1} - B^{2n-1}_k e_{2n-1}). \end{aligned} \tag{54}$$

After rearranging we get

$$\det \begin{bmatrix} A & -B \\ B & A \end{bmatrix} e_1 \wedge Je_1 \wedge \ldots \wedge e_{2n-1} \wedge Je_{2n-1}. \tag{55}$$

Noting that $\det \begin{bmatrix} I & 0 \\ -iI & I \end{bmatrix} = 1$ and $\det \begin{bmatrix} I & 0 \\ iI & I \end{bmatrix} = 1$ then

$$\begin{aligned} \det \begin{bmatrix} A & -B \\ B & A \end{bmatrix} &= \det \left(\begin{bmatrix} I & 0 \\ -iI & I \end{bmatrix} \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} I & 0 \\ iI & I \end{bmatrix} \right) \\ &= \det \begin{bmatrix} A - iB & -B \\ 0 & A + iB \end{bmatrix} \end{aligned} \tag{56}$$

and

$$\det \begin{bmatrix} A - iB & -B \\ 0 & A + iB \end{bmatrix} = |\det(A - iB)|^2 > 0. \tag{57}$$

Thus since the determinant is positive, the orientation is well defined. □

3.2 2-forms and skew-adjoint endomorphisms

We will show that 2-forms and skew adjoint endomorphisms can be identified using the metric.

Theorem 3.3. *Let $\alpha \in \Lambda^2 V^*$. Then it defines a skew adjoint endomorphism $\alpha^\sharp : V \to V$ given by $X \mapsto (X \lrcorner \alpha)^\sharp$, where $\sharp$ is the musical isomorphism with respect to the metric g.*

Proof. Let $\{e_i\}$ be an orthonormal basis for V and $\{e^j\}$ be the dual orthonormal basis for V^*, let $\alpha = \frac{1}{2}\alpha_{ij} e^i \wedge e^j$. Note that $e^i \wedge e^j$ can be identified as an endomorphism via the metric g by

$$\begin{aligned} e_k \mapsto ((e^i \wedge e^j)(e_k))^\sharp &= (e_k \lrcorner (e^i \wedge e^j))^\sharp \\ &= \langle e_i, e_k \rangle (e^j)^\sharp - \langle e_j, e_k \rangle (e^i)^\sharp. \end{aligned} \tag{58}$$

We extend this map linearly and the map $(\cdot \lrcorner (e^i \wedge e^j))^\sharp$ defines an endomorphism of

V. Note that if we swap e^i with e^j then $(e_k \lrcorner (e^j \wedge e^i))^\sharp = -(e_k \lrcorner (e^i \wedge e^j))^\sharp$. Thus for $\alpha = \frac{1}{2}\alpha_{ij} e^i \wedge e^j \in \Lambda^2 V^*$, $\alpha^\sharp$ is viewed as an endomorphism which is skew adjoint and thus for $X, Y \in V$, $\alpha^\sharp(X) = (X \lrcorner \alpha)^\sharp$ and

$$g(\alpha^\sharp(X), Y) = \alpha(X, Y) \tag{59}$$

as α is skew symmetric then

$$g(\alpha^\sharp(X), Y) + g(X, \alpha^\sharp(Y)) = 0. \quad \square$$

Next we define the $\sqrt{k}$-sphere in $\Lambda^2(V^*)$.

Definition 3.4. Let $\Lambda^2(V^*)$ the space of 2-forms, let $\langle \cdot, \cdot \rangle$ be the metric on $\Lambda^2(V^*)$ induced by the metric g on V and let $\{\alpha^i\}$ be an orthonormal basis for $\Lambda^2(V^*)$. Then for $k > 0$, $S_{\sqrt{k}}(\Lambda^2(V^*))$ is defined to be the set

$$\{\alpha = a_i \alpha^i \in \Lambda^2(V^*) \mid \langle \alpha, \alpha \rangle = k, i.e, \sum_i a_i^2 = k\}$$

3.3 The almost complex structure on the 2-sphere

We define the almost complex structure on the 2-sphere S^2. We choose the standard orientation on $\mathbb{R}^3$ with basis $\{e_1, e_2, e_3\}$ where $e_1 \times e_2 = e_3$.

Theorem 3.5. *Let $p \in S^2$, $Y_p \in T_pS^2$, r_p the outward radial vector of length one at the point p, then $J_p(Y_p) = r_p \times Y_p$ defines an almost complex structure on S^2.*

Proof. Since r_p is an outward radial vector of length one, it is orthogonal to the tangent plane T_pS^2. Then $r_p \times Y_p \in T_pS^2$ and $r_p \times (r_p \times Y_p) \in T_pS^2$. Hence $J_p : T_pS^2 \to T_pS^2$.

Using the identity of the iterated cross product on $\mathbb{R}^3$ then

$$\begin{aligned} J_p^2(Y_p) &= r_p \times (r_p \times Y_p) \\ &= \langle r_p, Y_p\rangle r_p - \langle r_p, r_p\rangle Y_p \\ &= -\langle r_p, r_p\rangle Y_p \qquad \text{since } r_p \text{ is orthogonal to } Y_p \\ &= -Y_p \qquad \text{since } \langle r_p, r_p\rangle = 1. \quad \square \end{aligned} \tag{60}$$

4 The vector bundle $E = \Lambda^2_- T^*M$

4.1 The almost complex structure on $S_{\sqrt{2}}(\Lambda^2_-(T^*M))$

Let M be an oriented 4-dimensional manifold with Riemnnian metric g. Let $\{e_0, e_1, e_2, e_3\}$ be a local oriented orthonormal frame for TM and let $\{e^0, e^1, e^2, e^3\}$ be the dual local orthonormal frame on T^*M. Hence on $\Lambda^2 T^*M$, $\{e^i \wedge e^j \;\; | \; i < j\}$ is a local orthonormal frame with the induced metric.

Let $E = \Lambda^2_- T^*M$ be the bundle of anti-self-dual (ASD) 2-forms. Recall that $\omega \in \Omega^2(M)$ is ASD iff

$$*\omega = -\omega \tag{61}$$

where $*$ is the Hodge star operator.

A local orthogonal frame on E is

$$\{\omega^1, \omega^2, \omega^3\} = \{e^0 \wedge e^1 - e^2 \wedge e^3, e^0 \wedge e^2 - e^3 \wedge e^1, e^0 \wedge e^3 - e^1 \wedge e^2\}.$$

Note that $|\omega^i|^2 = 2$. From (1) the local fibre coordinates on E will be (y_1, y_2, y_3) with respect to the frame $\{\omega^a\}$. Let $Z = S_{\sqrt{2}}(\Lambda^2_- T^*M) \subset E$ be the $\sqrt{2}$-sphere bundle. Since $\{\omega^a\}$ is an orthogonal frame, Z can be described locally as

$$Z = \{f = 0 \mid f = 2(y_1^2 + y_2^2 + y_3^3) - 2\}$$

then $y_1^2 + y_2^2 + y_3^3 = 1$ on Z.

Theorem 4.1. *Let $p \in M$, and $\omega \in Z_p$. Let $J := \omega^\sharp : T_pM \to T_pM$. Then J is a complex structure on T_pM.*

Proof. Let $\sharp$ be the musical isomorphism then we claim the complex structure J on T_pM is defined by

$$J = \omega^\sharp.$$

To show that J is a complex structure then we must show $J^2 = -id$.

Lemma 4.2. *The local frame $\{\omega^1, \omega^2, \omega^3\}$, when viewed as endomorphisms $\{(\omega^1)^\sharp, (\omega^2)^\sharp, (\omega^3)^\sharp\}$ with ω^k_{ab} the matrix representation of $(\omega^k)^\sharp$, satisfy the relations*

$$\begin{aligned} \omega^1_{ai}\omega^2_{ib} &= -\omega^3_{ab} & \omega^2_{bi}\omega^1_{ib} &= \omega^3_{ba} & \omega^1_{ai}\omega^1_{ib} &= -\delta_{ab} \\ \omega^2_{ai}\omega^3_{ib} &= -\omega^1_{ab} & \omega^3_{bi}\omega^2_{ia} &= \omega^1_{ba} & \omega^3_{ai}\omega^3_{ib} &= -\delta_{ab} \\ \omega^3_{ai}\omega^1_{ib} &= -\omega^2_{ab} & \omega^1_{bi}\omega^3_{ia} &= \omega^2_{ba} & \omega^2_{ai}\omega^2_{ib} &= -\delta_{ab}. \end{aligned} \tag{62}$$

Proof. Viewing the local frame $\{\omega^1, \omega^2, \omega^3\}$ as endomorphisms $\{(\omega^1)^\sharp, (\omega^2)^\sharp, (\omega^3)^\sharp\}$, letting ω^k_{ab} be the matrix representation of $(\omega^k)^\sharp$, we work out $\omega^k_{ia}\omega^l_{aj}$. The local frame $\{\omega^1, \omega^2, \omega^3\}$ is induced from the local frame $\{e_0, e_1, e_2, e_3\}$ on M, given by

$$\begin{aligned} \omega^1 &= e^0 \wedge e^1 - e^2 \wedge e^3 \\ \omega^2 &= e^0 \wedge e^2 - e^3 \wedge e^1 \\ \omega^3 &= e^0 \wedge e^3 - e^1 \wedge e^2. \end{aligned} \tag{63}$$

Working out $\omega^k_{ia}\omega^l_{aj}$, we have

$$\begin{aligned}
\omega^1_{ib}\omega^2_{bj} &= \Big((\delta^0_i\delta^1_b - \delta^0_b\delta^1_i) - (\delta^2_i\delta^3_b - \delta^2_b\delta^3_i)\Big)\Big((\delta^0_b\delta^2_j - \delta^0_j\delta^2_b) - (\delta^3_b\delta^1_j - \delta^3_j\delta^1_b)\Big) \\
&= (-\delta^0_i\delta^1_b\delta^3_j\delta^1_b - \delta^0_b\delta^1_i\delta^0_b\delta^2_j + \delta^2_i\delta^3_b\delta^3_b\delta^1_j + \delta^2_b\delta^3_i\delta^0_j\delta^2_b) \\
&= (-\delta^0_i\delta^3_j - \delta^1_i\delta^2_j + \delta^2_i\delta^1_j + \delta^3_i\delta^0_j) \\
&= -\omega^3_{ij}.
\end{aligned} \tag{64}$$

After cyclically permuting $1 \to 2 \to 3 \to 1$ in (64), we get

$$\begin{aligned}
\omega^1_{ib}\omega^3_{bj} &= -\omega^2_{ij}, \\
\omega^2_{ib}\omega^3_{bj} &= -\omega^1_{ij}.
\end{aligned} \tag{65}$$

We also have

$$\begin{aligned}
\omega^3_{ib}\omega^3_{bj} &= \Big((\delta^0_i\delta^3_b - \delta^0_b\delta^3_i) - (\delta^1_i\delta^2_b - \delta^1_b\delta^2_i)\Big)\Big((\delta^0_b\delta^3_j - \delta^0_j\delta^3_b) - (\delta^1_b\delta^2_j - \delta^1_j\delta^2_b)\Big) \\
&= -\delta^0_i\delta^3_b\delta^0_j\delta^3_b - \delta^0_b\delta^3_i\delta^0_b\delta^3_j - \delta^1_i\delta^2_b\delta^1_j\delta^2_b - \delta^1_b\delta^2_i\delta^1_b\delta^2_j \\
&= -\delta_{ij}.
\end{aligned} \tag{66}$$

Similarly $\omega^2_{ib}\omega^2_{bj} = \omega^1_{ib}\omega^1_{bj} = -\delta_{ij}$. Since the endomorphisms $\{\omega^k_{ij}\}$ are skew in i, j, and after swapping a with i and i with b we get

$$-\omega^3_{ab} = \omega^1_{ai}\omega^2_{ib} = \omega^2_{bi}\omega^1_{ia} = \omega^3_{ba}. \tag{67}$$

Similarly by cyclically permuting $1 \to 2 \to 3 \to 1$ in (67) we get

$$\omega^1_{ab} = \omega^3_{ai}\omega^2_{ib}, \omega^2_{ab} = \omega^3_{ai}\omega^1_{ib}. \quad \square$$

Let $(\omega^a)^\sharp = (J_a)_{ij}$. Then in a local orthonormal frame J_a as an endomorphism is

$(J_a)_{ij} := \omega^a_{ij}$. For $\omega = y_a\omega^a$, $J = \omega^\sharp = \sum_a y_a J_a$ then

$$\begin{aligned}
J^2 &= (\sum_a y_a J_a)^2 \\
&= (\sum_a y_a^2 J_a^2) + \sum_{a\neq b} y_a y_b J_a J_b) \\
&= (\sum_a y_a^2 J_a^2) + \sum_{a\leq b} y_a y_b (J_a J_b + J_b J_a) \\
&= (\sum_a y_a^2 J_a^2) + 0 \qquad \text{using the relations from Lemma 4.2} \\
&= -1. \quad \square
\end{aligned} \tag{68}$$

Note that using the local coordinates (1) for E, $\omega^\sharp(X) = \omega(X, e_j)g^{ji}e_i$. Since $\{e_0, e_1, e_2, e_3\}$ is an orthonormal frame $g^{ji} = \delta^{ji}$ thus

$$\omega^\sharp(X) = \omega(X, e_i)e_i.$$

Lemma 4.3. *The fundamental vector field $\xi \in \Gamma(VE)$ is orthogonal to TZ.*

Proof. As $Z = \{f = 0\}$ then $TZ = (\mathrm{grad}(f))^\perp$. Since $\mathrm{grad}(f) = \frac{1}{4}y_a\frac{\partial}{\partial y_a} = \frac{1}{4}\xi$ is in the same direction as the radius vector to the sphere, ξ is orthogonal to the tangent plane to Z. $\square$

From Lemma 4.3 the tangent space to Z at $\upsilon \in E_p$ is therefore

$$V_\upsilon Z = \{Y_\upsilon \in V_\upsilon E \mid \langle Y_\upsilon, \xi\rangle_{VE} = 0, \;\; \upsilon \in Z, \xi \in V_\upsilon E\}. \tag{69}$$

Given a connection ∇ on E, TE splits as $TE = HE \oplus VE$. As $Z \subset E$ we can restrict $\pi : E \to M$ to Z, $\pi_Z : Z \to M$. We thus have the induced map $(\pi_Z)_* : TZ \to TM$ with $VZ := \ker(\pi_Z)_*$. Thus we have the splitting $TZ = HZ \oplus VZ$. From (69) we have that $VE = VZ \oplus \langle\xi\rangle$ and $TE|_Z = HE|_Z \oplus VE|_Z$, so $HE|_Z = HZ$. As we have a connection on E we get an induced connection on Z since $Z \subset E$, and

$HZ \cong (\pi\big|_Z)^* TM \cong HE\big|_Z$.

We can define an almost complex structure J on $TZ \subset TE$ as follows;

Definition 4.4. Let $\omega \in Z_p$. On $H_\omega Z$, J is defined by $J(X^h) := ((X \lrcorner\, \omega)^\sharp)^h$, for $X \in T_pM$.

The next result shows that tautological 2-form from (38) can be used to represent $J(X^h)$.

Theorem 4.5. *Let Θ be the restriction of the tautological 2-form on $\Lambda^2 T^*M$ to Z. Let X^h be the horizontal lift of $X \in \Gamma(TM)$. Then*

$$J(X^h) = \Theta(X^h, e_i^h)e_i$$

where $\{e_0, e_1, e_2, e_3\}$ is a local orthonormal frame.

Proof. For $\{e_0, e_1, e_2, e_3\}$ a local orthonormal frame on TM, let $\{e_0^h, e_1^h, e_2^h, e_3^h\}$ be the induced orthonormal frame on HZ. Evaluating $J(X^h)$ at $\omega \in E_p$ we have

$$\begin{aligned} J(X^h) &= (\omega^\sharp(X))^h_{\pi(\omega)} \\ &= (\omega_{\pi(\omega)}(X_{\pi(\omega)}, (e_i)_{\pi(\omega)})(e_i)^h_{\pi(\omega)} \\ &= \omega_{\pi(\omega)}(\pi_* X^h_\omega, \pi_*(e_i)^h_{\pi(\omega)})(e_i)^h_{\pi(\omega)} \\ &= \Theta_\omega(X^h_{\pi(\omega)}, (e_i)^h_{\pi(\omega)})(e_i)^h_{\pi(\omega)}. \quad \square \end{aligned} \tag{70}$$

Theorem 4.6. *Let $\omega \in Z_p$, $Y^v \in V_\omega Z$, $\xi \in \Gamma(VE)$ the fundamental vector field. Then $J_\omega(Y^v) = \xi_\omega \times Y^v$ defines a complex structure on $V_\omega Z$.*

Proof. This follows from Theorem 3.5 since ξ_ω is orthogonal to Y^v and $|\xi_\omega|^2 = 1$. $\square$

4.2 The flow $\phi_t^{X^h}$ and the cross product

Lemma 4.7. *The vector bundle E is orientable.*

Proof. Let $\{\tilde{e_0}, \tilde{e_1}, \tilde{e_2}, \tilde{e_3}\}$ be another oriented orthonormal frame, where $\tilde{e}_i = P_{ki}e_k$ and $P \in SO(4)$. Recall that an orthogonal basis of E is given by $\{\omega^1, \omega^2, \omega^3\}$ where

$$\omega^i = e_0 \wedge e_i - *e_0 \wedge e_i = e_0 \wedge e_i - e_j \wedge e_k$$

and $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$. Computing $\tilde{\omega}^i$

$$\begin{aligned}
\tilde{\omega}^i &= \tilde{e}^0 \wedge \tilde{e}^i - *(\tilde{e}^0 \wedge \tilde{e}^i) \\
&= \tilde{e}^0 \wedge \tilde{e}^i - \tilde{e}^j \wedge \tilde{e}^k \\
&= P_{l0}e_l \wedge P_{ki}e_k - P_{lj}e_l \wedge P_{km}e_k \\
&= \frac{1}{2}((P_{l0}P_{ki} - P_{k0}P_{li}) - (P_{lj}P_{mk} - P_{kj}P_{lm}))e_l \wedge e_k. \qquad (71)
\end{aligned}$$

It follows that $\tilde{\omega}^i = Q_{ki}\omega^k$ for some orthogonal 3×3 matrix which is continously connected to the identity since P is. Thus $Q \in SO(3)$ and E is orientable. □

Lemma 4.8. *The flow $\phi_t^{X^h}$ preserves the cross product in the vertical direction.*

Proof. Let $U, W \in V_\omega E$, let $\{\frac{\partial}{\partial y_a}\}$ be induced from a local orthonormal frame on E for VE. Then $(\phi_t^{X^h})_*(\frac{\partial}{\partial y_a})$ is a basis for $V_{\phi_t^{X^h}(\omega)}E$ since $(\phi_t^{X^h})_*$ is an isomorphism. As $VE \cong \pi^*E$, it is orientable by Lemma 4.7. Denote $\times_\omega$ the cross product with respect the metric and orientation on $V_\omega E$ and $\times_{(\phi_t^{X^h}(\omega))}$ the repective cross product on $V_{\phi_t^{X^h}(\omega)}E$.

The metric on VE is denoted $\langle \cdot, \cdot \rangle_{VE}$ and we remove the VE in this computation for ease of notation.

Let ε_{def} be the standard permutation symbol. We compute:

$$\begin{aligned}(\phi_t^{X^h})_*(U \times_\omega W) &= \varepsilon_{def} U^d W^e \frac{\partial}{\partial y_f} \\ &= \varepsilon_{def} \langle U, \frac{\partial}{\partial y_d} \rangle \langle W, \frac{\partial}{\partial y_e} \rangle (\phi_t^{X^h})_* (\frac{\partial}{\partial y_f}) \\ &= \varepsilon_{def} \langle (\phi_t^{X^h})_* U, (\phi_t^{X^h})_* \frac{\partial}{\partial y_d} \rangle \langle (\phi_t^{X^h})_* W, (\phi_t^{X^h})_* \frac{\partial}{\partial y_e} \rangle (\phi_t^{X^h})_* \frac{\partial}{\partial y_f} \\ &\qquad \text{since } (\phi_t^{X^h})_* \text{ is an isometry} \\ &= (\phi_t^{X^h})_* U \times_{\phi_t^{X^h}(\omega)} (\phi_t^{X^h})_* W. \quad \square \end{aligned} \tag{72}$$

Lemma 4.9. *The flow preserves the complex structure J in the vertical direction. That is*

$$(\phi_t^{X^h})_* J_\omega (Y^v) = J_{\phi_t^{X^h}(\omega)} ((\phi_t^{X^h})_* Y^v). \tag{73}$$

Proof. Let $Z = S_{\sqrt{2}}(\Lambda^2_-(T^*M))$, the $\sqrt{2}$-sphere bundle, $\xi \in \Gamma(VE)$ the fundamental vector field and $Y^v \in \Gamma(VZ)$. Evaluating $(\phi_t^{X^h})_*(J_\omega((\phi_t^{X^h})_* Y^v))$ becomes

$$\begin{aligned}(\phi_t^{X^h})_*(J_\omega(Y^v)) &= (\phi_t^{X^h})_*(\xi_\omega \times_v Y^v) \\ &= (\phi_t^{X^h})_* \xi_\omega \times_{\phi_t^{X^h}(v)} (\phi_t^{X^h})_* Y^v \qquad \text{by Lemma 4.8} \\ &= (J_{\phi_t^{X^h}(\omega)})((\phi_t^{X^h})_* Y^v). \quad \square \end{aligned} \tag{74}$$

Theorem 4.10. *The Lie derivative $(L_{X^h} J)(Y^v)$ is zero.*

Proof. It follows from differentiating equation (73) in Lemma 4.9 with respect to t at $t = 0$. $\square$

4.3 Quaternions and cross products

The division algebra $\mathbb{H}$ of quaternions consists of

$$q = q_0 + q_1 i + q_2 j + q_3 k, \qquad q_0, q_1, q_2, q_3 \in \mathbb{R}$$

where

$$i^2 = -1, j^2 = -1, k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j. \tag{75}$$

The square of the norm of q is given by

$$\langle q, q \rangle = q_0^2 + q_1^2 + q_2^2 + q_3^2.$$

Then $\mathbb{H}$ is isomorphic as a real vector space to $\mathbb{R}^4$ via

$$q_0 + q_1 i + q_2 j + q_3 k \mapsto (q_0, q_1, q_2, q_3). \tag{76}$$

The space of imaginary quaternions is isomorphic to $\mathbb{R}^3$, where the imaginary part of the quaternion q is given by

$$\vec{q} = Im(q) = q_1 i + q_2 j + q_3 k,$$

and the map is given by

$$q_1 i + q_2 j + q_3 k \mapsto (q_1, q_2, q_3). \tag{77}$$

For $p = p_0 + p_1 i + p_2 j + p_3 k \in \mathbb{H}$, we define left and right multiplication by q, denoted $L_q(p), R_q(p)$ to be

$$L_q(p) = qp, \qquad R_q(p) = pq, \tag{78}$$

which can be represented in matrix form as

$$L_q(p) = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$R_q(p) = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} \tag{79}$$

We will see that $Im(L_{\vec{q}}(\vec{p}))$ and $Im(R_{\vec{q}}(\vec{p}))$ relate to the cross product on $\mathbb{R}^3$. Since $p = p_0 + \vec{p}$ and $q = q_0 + \vec{q}$, working out $R_q(p) = pq$ we get

$$R_q(p) = p_0q_0 - \langle \vec{p}, \vec{q} \rangle + p_0\vec{q} + q_0\vec{p} + \vec{p} \times \vec{q}. \tag{80}$$

Switching p and q yields $L_q(p) = q_0p_0 - \langle \vec{q}, \vec{p} \rangle + p_0\vec{q} + q_0\vec{p} + \vec{q} \times \vec{p}$. Thus we immediately see that

$$Im(L_{\vec{q}}(\vec{p})) = \vec{q} \times \vec{p}$$

and

$$Im(R_{\vec{q}}(\vec{p})) = \vec{p} \times \vec{q}.$$

Theorem 4.11. *For* $q = q_0 + \vec{q},\ p = p_0 + \vec{p} \in \mathbb{H}$

$$[R_q, R_p] = 2R_{\vec{p} \times \vec{q}}.$$

Proof. We have $R_qR_pu = (up)q = u(pq)$, for any $u \in \mathbb{H}$. Thus

$$R_qR_pu = u(p_0q_0 - \langle \vec{p}, \vec{q} \rangle + q_0p + p_0q + \vec{p} \times \vec{q}). \tag{81}$$

Then computing $[R_q, R_p]u$ yields

$$\begin{aligned}[R_q, R_p]u &= R_q R_p u - R_p R_q u \\ &= 2u(\vec{p} \times \vec{q}) \\ &= 2R_{\vec{p}\times\vec{q}}u \end{aligned} \tag{82}$$

thus $[R_q, R_p] = 2R_{\vec{p}\times\vec{q}}$. □

4.4 ASD 2-forms and quaternions

From Lemma 4.2, $\omega_p^\sharp$ defines a complex structure on T_pM and we get the following consequence.

Lemma 4.12. *The local frame $\{\omega^1, \omega^2, \omega^3\}$ satisfies the quaternionic relations in (75) as endomorphisms.*

Proof. Each $(\omega^i)^\sharp$ defines a complex structure by Lemma 4.2. We drop the $\sharp$ for convenience. Right and left multiplication by quaternions can be viewed as endomorphisms. Thus the map that sends $\{\omega^1, \omega^2, \omega^3\}$ to $\{-i, -j, -k\}$ is an algebra isomorphism. □

Theorem 4.13. *Identifying the local frame $\{\omega^1, \omega^2, \omega^3\}$ with endomorphisms, then*

$$\begin{aligned}[\omega^2, \omega^1] &= 2(\omega^2 \times \omega^1) = 2\omega^3, \\ [\omega^3, \omega^2] &= 2(\omega^3 \times \omega^2) = 2\omega^1, \\ [\omega^1, \omega^3] &= 2(\omega^1 \times \omega^3) = 2\omega^2. \end{aligned} \tag{83}$$

Proof. This is an immediate consequence of Lemma 4.12 and Theorem 4.11. □

For $\eta \in E_p = (\pi^*E)_\eta$ an ASD 2-form, η can be viewed as an endomorphism by

$\eta^\sharp$. Let $X, Y, Z, W \in \Gamma(TM)$. Recall the curvature $R(X,Y)$ satisfies

$$R(X,Y)(Z,W) = \langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$$

Theorem 4.14. *For $\eta \in E_p = (\pi^* E)_\eta$ viewed as an endomorphism and $R(X,Y)$ the curvature*

$$(\eta \times R(X,Y)\eta)(V,T)) = -R(X,Y,\eta V,\eta T) + R(X,Y,V,T). \tag{84}$$

Proof. Noting the curvature is a derivation and acts on 2-forms we get $(R(X,Y)\eta)(V) = R(X,Y)(\eta V) - \eta(R(X,Y)V)$.
Using this, the skew-adjointness of η and $\eta^2 = -\mathrm{id}$, the commutator $[\eta, R(X,Y)\eta](V,T)$ becomes

$$\begin{aligned}
[\eta, R(X,Y)\eta](V,T) &= \langle \eta((R(X,Y)\eta)(V)), T\rangle - \langle (R(X,Y)\eta)(\eta V), T\rangle \\
&= -\langle ((R(X,Y)\eta)(V)), \eta T\rangle - \langle (R(X,Y)\eta)(\eta V), T\rangle \\
&= -\langle R(X,Y)(\eta V) - \eta(R(X,Y)V), \eta T\rangle \\
&\quad - \langle R(X,Y)(\eta^2 V) - \eta R(X,Y)(\eta V), T\rangle \\
&= (-R(X,Y,\eta V,\eta T) + R(X,Y,V,T) \\
&\quad + R(X,Y,V,T) - R(X,Y,\eta V,\eta T)) \\
&= 2R(X,Y,V,T) - 2R(X,Y\eta V,\eta T).
\end{aligned} \tag{85}$$

The commutator can be identified with twice the cross product by Theorem 4.13, thus

$$(\eta \times R(X,Y)\eta)(V,T)) = -R(X,Y,\eta V,\eta T) + R(X,Y,V,T). \quad \square \tag{86}$$

5 Calculation of the Nijenhuis tensor

First we define the Nijenhuis tensor.

Definition 5.1. Let $U, V \in \Gamma(TM)$, let J be an almost complex structure. The Nijenhuis tensor is defined as

$$N(U,V) = [U,V] + J[JU,V] + J[U,JV] - [JU,JV].$$

Definition 5.2. We say an almost complex structure is integrable if $N(U,V) = 0$, otherwise it is not integrable.

The Lie derivative of J along U applied to V and the the Lie derivative of J along JU applied to V is given by

$$\begin{aligned}(L_U J)(V) &= L_U(JV) - J(L_U V) \\ &= [U,JV] - J[U,V] \\ (L_{JU} J)(V) &= L_{JU}(JV) - J(L_{JU} V) \\ &= [JU,JV] - J[JU,V]. \end{aligned} \tag{87}$$

Hence the Nijenhuis tensor can also be written as

$$N(U,V) = J((L_U J)(V)) - (L_{JU} J)(V).$$

5.1 Horizontal part of the Nijenhuis tensor of a horizontal and vertical vector

Let $X^h \in \Gamma(HE)$ be a horizontal lift of $X \in \Gamma(TM)$. For the frame $\{e_0, e_1, e_2, e_3\}$ on M, e_i^h is the horizontal lift of e_i. Let $Y^v \in \Gamma(VZ), \eta \in \Gamma(E)$, and let $\xi \in \Gamma(VE)$ be the fundamental vector field. Denote $L_{X^h} Y^v$ the Lie derivative of Y^v

along X^h. The terms of the Nijenhuis tensor for one horizontal lift and one vertical vector field are $[X^h, Y^v], J[JX^h, Y^v], J[X^h, JY^v], [JX^h, JY^v]$. We will look at the terms individually and work them out using the tautological 2-form and Theorem 4.5. The terms $[JX^h, JY^v], J[JX^h, Y^v]$ are

$$\begin{aligned} &[JX^h, JY^v] \\ &= [\Theta(X^h, e_i^h)e_i^h, JY^v] \\ &= -(JY^v)(\Theta(X^h, e_i^h))e_i^h + \Theta(X^h, e_i^h)[e_i^h, JY^v], \end{aligned} \tag{88}$$

$$\begin{aligned} &J[JX^h, Y^v] \\ &= J[\Theta(X^h, e_i^h)e_i^h, Y^v] \\ &= J(-Y^v(\Theta(X^h, e_i^h))e_i^h + \Theta(X^h, e_i^h)[e_i^h, Y^v]) \\ &= -Y^v(\Theta(X^h, e_i^h))\Theta(e_i^h, e_l^h)e_l^h - \Theta(X^h, e_i^h)J[e_i^h, Y^v]. \end{aligned} \tag{89}$$

Since $\pi_*[X^h, Y^v] = [\pi_* X^h, \pi_* Y^v] = 0$ then $[X^h, Y^v], [e_i^h, Y^v] \in \Gamma(VE)$ are vertical vector fields. Since $JY^v \in \Gamma(VE)$ is a vertical vector field then so are $[e_i^h, JY^v], [X^h, JY^v]$ and consequently also $J[e_i^h, JY^v], J[e_i^h, Y^v], J[X^h, JY^v]$.

Theorem 5.3. *The horizontal component of $N(X^h, Y^v)$ vanishes:*

$$-Y^v(\Theta(X^h, e_i^h)\Theta(e_i^h, e_l^h)e_l^h + (JY^v)(\Theta(X^h, e_l^h))e_l^h = 0 \tag{90}$$

Proof. We let $Y^v = v_a \frac{\partial}{\partial_{y_a}} \in \Gamma(VE)$ and let $\eta = y_b \omega^b \in \Gamma(E)$.
Calculating JY^v via the cross product from Theorem 4.6 and using the identification of VE with $\pi^* E$ in Corollary 2.4, recalling the local frame of E is $\{\omega^1, \omega^2, \omega^3\}$ then computing JY^v we have

$$\begin{aligned}JY^v &= (\xi \times Y^v)\\ &= (y_2v_3 - y_3v_2)\frac{\partial}{\partial y_1} + (y_3v_1 - y_1v_3)\frac{\partial}{\partial y_2} + (y_1v_2 - y_2v_1)\frac{\partial}{\partial y_3}.\end{aligned} \tag{91}$$

We will expand $Y^v(\Theta(X^h, e_i^h))$ and evaluate it at η. Expanding $Y^v(\Theta(X^h, e_i^h))$ and using (38) yields

$$\begin{aligned}&Y_\eta^v(\Theta(X^h, e_i^h))\\ &= v_a \frac{\partial}{\partial y_a}\Big|_\eta (y_1\omega^1 + y_2\omega^2 + y_3\omega^3)(X, e_i)\\ &= v_a\omega^a_{\pi(\eta)}(X, e_i).\end{aligned} \tag{92}$$

Hence $Y_\eta^v(\Theta(X^h, e_i^h))\Theta_\eta(e_i^h, e_l^h)$ is equal to

$$\begin{aligned}Y_\eta^v(\Theta(X^h, e_i^h)\Theta_\eta(e_i^h, e_l^h) &= v_a\omega^a_{\pi(\eta)}(X, e_i)\eta_{\pi(\eta)}(e_i, e_l)\\ &= v_a\omega^a_{\pi(\eta)}(X, e_i)y_b\omega^b_{\pi(\eta)}(e_i, e_l).\end{aligned} \tag{93}$$

Similarly expanding $(J_\eta Y^v)(\Theta(X^h, e_l^h))e_l^h$ using (38) and equation (91) and the method from equation (92) we get

$$\begin{aligned}(J_\eta Y^v)(\Theta(X^h, e_l^h)) &= (y_2v_3 - y_3v_2)\omega^1_{\pi(\eta)}(X, e_l)\\ &+ (y_3v_1 - y_1v_3)\omega^2_{\pi(\eta)}(X, e_l) + (y_1v_2 - y_2v_1)\omega^3_{\pi(\eta)}(X, e_l).\end{aligned} \tag{94}$$

By (69), the condition on the vertical tangent space of $Z \subset E$ is that $\langle Y^v, \xi\rangle_{VE} = 0$ for all $Y^v \in \Gamma(VZ)$. Expanding $\langle Y^v, \xi\rangle_{VE} = 0$ using the local frame $\{\frac{\partial}{\partial y_i}\}$ for VE we get

$$\langle Y^v, \xi_\eta\rangle_{VE} = \langle v_a\frac{\partial}{\partial y_a}\Big|_\eta, y_b\frac{\partial}{\partial y_b}\Big|_\eta\rangle_{VE} = \sum_{a=1}^{3} y_a v_a = 0 \tag{95}$$

We now substitute the computations from (92),(93),(94) into

$$(JY^v)(\Theta(X^h, e_l^h)e_l^h - Y^v(\Theta(X^h, e_i^h))\Theta(e_i^h, e_l^h))e_l^h$$

and dropping the $\pi(\eta)$ from the above expression and evaluating the above expression at η, we get

$$
\begin{aligned}
&Y_\eta^v(\Theta(X^h, e_i^h))\Theta_\eta(e_i^h, e_l^h) - (J_\eta Y^v)(\Theta(X^h, e_l^h)) \\
&= v_1 y_1 \omega^1(X, e_i)\omega^1(e_i, e_l) \\
&\quad + v_1 y_2 \omega^1(X, e_i)\omega^2(e_i, e_l) + v_1 y_3 \omega^1(X, e_i)\omega^3(e_i, e_l) \\
&\quad + v_2 y_1 \omega^2(X, e_i)\omega^1(e_i, e_l) + v_2 y_2 \omega^2(X, e_i)\omega^2(e_i, e_l) + v_2 y_3 \omega^2(X, e_i)\omega^3(e_i, e_l) \\
&\quad + v_3 y_1 \omega^3(X, e_i)\omega^1(e_i, e_l) + v_3 y_2 \omega^3(X, e_i)\omega^2(e_i, e_l) + v_3 y_3 \omega^3(X, e_i)\omega^3(e_i, e_l) \\
&\quad - [(y_2 v_3 - y_3 v_2)\omega^1(X, e_l) + (y_3 v_1 - y_1 v_3)\omega^2(X, e_l) + (y_1 v_2 - y_2 v_1)\omega^3(X, e_l)].
\end{aligned} \tag{96}
$$

Given that we have a metric g on Z then $\omega(X, Y) = g(JX, Y)$ for $X, Y \in T_pM$ since ω is a complex structure. We work out $\omega(X, e_i)\omega(e_i, e_l)$,

$$
\begin{aligned}
\omega(X, e_i)\omega(e_i, e_l) &= g(JX, e_i)g(Je_i, e_l) \\
&= -g(JX, e_i)g(e_i, Je_l) \\
&= -g(JX, Je_l) \\
&= -g(X, e_l).
\end{aligned} \tag{97}
$$

Substituting the identity $\omega(X, e_i)\omega(e_i, e_l) = -g(X, e_l)$ from (97) into equation (96) and simplifying, we get

$$
\begin{aligned}
&Y^v_\eta(\Theta(X^h, e^h_i))\Theta_\eta(e^h_i, e^h_l) - (J_\eta Y^v)(\Theta(X^h, e^h_l)\\
&= -v_1y_1g(X, e_l) + v_1y_2\omega^1(X, e_i)\omega^2(e_i, e_l) + v_1y_3\omega^1(X, e_i)\omega^3(e_i, e_l)\\
&\quad + v_2y_1\omega^2(X, e_i)\omega^1(e_i, e_l) - v_2y_2g(X, e_l) + v_2y_3\omega^2(X, e_i)\omega^3(e_i, e_l)\\
&\quad + v_3y_1\omega^3(X, e_i)\omega^1(e_i, e_l) + v_3y_2\omega^3(X, e_i)\omega^2(e_i, e_l) - v_3y_3g(X, e_l)\\
&\quad - [(y_2v_3 - y_3v_2)\omega^1(X, e_l) + (y_3v_1 - y_1v_3)\omega^2(X, e_l) + (y_1v_2 - y_2v_1)\omega^3(X, e_l)].
\end{aligned} \tag{98}
$$

From the first, fifth, and ninth terms in (98) we see that

$$
\begin{aligned}
&- v_1y_1g(X, e_l) - v_2y_2g(X, e_l) - v_3y_3g(X, e_l)\\
&= -(\sum_{a=1} v_a y_a)g(X, e_l)\\
&= 0 \text{ from (95).}
\end{aligned} \tag{99}
$$

We therefore have

$$
\begin{aligned}
&Y^v_\eta(\Theta(X^h, e^h_i))\Theta_\eta(e^h_i, e^h_l) - (J_\eta Y^v)(\Theta(X^h, e^h_l))\\
&= v_1y_2\omega^1(X, e_i)\omega^2(e_i, e_l) + v_1y_3\omega^1(X, e_i)\omega^3(e_i, e_l)\\
&\quad + v_2y_1\omega^2(X, e_i)\omega^1(e_i, e_l) + v_2y_3\omega^2(X, e_i)\omega^3(e_i, e_l)\\
&\quad + v_3y_1\omega^3(X, e_i)\omega^1(e_i, e_l) + v_3y_2\omega^3(X, e_i)\omega^2(e_i, e_l)\\
&\quad - [(y_2v_3 - y_3v_2)\omega^1 + (y_3v_1 - y_1v_3)\omega^2 + (y_1v_2 - y_2v_1)\omega^3].
\end{aligned} \tag{100}
$$

Note that by identifying the local frame $\{\omega^1, \omega^2, \omega^3\}$ with the quaternions in

Theorem 4.13, we have the quaternionic relations

$$\begin{aligned}
\omega^1(X, e_i)\omega^2(e_i, e_l) &= -\omega^3(X, e_l), \\
-\omega^3(X, e_i)\omega^1(e_i, e_l) &= \omega^2(X, e_l), \\
-\omega^2(X, e_i)\omega^3(e_i, e_l) &= \omega^1(X, e_l),
\end{aligned} \tag{101}$$

and substituting the relations (101) into (100) and simplifying yields

$$\begin{aligned}
&Y^v_\eta(\Theta(X^h, e^h_i))\Theta_\eta(e^h_i, e^h_l) - (J_\eta Y^v)(\Theta(X^h, e^h_l)) \\
&= -v_1 y_2 \omega^3(X, e_l) + v_1 y_3 \omega^2(X, e_l) \\
&\qquad + v_2 y_1 \omega^3(X, e_l) - v_2 y_3 \omega^1(X, e_l) \\
&\qquad - v_3 y_1 \omega^2(X, e_l) + v_3 y_2 \omega^1(X, e_l) \\
&\qquad - [(y_2 v_3 - y_3 v_2)\omega^1(X, e_l) + (y_3 v_1 - y_1 v_3)\omega^2(X, e_l) + (y_1 v_2 - y_2 v_1)\omega^3(X, e_l)] \\
&= 0. \quad \square
\end{aligned} \tag{102}$$

5.2 Vertical part of the Nijenhuis tensor of a horizontal and vertical vector

Let X^h be a horizontal lift and Y^v be a vertical vector field. Using (88),(89),(87) and Theorem 5.3 we can write

$$
\begin{aligned}
N(X^h, Y^v) &= [X^h, Y^v] + J[X^h, JY^v] - [JX^h, JY^v] + J[X^h, JY^v] \\
N(X^h, Y^v) &= [X^h, Y^v] + J[X^h, JY^v] \\
&\quad - (-(JY^v)(\Theta(X^h, e_l^h))e_l^h + \Theta(X^h, e_i^h)[e_i^h, JY^v]) \\
&\quad + (-Y^v(\Theta(X^h, e_i^h))\Theta(e_i^h, e_l^h)e_l^h - \Theta(X^h, e_i^h)J[e_i^h, Y^v]) \\
&= J((L_{X^h}J)(Y^v)) - \Theta(X^h, e_i^h)(L_{e_i^h}J)(Y^v) \\
&\quad - Y^v(\Theta(X^h, e_i^h))\Theta(e_i^h, e_l^h)e_l^h + (JY^v)(\Theta(X^h, e_l^h))e_l^h. \qquad (103)
\end{aligned}
$$

As

$$-Y^v(\Theta(X^h, e_i^h))\Theta(e_i^h, e_l^h)e_l^h + (JY^v)(\Theta(X^h, e_l^h))e_l^h = 0 \qquad (104)$$

by (87) and using that

$$0 = L_{X^h}(-1) = L_{X^h}J^2 = (L_{X^h}J)J + JL_{X^h}J \qquad (105)$$

we get

$$
\begin{aligned}
N(X^h, Y^v) =&(J((L_{X^h}J)Y^v) - \Theta(X^h, e_i^h)((L_{e_i^h}J)Y^v) \\
&= -(L_{X^h}J)(JY^v) - \Theta(X^h, e_i^h)((L_{e_i^h}J)Y^v). \qquad (106)
\end{aligned}
$$

Note that as the horizontal component of $N(X^h, Y^v)$ is zero then equation (103) is purely vertical.

To show that $N(X^h, Y^v)$ vanishes it suffices to show that $(L_{X^h}J)(JY^v) = 0$ and $(L_{e_i^h}J)(Y^v) = 0$.

We have shown previously in Theorem 4.10 that for any vertical vector field $S^v \in$

$\Gamma(VZ)$

$$(L_{X^h}J)(S^v) = 0.$$

Thus $(L_{X^h}J)(JY^v) = 0$ and $(L_{e_i^h}J)(Y^v) = 0$. Thus $N(X^h, Y^v) = 0$.

5.3 Nijenhuis tensor of two horizontal vectors

Let X, Y, V, T be vector fields on our oriented 4-manifold M and X^h, Y^h, V^h, T^h the respective horizontal lifts to $E = \Lambda^2_- T^*M$. Let $\{e^0, e^1, e^2, e^3\}$ be a local orthonormal frame on M. By definition the Nijenhuis tensor of two horizontal vector fields is

$$\begin{aligned}
&N(X^h, Y^h) \\
&= [X^h, Y^h] + J[JX^h, Y^h] + J[X^h, JY^h] - [JX^h, JY^h]. \qquad (107)
\end{aligned}$$

Using our identity for $J(X^h)$ in Theorem 4.5, this becomes

$$\begin{aligned}
N(X^h, Y^h) = [X^h, Y^h]& \\
&+ J[\Theta(X^h, e^h_k)e^h_k, Y^h] + J[X^h, \Theta(Y^h, e^h_k)e^h_k] \\
&\quad - [\Theta(X^h, e^h_k)e^h_k, \Theta(Y^h, e^h_l)e^h_l]. \qquad (108)
\end{aligned}$$

Next, expanding the Lie brackets yields

$$\begin{aligned}
N(X^h, Y^h) = [X^h, Y^h]& \\
&+ J(\Theta(X^h, e^h_k)[e^h_k, Y^h] - Y^h(\Theta(X^h, e^h_k))e^h_k) \\
&\quad + J(\Theta(Y^h, e^h_k)[X^h, e^h_k] + X^h(\Theta(Y^h, e^h_k))e^h_k) \\
&\quad - \Theta(X^h, e^h_k)\Theta(Y^h, e^h_l)[e^h_k, e^h_l] \\
&\quad - \Theta(X^h, e^h_k)(e^h_k\Theta(Y^h, e^h_l))e^h_l \\
&\quad + \Theta(Y^h, e^h_l)(e^h_l\Theta(X^h, e^h_k))e^h_k. \qquad (109)
\end{aligned}$$

Theorem 5.4. *Let* $\{\omega^1, \omega^2, \omega^3\} = \{e^0 \wedge e^1 - e^2 \wedge e^3, e^0 \wedge e^2 - e^3 \wedge e^1, e^0 \wedge e^3 - e^1 \wedge e^2\}$ *be the local frame for* E*. Let* $\eta = \sum_i y_i \omega^i$ *where the* y_i *are smooth. For* $X^h, Y^h \in \Gamma(HE)$ *horizontal lifts of* $X, Y \in \Gamma(TM)$ *respectively,* $\eta \in E_p$ *then* $[X^h, Y^h]_\eta - [X, Y]^h_\eta = -R(X, Y)\eta$*.*

Proof. As $[X^h, Y^h]_\eta - [X, Y]^h_\eta \in V_\eta E$ from Theorem 2.18 using the identification of $VE \cong \pi^* E$ from Theorem 2.3

$$[X^h, Y^h]_\eta - [X, Y]^h_\eta = -y_i R(X_p, Y_p)^i_j \omega^j = -R(X_p, Y_p)\eta_{\pi(\eta)}. \quad \square$$

Evaluating $N(X^h, Y^h)$ at η, using Theorem 4.5 and using the identity $[X^h, Y^h]^h_\eta - [X, Y]^h_\eta = -R_p(X, Y)\eta$ from Theorem 5.4 in (109) and dropping the $\pi(\eta)$ from $\eta_{\pi(\eta)}$ for ease of notation then

$$\begin{aligned} N(X^h, Y^h) = [X, Y]^h - R(X, Y)\eta \\ &+ \Theta(X^h, e^h_k)\Theta([e_k, Y]^h, e^h_l)e^h_l - Y^h(\Theta(X^h, e^h_k))\Theta(e^h_k, e^h_l)e^h_l \\ &\quad + \Theta(Y^h, e^h_k)\Theta(([X, e_k]^h, e^h_l)e^h_l + X^h(\Theta(Y^h, e^h_k))\Theta(e^h_k, e^h_l)e^h_l \\ &\quad - \Theta(X^h, e^h_k)\Theta(Y^h, e^h_l)[e_k, e_l]^h \\ &\quad - \Theta(X^h, e^h_k)(e^h_k\Theta(Y^h, e^h_l))e^h_l \\ &\quad + \Theta(Y^h, e^h_l)(e^h_l\Theta(X^h, e^h_k))e^h_k \\ &\quad + \Theta(X^h, e^h_k)JR(e_k, Y)\eta \\ &\quad - \Theta(X^h, e^h_k)\Theta(Y^h, e^h_l)R(e_k, e_l)\eta \\ &\quad + \Theta(e^h_k, Y^h)JR(e_k, X)\eta. \end{aligned} \tag{110}$$

Using the tautological 2-form from (38), we compute $e^h_m(\Theta(e^h_k, e^h_l))$ and get

$$e^h_m(\Theta(e^h_k, e^h_l)) = \Theta((\nabla_{e_m} e_k)^h, e^h_l) + \Theta(e^h_k, (\nabla_{e_m} e_l)^h). \tag{111}$$

Recall the connection on E is induced from the Levi-Civita connection ∇ on TM. We substitute the identity $[X,Y] = \nabla_X Y - \nabla_Y X$ and (111) into equation (110) to get

$$
\begin{aligned}
N(X^h, Y^h) = [X,Y]^h - R(X,Y)\eta \\
&+ \Theta(X^h, e_k^h)\Theta((\nabla_{e_k} Y - \nabla_Y e_k)^h, e_l^h)e_l^h - Y^h(\Theta(X^h, e_k^h))\Theta(e_k^h, e_l^h)e_l^h \\
&+ \Theta(Y^h, e_k^h)\Theta((\nabla_X e_k - \nabla_{e_k} X)^h, e_l^h)e_l^h + X^h(\Theta(Y^h, e_k^h))\Theta(e_k^h, e_l^h)e_l^h \\
&- \Theta(X^h, e_k^h)\Theta(Y^h, e_l^h)(\nabla_{e_k} e_l - \nabla_{e_l} e_k)^h \\
&- \Theta(X^h, e_k^h)\Theta((\nabla_{e_k} Y)^h, e_l^h)e_l^h - \Theta(X^h, e_k^h)\Theta(Y^h, (\nabla_{e_k} e_l)^h)e_l^h \\
&+ \Theta(Y^h, e_l^h)\Theta((\nabla_{e_l} X)^h, e_k^h)e_k^h + \Theta(Y^h, e_l^h)\Theta(X^h, (\nabla_{e_l} e_k)^h)e_k^h \\
&+ \Theta(X^h, e_k^h)JR(e_k, Y)\eta \\
&- \Theta(X^h, e_k^h)\Theta(Y^h, e_l^h)R(e_k, e_l)\eta \\
&+ \Theta(e_k^h, Y^h)JR(e_k, X)\eta.
\end{aligned}
\tag{112}
$$

Using that the horizontal lift is linear then

$$
\begin{aligned}
(\nabla_{e_k} e_l - \nabla_{e_l} e_k)^h &= (\nabla_{e_k} e_l)^h - (\nabla_{e_l} e_k)^h, \\
(\nabla_{e_k} Y - \nabla_Y e_k)^h &= (\nabla_{e_k} Y)^h - (\nabla_Y e_k)^h, \\
(\nabla_X e_k - \nabla_{e_k} X)^h &= (\nabla_X e_k)^h - (\nabla_{e_k} X)^h.
\end{aligned}
\tag{113}
$$

Substituting (113) into equation (112) becomes

$$
\begin{aligned}
N(X^h, Y^h) = [X, Y]^h - R(X, Y)\eta \\
&+ \Theta(X^h, e_k^h)\Theta((-\nabla_Y e_k)^h, e_l^h)e_l^h - Y^h(\Theta(X^h, e_k^h))\Theta(e_k^h, e_l^h)e_l^h \\
&+ \Theta(Y^h, e_k^h)\Theta((\nabla_X e_k)^h, e_l^h)e_l^h + X^h(\Theta(Y^h, e_k^h))\Theta(e_k^h, e_l^h)e_l^h \\
&- \Theta(X^h, e_k^h)\Theta(Y^h, e_l^h)((\nabla_{e_k} e_l)^h - (\nabla_{e_l} e_k)^h) \\
&- \Theta(X^h, e_k^h)\Theta(Y^h, (\nabla_{e_k} e_l)^h)e_l^h \\
&+ \Theta(Y^h, e_l^h)\Theta(X^h, (\nabla_{e_l} e_k)^h)e_l^h \\
&+ \Theta(X^h, e_k^h)JR(e_k, Y)\eta \\
&- \Theta(X^h, e_k^h)\Theta(Y^h, e_l^h)R(e_k, e_l)\eta \\
&+ \Theta(e_k^h, Y^h)JR(e_k, X)\eta.
\end{aligned}
\tag{114}
$$

Rearranging some terms in equation (114) yields

$$
\begin{aligned}
N(X^h, Y^h) = [X, Y]^h \\
&+ \Theta(X^h, e_k^h)\Theta((-\nabla_Y e_k)^h, e_l^h)e_l^h - Y^h(\Theta(X^h, e_k^h))\Theta(e_k^h, e_l^h)e_l^h \\
&+ \Theta(Y^h, e_k^h)\Theta((\nabla_X e_k)^h, e_l^h)e_l^h + X^h(\Theta(Y^h, e_k^h))\Theta(e_k^h, e_l^h)e_l^h \\
&- \Theta(X^h, e_k^h)\Theta(Y^h, e_l^h)(\nabla_{e_k} e_l)^h - \Theta(X^h, e_k^h)\Theta(Y^h, (\nabla_{e_k} e_l)^h)e_l^h \\
&+ \Theta(Y^h, e_l^h)\Theta(X^h, (\nabla_{e_l} e_k)^h)e_l^h + \Theta(X^h, e_k^h)\Theta(Y^h, e_l^h)(\nabla_{e_l} e_k)^h \\
&+ R(X, Y)\eta \\
&+ \Theta(X^h, e_k^h)JR(e_k, Y)\eta \\
&- \Theta(X^h, e_k^h)\Theta(Y^h, e_l^h)R(e_k, e_l)\eta \\
&+ \Theta(e_k^h, Y^h)JR(e_k, X)\eta.
\end{aligned}
\tag{115}
$$

We note that from Theorem 4.5 that $J^2(X^h) = -X^h$ and $-X^h = \Theta(X^h, e_i^h)\Theta(e_i^h, e_j^h)e_j^h$. Expanding X^h in terms of the frame $\{e_i^h\}$ using the metric g on M then, $X^h =$

$g_{\pi(\eta)}(\pi_* X^h, \pi_* e_j^h)e_j^h = g_{\pi(\eta)}(X, e_j)e_j^h$. And $J^2(X^h) = -X^h$ becomes

$$-g_{\pi(\eta)}(X, e_j) = \Theta(x^h, e_i^h)\Theta(e_i^h, e_j^h). \tag{116}$$

Theorem 5.5. *The expression $N(X^h, Y^h)$ at η of equation (115) can be reduced to*

$$\begin{aligned} N(X^h, Y^h)_\eta = \Theta_\eta(X^h, e_k^h)&J(R(e_k, Y)\eta) \\ &- \Theta_\eta(X^h, e_k^h)\Theta_\eta(Y^h, e_l^h)R(e_k, e_l)\eta \\ &+ \Theta_\eta(e_k^h, Y^h)J(R(e_k, X)\eta) \\ &+ R(X, Y)\eta. \end{aligned} \tag{117}$$

Proof. The proof will follow from two lemmas.

Lemma 5.6. *The fourth term and fifth term in equation (115) vanish:*

$$0 = -\Theta(X^h, e_k^h)\Theta(Y^h, e_l^h)(\nabla_{e_k} e_l)^h - \Theta(X^h, e_k^h)\Theta(Y^h, (\nabla_{e_k} e_l)^h)e_l^h \tag{118}$$

$$0 = \Theta(Y^h, e_l^h)\Theta(X^h, (\nabla_{e_l} e_k)^h)e_l^h + \Theta(X^h, e_k^h)\Theta(Y^h, e_l^h)(\nabla_{e_l} e_k)^h. \tag{119}$$

Proof. Looking at the term

$$-\Theta(X^h, e_k^h)\Theta(Y^h, e_l^h)(\nabla_{e_k} e_l)^h - \Theta(X^h, e_k^h)(\Theta(Y^h, (\nabla_{e_k} e_l)^h))e_l^h,$$

and substituting $g(\nabla_{e_k} e_l, e_m) e_m^h = (\nabla_{e_k} e_l)^h$ yields

$$\begin{aligned}
&- \Theta(X^h, e_k^h)\Theta(Y^h, e_l^h)(\nabla_{e_k} e_l)^h - \Theta(X^h, e_k^h)\Theta(Y^h, (\nabla_{e_k} e_l)^h) e_l^h \\
&= -\Theta(X^h, e_k^h)\Theta(Y^h, e_l^h) g(\nabla_{e_k} e_l, e_m) e_m^h \\
&\qquad - \Theta(X^h, e_k^h)\Theta(Y^h, g(\nabla_{e_k} e_l, e_m) e_m^h) e_l^h \\
&= -\Theta(X^h, e_k^h)\Theta(Y^h, e_l^h) g(\nabla_{e_k} e_l, e_m) e_m^h \\
&\qquad - \Theta(X^h, e_k^h)\Theta(Y^h, e_l^h) g(\nabla_{e_k} e_m, e_l) e_m^h \qquad \text{Relabelling } l \to m, m \to l \\
&= -e_k(g(e_l, e_m))\Theta(X^h, e_k^h)\Theta(Y^h, e_l^h) \\
&= 0. \qquad (120)
\end{aligned}$$

Similarly

$$\begin{aligned}
&\Theta(Y^h, e_l^h)\Theta(X^h, (\nabla_{e_l} e_k)^h) e_l^h + \Theta(X^h, e_k^h)\Theta(Y^h, e_l^h)(\nabla_{e_l} e_k)^h \\
&= \Theta(X^h, e_k^h)\Theta(Y^h, e_l^h) e_l(g(e_k, e_m)) \\
&= 0. \qquad (121)
\end{aligned}$$

We have shown that the fourth and fifth term in equation (115) vanish. □

Lemma 5.7. *The first three terms in equation (115)vanish:*

$$\begin{aligned}
0 =& [X, Y]^h \\
&+ \Theta(X^h, e_k^h)\Theta((-\nabla_Y e_k)^h, e_l^h) e_l^h - Y^h(\Theta(X^h, e_k^h))\Theta(e_k^h, e_l^h) e_l^h \\
&\quad + \Theta(Y^h, e_k^h)\Theta((\nabla_X e_k)^h, e_l^h) e_l^h + X^h(\Theta(Y^h, e_k^h))\Theta(e_k^h, e_l^h) e_l^h \qquad (122)
\end{aligned}$$

Proof. Looking at second term in equation (115), $\Theta(X^h, e_k^h)\Theta(-\nabla_Y e_k)^h, e_l^h) e_l^h$, using

Theorem 2.20 and (116) in $\Theta(X^h, e_k^h)\Theta((-\nabla_Y e_k)^h, e_l^h)e_l^h$, we have

$$\begin{aligned}
&\Theta(X^h, e_k^h)\Theta((-\nabla_Y e_k)^h, e_l^h)e_l^h \\
&= -\Theta(X^h, e_k^h)(Y^h(\Theta(e_k^h, e_l^h))e_l^h - \Theta(e_k^h, (\nabla_Y e_l)^h)e_l^h) \qquad \text{by Theorem (2.20)} \\
&= -g(X, \nabla_Y e_l)e_l^h - \Theta(X^h, e_k^h)(Y^h\Theta(e_k^h, e_l^h))e_l^h \qquad \text{by equation (116)} \\
&= -Y(g(X, e_l))e_l^h + g(\nabla_Y X, e_l)e_l^h - \Theta(X^h, e_k^h)(Y^h\Theta(e_k^h, e_l^h))e_l^h \\
&= -Y(g(X, e_l))e_l^h + (\nabla_Y X)^h - Y^h(\Theta(X^h, e_k^h)\Theta(e_k^h, e_l^h)e_l^h) \\
&\qquad + Y^h(\Theta(X^h, e_k^h))\Theta(e_k^h, e_l^h)e_l^h \qquad \text{by equation (116) and } Y^h(f\circ\pi) = (Yf)\circ\pi \\
&= (\nabla_Y X)^h + (Y^h(\Theta(X^h, e_k^h)))\Theta(e_k^h, e_l^h)e_l^h.
\end{aligned} \tag{123}$$

Using Theorem 2.20 and (116) in $\Theta_\eta(Y^h, e_k^h)\Theta_\eta((\nabla_X e_k)^h, e_l^h)e_l^h$, interchanging X and Y in (123) and changing the sign we get

$$\begin{aligned}
&\Theta(Y^h, e_k^h)\Theta((\nabla_X e_k)^h, e_l^h)e_l^h \\
&= -(\nabla_X Y)^h - (X^h\Theta(Y^h, e_k^h))\Theta(e_k^h, e_l^h)e_l^h.
\end{aligned} \tag{124}$$

From equations (123) and (124) we have

$$\begin{aligned}
\Theta(X^h, e_k^h)\Theta((-\nabla_Y e_k)^h, e_l^h)e_l^h &= (\nabla_Y X)^h + (Y^h(\Theta(X^h, e_k^h))\Theta(e_k^h, e_l^h)e_l^h \\
\Theta(Y^h, e_k^h)\Theta((\nabla_X e_k)^h, e_l^h)e_l^h &= -(\nabla_X Y)^h - (X^h(\Theta(Y^h, e_k^h))\Theta(e_k^h, e_l^h)e_l^h.
\end{aligned} \tag{125}$$

Substituting (125) and substituting $[X,Y]^h = (\nabla_X Y)^h - (\nabla_Y X)^h$ we obtain

$$
\begin{aligned}
&[X,Y]^h \\
&\quad + \Theta(X^h, e_k^h)\Theta((-\nabla_Y e_k)^h, e_l^h)e_l^h - Y^h(\Theta(X^h, e_k^h))\Theta(e_k^h, e_l^h)e_l^h \\
&\quad + \Theta(Y^h, e_k^h)\Theta((\nabla_X e_k)^h, e_l^h)e_l^h + X^h(\Theta(Y^h, e_k^h))\Theta(e_k^h, e_l^h)e_l^h \\
&= (\nabla_X Y)^h - (\nabla_Y X)^h \\
&\quad + (\nabla_Y X)^h + Y^h(\Theta(X^h, e_k^h))\Theta(e_k^h, e_l^h)e_l^h - Y^h(\Theta(X^h, e_k^h))\Theta(e_k^h, e_l^h)e_l^h \\
&\quad - (\nabla_X Y)^h - (X^h\Theta(Y^h, e_k^h))\Theta(e_k^h, e_l^h)e_l^h + X^h(\Theta(Y^h, e_k^h))\Theta(e_k^h, e_l^h)e_l^h) \\
&= 0 \quad \square
\end{aligned} \tag{126}
$$

From Lemma 5.7 and Lemma 5.6 our expression of $N(X^h, Y^h)_\eta$ in equation (115) reduces to

$$
\begin{aligned}
N(X^h, Y^h)_\eta = \Theta_\eta(X^h, e_k^h)&J(R(e_k, Y)\eta) \\
&- \Theta_\eta(X^h, e_k^h)\Theta_\eta(Y^h, e_l^h)R(e_k, e_l)\eta \\
&+ \Theta_\eta(e_k^h, Y^h)J(R(e_k, X)\eta) \\
&+ R(X,Y)\eta.
\end{aligned} \tag{127}
$$

$\square$

5.4 $N(X^h, Y^h)$ and Weyl curvature

Let X, Y, V, T be vector fields on M and $\eta \in E$. From Theorem 4.5 we have that

$$
\begin{aligned}
N(X^h, Y^h)_\eta = \Theta_\eta(X^h, e_k^h)&J(R(e_k, Y)\eta) \\
&- \Theta_\eta(X^h, e_k^h)\Theta_\eta(Y^h, e_l^h)R(e_k, e_l)\eta \\
&+ \Theta_\eta(e_k^h, Y^h)J(R(e_k, X)\eta) \\
&+ R(X,Y)\eta.
\end{aligned} \tag{128}
$$

Since the $\Theta(U, V)$ are functions we can move them inside the curvature operator and using Theorem 4.5

$$
\begin{aligned}
N(X^h, Y^h)_\eta = J(R(\Theta_\eta(X^h, e_k^h)e_k, Y)\eta) \\
- R(\Theta_\eta(X^h, e_k^h)e_k, \Theta_\eta(Y^h, e_l^h)e_l)\eta \\
+ J(R(\Theta_\eta(e_k^h, Y^h)e_k, X)\eta) \\
+ R(X, Y)\eta
\end{aligned} \tag{129}
$$

Using that $\Theta_\eta(X^h, e_k^h)e_k = \eta(X, e_k)e_k$ and viewing η as an endomorphism, thus $\eta(X, e_k)e_k = g(\eta(X), e_k)e_k = \eta X$. We then get

$$
\begin{aligned}
N(X^h, Y^h)_\eta = J(R(\eta X, Y)\eta) \\
- R(\eta X, \eta Y)\eta \\
- J(R(\eta Y, X)\eta) \\
+ R(X, Y)\eta.
\end{aligned} \tag{130}
$$

Note that the terms such as $R(\eta X, Y)\eta$ in (130) are identified as vertical tangent vectors of E via $VE \cong \pi^* E$. The J is acting on vertical vectors in (129). Recall that $J(Y^v) = Y^v \times \xi$, under the identification of $V_\eta E \cong E_{\pi(\eta)}$ in Corollary 2.4. Hence $J(Y^v) = l(\lambda) \times l(\eta) = l(\eta \times \lambda)$ where $\lambda = v_i\omega^i, \eta = y_i\omega^i \in \Gamma(E)$ then $J(\lambda) = \eta \times \lambda$. Thus,

$$J(R(X, Y)\eta_{\pi(\eta)}) = \frac{1}{2}[\eta_{\pi(\eta)}, R(X, Y)\eta_{\pi(\eta)}] = \eta_{\pi(\eta)} \times R(X, Y)\eta_{\pi(\eta)} \tag{131}$$

when viewing $\eta_{\pi(\eta)}$ as an endomorphism. Note that $R(X, Y)$ is an operator, a derivation, and η is an operator as an endomorphism. Thus the commutator is a Lie bracket of operators that act on a vector field. Substituting (131) into (129), then (129) becomes

$$
\begin{aligned}
N(X^h, Y^h)_\eta(V,T) = R(X,Y,V,\eta(T))& \\
&+ R(X,Y,\eta(V),T) \\
&+ R(X,\eta(Y),V,T) \\
&+ R(\eta(X),Y,V,T) \\
&- R(\eta(X),\eta(Y),\eta(V),T) \\
&- R(X,\eta(Y),\eta(V),\eta(T)) \\
&- R(\eta(X),Y,\eta(V),\eta(T)) \\
&- R(\eta(X),\eta(Y),V,\eta(T)).
\end{aligned} \tag{132}
$$

Since the Nijenhuis tensor reduced to only terms involving curvature, we can decompose the Nijenhuis tensor further, by using curvature decomposition and the Kulkarni-Nomizu product.

Definition 5.8. For $X, Y, V, T \in \Gamma(TM)$, g the metric on M, then the Kulkarni-Nomizu product of g with a symmetric 2-tensor S is defined as

$$
\begin{aligned}
&(g \owedge S)(X,Y,V,T) = \\
&\langle X,T\rangle S(Y,V) + \langle Y,V\rangle S(X,T) - \langle X,V\rangle S(Y,T) - \langle Y,T\rangle S(X,V).
\end{aligned} \tag{133}
$$

Let K be the space of curvature tensors, g the metric on M, $S^2(\Lambda^2 T^*M)$ the symmetric 2-tensors and W the Weyl curvature. The space of curvature tensors splits as

$$\Gamma(K) = \Gamma(C^\infty(M)) \oplus \Gamma(S^2(\Lambda^2(T^*M))) \oplus \Gamma(W).$$

The curvature decomposition is given by

$$R = C_n R_g \owedge g \oplus \tilde{C}_n g \owedge Ric^\circ \oplus W = (g \owedge S) \oplus W.$$

The constants $C_n, \tilde{C}_n$ depend on the dimension of M, R_g is the scalar curvature and S is the Schouten tensor. The Schouten tensor is a linear combination of the scalar and trace free Ricci curvature tensors. See [4] for more details and proofs.

The Nijenhuis tensor (132) is linear in $R(X,Y)$. Applying this decomposition to the right hand side of (132) we get

$$\begin{aligned}
&\langle X,\eta(T)\rangle S(Y,V)+\langle Y,V\rangle S(X,\eta(T))-\langle X,V\rangle S(Y,\eta(T))-\langle Y,\eta(T)\rangle S(X,V)\\
&+\langle X,T\rangle S(Y,\eta(V))+\langle Y,\eta(V)\rangle S(X,T)-\langle X,\eta(V)\rangle S(Y,T)-\langle Y,T\rangle S(X,\eta(V))\\
&+\langle X,T\rangle S(\eta(Y),V)+\langle \eta(Y),V\rangle S(X,T)-\langle X,V\rangle S(\eta(Y),T)-\langle \eta(Y),T\rangle S(X,V)\\
&+\langle \eta(X),T\rangle S(Y,V)+\langle Y,V\rangle S(\eta(X),T)-\langle \eta(X),V\rangle S(Y,T)-\langle Y,T\rangle S(\eta(X),V)\\
&-\Big[\langle \eta(X),T\rangle S(\eta(Y),\eta(V))\\
&+\langle \eta(Y),\eta(V)\rangle S(\eta(X),T)-\langle \eta(X),\eta(V)\rangle S(\eta(Y),T)-\langle \eta(Y),T\rangle S(\eta(X),\eta(V))\Big]\\
&-\Big[\langle X,\eta(T)\rangle S(\eta(Y),\eta(V))\\
&+\langle \eta(Y),\eta(V)\rangle S(X,\eta(T))-\langle X,\eta(V)\rangle S(\eta(Y),\eta(T))-\langle \eta(Y),\eta(T)\rangle S(X,\eta(V))\Big]\\
&-\Big[\langle \eta(X),\eta(T)\rangle S(Y,\eta(V))\\
&+\langle Y,\eta(V)\rangle S(\eta(X),\eta(T))-\langle \eta(X),\eta(V)\rangle S(Y,\eta(T))-\langle Y,\eta(T)\rangle S(\eta(X),\eta(V))\Big]\\
&-\Big[\langle \eta(X),\eta(T)\rangle S(\eta(Y),V)\\
&+\langle \eta(Y),V\rangle S(\eta(X),\eta(T))-\langle \eta(X),V\rangle S(\eta(Y),\eta(T))-\langle \eta(Y),\eta(T)\rangle S(\eta(X),V)\Big].
\end{aligned} \tag{134}$$

Using the identities coming from the orthogonal complex structure $J = (X \lrcorner \eta)^\sharp$ where η is viewed as an endomorphism, we have $\langle X, \eta Y\rangle = -\langle \eta X, Y\rangle$ and $\langle \eta X, \eta Y\rangle = \langle X, Y\rangle$. Then all terms cancel out in (134). This implies that the $(g \owedge S)(X,Y,V,T)$ component of $N(X^h, Y^h)$ vanishes and thus $N(X^h, Y^h)(V,T)$ only depends on the Weyl curvature component.

Thus, from (132), the Nijenhuis tensor on horizontal vector fields can thus be rewritten as

$$\begin{aligned} N(X^h, Y^h)_\eta(V,T) = W(&X, Y, V, \eta(T)) \\ &+ W(X, Y, \eta(V), T) \\ &+ W(X, \eta(Y), V, T) \\ &+ W(\eta(X), Y, V, T) \\ &- W(\eta(X), \eta(Y), \eta(V), T) \\ &- W(X, \eta(Y), \eta(V), \eta(T)) \\ &- W(\eta(X), Y, \eta(V), \eta(T)) \\ &- W(\eta(X), \eta(Y), V, \eta(T)), \end{aligned} \tag{135}$$

where W is the Weyl curvature.

The Weyl curvature decomposes into self-dual and anti-self-dual components, $W = W_+ \oplus W_-$, where W_+ is the self-dual part and W_- is the anti-self-dual part.

Lemma 5.9. *The expression $N(X^h, Y^h)(V,T)$ only depends on the anti-self-dual part of the Weyl curvature W_-.*

Proof. We note that $W(X, Y, V, T)$ can be rewritten as $\langle W(X \wedge Y), V \wedge T \rangle$. Using this we rewrite (135) of $N(X^h, Y^h)(V,T)$ as

$$
\begin{aligned}
N(X^h, Y^h)(V,T) = {} & \langle W(X \wedge Y), V \wedge \eta(T) \rangle \\
& + \langle W(X \wedge Y), \eta(V) \wedge T \rangle \\
& + \langle W(X \wedge \eta(Y)), V \wedge T \rangle \\
& + \langle W(\eta(X) \wedge Y), V \wedge T \rangle \\
& - \langle W(\eta(X) \wedge \eta(Y)), \eta(V) \wedge T \rangle \\
& - \langle W(X \wedge \eta(Y)), \eta(V) \wedge \eta(T) \rangle \\
& - \langle W(\eta(X) \wedge Y), \eta(V) \wedge \eta(T) \rangle \\
& - \langle W(\eta(X) \wedge \eta(Y)), V \wedge \eta(T) \rangle.
\end{aligned} \tag{136}
$$

Next gathering like terms in the $N(X^h, Y^h)(V,T)$ equation we obtain

$$
\begin{aligned}
N(X^h, Y^h)_\eta(V,T) = {} & \langle W(X \wedge Y) - W(\eta X \wedge \eta Y), V \wedge \eta(T) \rangle \\
& + \langle W(X \wedge Y) - W(\eta X \wedge \eta Y), \eta(V) \wedge T \rangle \\
& + \langle W(X \wedge \eta(Y)), V \wedge T - \eta(V) \wedge \eta(T) \rangle \\
& + \langle W(\eta(X) \wedge Y), V \wedge T - \eta(V) \wedge \eta(T) \rangle.
\end{aligned} \tag{137}
$$

Since the Weyl curvature is linear we get

$$
\begin{aligned}
N(X^h, Y^h)_\eta(V,T) = {} & \langle W(X \wedge Y - \eta X \wedge \eta Y), V \wedge \eta(T) + \eta(V) \wedge T \rangle \\
& + \langle W(X \wedge \eta(Y) + \eta(X) \wedge Y), V \wedge T - \eta(V) \wedge \eta(T) \rangle.
\end{aligned} \tag{138}
$$

Using the fact that $V \wedge \eta(T) + \eta(T) \wedge T$ and $V \wedge T - \eta(V) \wedge \eta(T)$ are ASD, equation

(138) becomes:

$$
\begin{aligned}
&\langle W(X \wedge Y - \eta X \wedge \eta Y), V \wedge \eta(T) + \eta(V) \wedge T \rangle \\
&\qquad + \langle W(X \wedge \eta(Y)) + \eta(X) \wedge Y), V \wedge T - \eta(V) \wedge \eta(T) \rangle \\
&= \langle W_-(X \wedge Y - \eta X \wedge \eta Y), (V \wedge \eta(T) + \eta(V) \wedge T) \rangle \\
&\qquad + \langle W_-(X \wedge \eta(Y)) + \eta(X) \wedge Y), (V \wedge T - \eta(V) \wedge \eta(T)) \rangle.
\end{aligned} \tag{139}
$$

Hence we see that W_+ does not contribute to (139) and the Nijenhuis tensor has terms only with the anti-self-dual Weyl curvature part. □

We define some notation we will use next. We let

$$
\begin{aligned}
&(W)_{0123} \\
&= W(e_0, e_1, e_2, e_3) \\
&= \langle W(e_0 \wedge e_1), e_2 \wedge e_3 \rangle.
\end{aligned} \tag{140}
$$

The Weyl curvature shares the same symmetries as the Riemannian curvature tensor and it is trace free, that is $\sum_i W_{ijki} = 0$.

Using our local oriented orthonormal frame $\{e_0, e_1, e_2, e_3\}$ for our manifold M, we have $e_0 \wedge e_1 \wedge e_2 \wedge e_3 = \mu$. Since X, Y, V, T are arbitrary, letting $X = e_0, Y = e_2, e_1 = \eta X, e_3 = -\eta Y$, η in this context represents our orthogonal complex structure, $(X \lrcorner \omega^1)^\sharp$, that sends $e_0 \to e_1, e_1 \to -e_0, e_2 \to -e_3, e_3 \to e_2$. We denote this complex structure ω^1. Since η is anti-self-dual, we get that the orientation induced by η is $X \wedge \eta X \wedge Y \wedge \eta Y = -\mu$. For our choice of the complex structure that sends

$e_0 \to e_1, e_2 \to e_3$ which is ω^1 then the local frame $\{\omega^1, \omega^2, \omega^3\}$ will be described by

$$\begin{aligned}
\omega^1 &= e^0 \wedge e^1 - e^2 \wedge e^3 \\
&= e^0 \wedge \eta e^0 + e^2 \wedge \eta e^2 \\
&= X \wedge \eta X + Y \wedge \eta Y \\
\omega^2 &= e^0 \wedge e^2 - e^3 \wedge e^1 \\
&= e^0 \wedge e^2 - (-\eta e^2) \wedge \eta e^0 \\
&= X \wedge Y - \eta X \wedge \eta Y \\
\omega^3 &= e^0 \wedge e^3 - e^1 \wedge e^2 \\
&= e^0 \wedge (-\eta e^2) - \eta e^0 \wedge e^2 \\
&= -(X \wedge \eta Y + \eta X \wedge Y). \qquad (141)
\end{aligned}$$

The tensor W_- can be represented in terms of the frame (141) as

$$W_- = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} = \begin{bmatrix} W_-(\omega^1, \omega^1) & W_-(\omega^1, \omega^2) & W_-(\omega^1, \omega^3) \\ W_-(\omega^2, \omega^1) & W_-(\omega^2, \omega^2) & W_-(\omega^2, \omega^3) \\ W_-(\omega^3, \omega^1) & W_-(\omega^3, \omega^2) & W_-(\omega^3, \omega^3) \end{bmatrix} \qquad (142)$$

We will show all the components of the matrix vanish iff $N(X^h, Y^h) = 0$ for all X^h, Y^h.

Lemma 5.10. *We have $a + b + c = 0$ in the W_- matrix (142).*

Proof. This lemma will always be true since the Weyl curvature is trace free. We have

$$
\begin{aligned}
&a+b+c \\
&= (W_-)(\omega^1,\omega^1) + (W_-)(\omega^2,\omega^2) + (W_-)(\omega^3,\omega^3) \\
&= (W_-)_{(01-23)(01-23)} + (W_-)_{(02-31)(02-31)} + (W_-)_{(03-12)(03-12)} \\
&= (W_-)_{0101} + (W_-)_{2323} + (W_-)_{0202} \\
&\qquad + (W_-)_{3131} + (W_-)_{0303} + (W_-)_{1212} \\
&\qquad - 2((W_-)_{0123} + (W_-)_{0231} + (W_-)_{0312}).
\end{aligned} \tag{143}
$$

As W_- shares the same symmetries as the Riemannian curvature tensor then by the Bianchi identity for the Weyl curvature $(W_-)_{0123} + (W_-)_{0231} + (W_-)_{0312} = 0$. As W_- is trace free, and $(W_-)_{0000} = 0$ then

$$
\begin{aligned}
(W_-)_{0101} + (W_-)_{0202} + (W_-)_{0303} &= -(W_-)_{0110} - (W_-)_{0220} - (W_-)_{0330} - (W_-)_{0000} \\
&= -\sum_{i=0}^{4} (W_-)_{0ii0} \\
&= 0.
\end{aligned} \tag{144}
$$

We then have left to show that

$$
(W_-)_{2323} + (W_-)_{3131} + (W_-)_{1212} = 0. \tag{145}
$$

By using the identity $(W_-)* = *(W_-)$, and that the Hodge star is an isometry we will rearrange $(W_-)_{2323}$ as follows

$$\begin{aligned}(W_-)_{2323} &= \langle (W_-)(e^2 \wedge e^3), e^2 \wedge e^3 \rangle \\ &= \langle *(W_-)(e^2 \wedge e^3), *(e^2 \wedge e^3) \rangle \\ &= \langle (W_-) * (e^2 \wedge e^3), *(e^2 \wedge e^3) \rangle \\ &= \langle (W_-)((e^0 \wedge e^1)), (e^0 \wedge e^1) \rangle \\ &= (W_-)_{0101}. \end{aligned} \tag{146}$$

Substituting (146) into (145) we get

$$\begin{aligned}&(W_-)_{2323} + (W_-)_{3131} + (W_-)_{1212} \\ &= (W_-)_{0101} + (W_-)_{3131} + (W_-)_{1212} \\ &= -\sum_{i=0}^{4} (W_-)_{1ii1} \qquad \text{using the technique in (144)} \\ &= 0. \end{aligned} \tag{147}$$

Thus (W_-) is trace free, so $a + b + c = 0$. □

Lemma 5.11. *We have $f = 0$ in the W_- matrix (142) if $N(X^h, Y^h) = 0$ for all X^h, Y^h.*

Proof. Using the Nijenhuis Weyl equation (138), and substituting $(X, Y, V, T) = (e^0, e^2, e^0, e^2)$ and using that W_- is a self-adjoint operator on $\Lambda^2 T^* M$ and taking the complex structure $\eta = \omega^1$ then

$$\begin{aligned}0 = N(X^h, Y^h)(V, T) &= 2\langle W_-(e^0 \wedge e^2 - \eta e^0 \wedge \eta e^2), e^0 \wedge \eta e^2 + \eta e^0 \wedge e^2 \rangle \\ &= 2W_-(\omega^2, \omega^3) \\ &= -2f. \quad \square \end{aligned} \tag{148}$$

Lemma 5.12. *We have $b - c = 0$ in the W_- matrix (142) if $N(X^h, Y^h) = 0$ for all X^h, Y^h.*

Proof. Using the Nijenhuis Weyl Equation (138), and subsituting $(X, Y, V, \eta T) = (e^0, e^2, e^0, e^2)$, using the complex structure $\eta = \omega^1$ then we get

$$\begin{aligned} 0 = N(X^h, Y^h)(V, T) &= \langle W_-(e^0 \wedge e^2 - \eta e^0 \wedge \eta e^2), e^0 \wedge e^2 - \eta e^0 \wedge \eta e^2 \rangle \\ &+ \langle W_-(e^0 \wedge \eta e^2 + \eta e^0 \wedge e^2), e^0 \wedge (-\eta e^2) - \eta e^0 \wedge e^2 \rangle \\ &= W_-(\omega^2, \omega^2) - W_-(\omega^3, \omega^3) \\ &= b - c. \quad \Box \end{aligned} \tag{149}$$

Lemma 5.13. *We have $d, e, a, b, c = 0$ in the matrix (142) of W_- if $N(X^h, Y^h) = 0$ for all X^h, Y^h.*

Proof. The Nijenhuis Weyl eqaution (138) is satisfied for all $\eta \in \Lambda^2_-(T^*M)$ such that $|\eta|^2 = 2$.

Consider the complex structures defined by ω^2, namely

$$\begin{aligned} e^0 &\to e^2 \\ e^2 &\to -e^0 \\ e^3 &\to -e^1 \\ e^1 &\to e^3 \end{aligned} \tag{150}$$

and by ω^3, namely

$$\begin{aligned} e^0 &\to e^3 \\ e^3 &\to -e^0 \\ e^1 &\to -e^2 \\ e^2 &\to e^1. \end{aligned} \tag{151}$$

This has the effect of permuting the anti-self-dual frame in (141) by sending $(\omega^1, \omega^2, \omega^3) \to (\omega^2, \omega^3, \omega^1)$ for the complex structure ω^2 and by sending $(\omega^1, \omega^2, \omega^3) \to (\omega^3, \omega^1, \omega^2)$ for the complex structure ω^3 respectively. Hence we get a new set of Weyl equations by permuting the anti-self-dual basis vectors in the Weyl tensor. For the complex structure ω^2 and following what we did in Lemma 5.11 and Lemma 5.12 we get

$$\begin{aligned} &W_-(\omega^3, \omega^3) - W_-(\omega^1, \omega^1) = c - a = 0 \\ &W_-(\omega^3, \omega^1) = 0 = e. \end{aligned} \tag{152}$$

For the complex structure ω^3 and following what we did in Lemma 5.11 and Lemma 5.12 we get

$$\begin{aligned} &W_-(\omega^1, \omega^1) - W_-(\omega^2, \omega^2) = a - b = 0 \\ &W_-(\omega^1, \omega^2) = 0 = d. \quad \square \end{aligned} \tag{153}$$

Theorem 5.14. *For $N(X^h, Y^h) = 0$ for all X^h, Y^h iff all the components of W_- vanish.*

Proof. Combining Lemma 5.11, Lemma 5.12, Lemma 5.13 we get $a = b = c = d = e = f = 0$ in the W_- matrix (142) and hence for $N(X^h, Y^h)$ to vanish W_- must vanish. By (142) if $a = b = c = d = e = f = 0$ then $N(X^h, Y^h) = 0$ for all X^h, Y^h. $\square$

5.5 Nijenhuis tensor of two vertical vectors

Let $X^v, Y^v \in \Gamma(VE)$. By definition the Nijenhuis tensor of two vertical vector fields is

$$N(X^v, Y^v) = [X^v, Y^v] + J[JX^v, Y^v] + J[X^v, JY^v] - [JX^v, JY^v]. \tag{154}$$

Using that $JX^v = \xi \times X^v$ from Definition 4.6, $\xi \in \Gamma(VE)$ is orthogonal to TZ in (69) and noting that $N(X^v, Y^v)$ is a vertical vector field and that the fibre of E is isomorphic to $\mathbb{R}^3$ then all of our operations in this section can be regarded as in $\mathbb{R}^3$. We will use the usual Euclidean connection for this section. Hence, for this section let ∇ be the standard connection on $\mathbb{R}^3$. On $\mathbb{R}^3$ it is well known that $\mathbb{R}^3 \cong so(3)$ and the Lie bracket on the Lie algebra $so(3)$ corresponds to twice the cross product on $\mathbb{R}^3$. Hence using this correspondence and that $[X,Y] = \nabla_X Y - \nabla_Y X$ then $N(X^v, Y^v)$ becomes

$$\begin{aligned} N(X^v,Y^v) &= [X^v,Y^v] + \xi \times [\xi \times X^v, Y^v] + \xi \times [X^v, \xi \times Y^v] - [\xi \times X^v, \xi \times Y^v] \\ &= \nabla_{X^v} Y^v - \nabla_{Y^v} X^v \\ &\quad + \xi \times (\nabla_{\xi \times X^v} Y^v - \nabla_{Y^v}(\xi \times X^v)) \\ &\quad + \xi \times (\nabla_{X^v}(\xi \times Y^v) - \nabla_{\xi \times Y^v} X^v) \\ &\quad - (\nabla_{\xi \times X^v}(\xi \times Y^v) - \nabla_{\xi \times Y^v}(\xi \times X^v)). \end{aligned} \tag{155}$$

For $C, A, B \in \Gamma(T\mathbb{R}^3)$, and letting $\langle , \rangle$ be the metric on $\mathbb{R}^3$, the iterated cross product identity is

$$A \times (B \times C) = \langle A, C \rangle B - \langle A, B \rangle C. \tag{156}$$

For $C, A, B \in \Gamma(T\mathbb{R}^3)$, and letting $\langle , \rangle$ be the metric on $\mathbb{R}^3$, the Euclidean connection acting on the cross product satisfies

$$\nabla_C (A \times B) = (\nabla_C A) \times B + A \times (\nabla_C B). \tag{157}$$

Theorem 5.15. *The Nijenhuis tensor of two vertical vector fields vanishes.*

Proof. Expanding $N(X^v, Y^v)$ as in equation (155) using (157) then

$$
\begin{aligned}
N(X^v, Y^v) = \nabla_{X^v} Y^v - \nabla_{Y^v} X^v \\
&+ \xi \times \nabla_{\xi \times X^v} Y^v \\
&- \xi \times ((\nabla_{Y^v} \xi \times X^v) + \xi \times (\nabla_{Y^v} X^v)) \\
&+ \xi \times ((\nabla_{X^v} \xi) \times Y^v + \xi \times \nabla_{X^v} Y^v) \\
&- \xi \times \nabla_{\xi \times Y^v} X^v \\
&- (\nabla_{\xi \times X^v} \xi) \times Y^v - \xi \times (\nabla_{\xi \times X^v} Y^v) \\
&+ (\nabla_{\xi \times Y^v}) \xi \times X^v + \xi \times (\nabla_{\xi \times Y^v} X^v). \qquad (158)
\end{aligned}
$$

Lemma 5.16. *For $\xi = y_1 \frac{\partial}{\partial y^1} + y_2 \frac{\partial}{\partial y^2} + y_3 \frac{\partial}{\partial y^3}$, $A = A_1 \frac{\partial}{\partial y^1} + A_2 \frac{\partial}{\partial y^2} + A_3 \frac{\partial}{\partial y^3}$ and for ∇ the Euclidean connection then*

$$
\nabla_A \xi = A. \qquad (159)
$$

Proof. Evaluating $\nabla_A \xi$ we get

$$
\begin{aligned}
\nabla_A \xi &= \left(A_1 \nabla_{\frac{\partial}{\partial y^1}} + A_2 \nabla_{\frac{\partial}{\partial y^2}} + A_3 \nabla_{\frac{\partial}{\partial y^3}} \right) \left(y_1 \frac{\partial}{\partial y^1} + y_2 \frac{\partial}{\partial y^2} + y_3 \frac{\partial}{\partial y^3} \right) \\
&= A_1 \frac{\partial}{\partial y^1} + A_2 \frac{\partial}{\partial y^2} + A_3 \frac{\partial}{\partial y^3} \\
&= A. \quad \square \qquad (160)
\end{aligned}
$$

Using Lemma 5.16 in (158), and

$$
|\xi|^2 = \sum_{i,j=1}^{3} y_i y_j \langle \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \rangle = \sum_{i=1}^{3} y_i^2 \langle \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_i} \rangle = \sum_{i=1}^{3} y_i^2 = 1
$$

we get

$$
\begin{aligned}
N(X^v, Y^v) = \nabla_{X^v} Y^v - \nabla_{Y^v} X^v \\
&- (\langle \xi, X^v \rangle \nabla_{Y^v} \xi - \langle \xi, \nabla_{Y^v} \xi \rangle X^v \\
&+ (\langle \xi, Y^v \rangle \nabla_{X^v} \xi - \langle \xi, \nabla_{X^v} \xi \rangle Y^v \\
&- (\langle \xi, \nabla_{Y^v} X^v \rangle \xi - \langle \xi, \xi \rangle \nabla_{Y^v} X^v \\
&+ (\langle \xi, \nabla_{X^v} Y^v \rangle \xi - \langle \xi, \xi \rangle \nabla_{X^v} Y^v \\
&- ((\xi \times X^v) \times Y^v) \\
&+ ((\xi \times Y^v) \times X^v).
\end{aligned} \tag{161}
$$

Simplifying using the identity (156), equation (161) becomes

$$
\begin{aligned}
N(X^v, Y^v) = \nabla_{X^v} Y^v - \nabla_{Y^v} X^v \\
&- (\langle \xi, X^v \rangle Y^v - \langle \xi, Y^v \rangle X^v) \\
&+ (\langle \xi, Y^v \rangle X^v - \langle \xi, X^v \rangle Y^v) \\
&- (\langle \xi, \nabla_{Y^v} X^v \rangle \xi - \nabla_{Y^v} X^v) \\
&+ (\langle \xi, \nabla_{X^v} Y^v \rangle \xi - \nabla_{X^v} Y^v) \\
&- (\langle \xi, Y^v \rangle X^v - \langle X^v, Y^v \rangle \xi) \\
&+ (\langle \xi, X^v \rangle Y^v - \langle X^v, Y^v \rangle \xi).
\end{aligned} \tag{162}
$$

Cancelling terms in equation (162) and using that ξ is orthogonal to Y^v and X^v then equation (162) becomes

$$
\begin{aligned}
N(X^v, Y^v) &= -\langle \xi, \nabla_{Y^v} X^v \rangle \xi \\
&\quad + \langle \xi, \nabla_{X^v} Y^v \rangle \xi \\
&= \langle \xi, [X^v, Y^v] \rangle \xi.
\end{aligned} \tag{163}
$$

Since $[X^v, Y^v] \in TZ$ and ξ is orthogonal to TZ, hence ξ is orthogonal $[X^v, Y^v]$ then

$\langle \xi, [X^v, Y^v] \rangle = 0$ and $N(X^v, Y^v) = 0$. $\square$

6 Pullback of self-dual connections on vector bundles

Recall that $Z \subset E = \Lambda^2_- T^*M$ where $Z = S_{\sqrt{2}}(\Lambda^2_-(T^*M))$. Let B be a vector bundle over M and consider a connection ∇^B on B. Let $\pi : Z \to M$ be the projection map. We will look at the pullback of 2-forms to Z to show that self-dual instantons on B, Definition 6.6, correspond to holomorphic structures on $\pi^* B$.

Definition 6.1. For $\eta \in \Omega^2(M)$ we define η_+ to be the self-dual part of η and η_- to be the anti-self-dual part of η.

Lemma 6.2. *For λ a constant. For $\omega \in E$ $\eta \in \Omega^2(M)$ identified with an endomorphisms. Then*

$$(\omega(\eta^+ + \eta^-))_{kc} = ((\eta^+ - \eta^-)\omega)_{kc} + \langle -\eta, \omega \rangle \delta_{kc},$$

where $\eta\omega$ is composition of endomorphisms.

Proof. Let μ be the volume form and $(*\omega)_{ab} = \frac{1}{2}\omega_{pq}\mu_{pqab}$. We use $\mu_{ijkp}\mu_{abcp} = (\delta_{ia}\delta_{jb}\delta_{kc} + \delta_{ib}\delta_{jc}\delta_{ka} + \delta_{ic}\delta_{ja}\delta_{kb} - \delta_{ia}\delta_{jc}\delta_{kb} - \delta_{ib}\delta_{ja}\delta_{kc} - \delta_{ic}\delta_{jb}\delta_{ka})$. This follows from taking the inner product of $e_k \lrcorner e_j \lrcorner e_i \lrcorner \mu = \mu_{ijkp} e^p$ with $e_c \lrcorner e_b \lrcorner e_a \lrcorner \mu = \mu_{abcq} e^q$, the duality between the interior/exterior product and the Hodge star isomorphism.

$$
\begin{aligned}
(\omega\eta)_{kc} &= \omega_{kp}\eta_{pc} \\
&= \omega_{kp}(\eta^{+}_{pc} + \eta^{-}_{pc}) \\
&= \frac{1}{2}(\omega_{kp}\mu_{abpc}\eta^{+}_{ab} - \omega_{kp}\mu_{abpc}\eta^{-}_{ab}) \\
&= \frac{1}{4}(-\omega_{ij}\mu_{ijkp}\mu_{abpc}\eta^{+}_{ab} + \omega_{ij}\mu_{ijkp}\mu_{abpc}\eta^{-}_{ab}) \text{ , since } (*\omega)_{ab} = \omega_{pq}\mu_{pqab} \\
&= \frac{1}{4}(\omega_{ij}\mu_{ijkp}\mu_{abcp}\eta^{+}_{ab} - \omega_{ij}\mu_{ijkp}\mu_{abcp}\eta^{-}_{ab}) \\
&= \frac{1}{4}(\omega_{ij}\eta^{+}_{ab} - \omega_{ij}\eta^{-}_{ab})\mu_{ijkp}\mu_{abcp} \\
&= \frac{1}{4}(\omega_{ij}\eta^{+}_{ab} - \omega_{ij}\eta^{-}_{ab})(\delta_{ia}\delta_{jb}\delta_{kc} + \delta_{ib}\delta_{jc}\delta_{ka} + \delta_{ic}\delta_{ja}\delta_{kb} - \delta_{ia}\delta_{jc}\delta_{kb} - \delta_{ib}\delta_{ja}\delta_{kc} - \delta_{ic}\delta_{jb}\delta_{ka}) \\
&= \frac{1}{2}(\omega_{ab}\eta^{+}_{ab}\delta_{kc} + \omega_{bc}\eta^{+}_{ab}\delta_{ka} + \omega_{ca}\eta^{+}_{ab}\delta_{kb}) \\
&\quad - \frac{1}{2}(\omega_{ab}\eta^{-}_{ab}\delta_{kc} + \omega_{bc}\eta^{-}_{ab}\delta_{ka} + \omega_{ca}\eta^{-}_{ab}\delta_{kb}) \\
&= \frac{1}{2}(\omega_{ab}\eta^{+}_{ab}\delta_{kc} + \omega_{bc}\eta^{+}_{kb} + \omega_{ca}\eta^{+}_{ak} \\
&\quad - \frac{1}{2}(\omega_{ab}\eta^{-}_{ab}\delta_{kc} + \omega_{bc}\eta^{-}_{kb} + \omega_{ca}\eta^{-}_{ak}) \\
&= ((\eta^{+} - \eta^{-})\omega)_{kc} + \langle \eta^{+} - \eta^{-}, \omega\rangle\delta_{kc} \\
&= ((\eta^{+} - \eta^{-})\omega)_{kc} + \langle -\eta^{-}, \omega\rangle\delta_{kc} \text{ since } \omega \text{ is orthogonal to } \eta^{+} \\
&= ((\eta^{+} - \eta^{-})\omega)_{kc} - \langle \eta, \omega\rangle\delta_{kc} \quad \square
\end{aligned}
\tag{164}
$$

Lemma 6.3. *For $\omega \in E$, λ a constant, $\eta \in \Omega^2(M)$ identified with endomorphisms then $\eta = \eta^+ + \lambda\omega$ iff $\eta\omega = \omega\eta$, where $\lambda = \frac{1}{2}\langle \eta, \omega\rangle$.*

Proof. Assume that $\eta = \lambda\omega + \eta^+$. Then by Lemma 6.2 we get $\omega\eta^+ = \eta^+\omega - \langle \eta^+, \omega\rangle I = \eta^+\omega$ since ω is orthogonal to η^-. Thus $\eta\omega = \omega\eta$. Assume that $\eta\omega = \omega\eta$ then $\omega(\eta^+ + \eta^-) = (\eta^+ + \eta^-)\omega$ and using Lemma 6.2 we get $2\eta^-\omega = -\langle \eta, \omega\rangle I$. Rearranging we get

$$\begin{aligned} 2\eta^- \omega &= \langle \eta, \omega \rangle I \\ \eta^- \omega\omega &= -\frac{1}{2}\langle \eta, \omega \rangle \omega \\ \eta^- &= \frac{1}{2}\langle \eta, \omega \rangle \omega \end{aligned} \tag{165}$$

Hence $\eta = \eta^+ + \eta^- = \eta^+ + \frac{1}{2}\langle \eta, \omega \rangle \omega$. □

The following lemma uses complex differential geometry. For an introduction and its notation see [10].

Lemma 6.4. *For $\eta \in \Omega^2(M)$, $X, Y \in \Gamma(TZ)$, J an almost complex structure. Then*

$$\eta(JX, JY) = \eta(X, Y) \qquad \textit{iff } \eta \textit{ is of type } (1,1).$$

Proof. As $X, Y \in \Gamma(TZ)$ then $X = X^{(1,0)} + X^{(0,1)}$ and $Y = Y^{(1,0)} + Y^{(0,1)}$, then $\eta(JX, JY)$ becomes

$$\begin{aligned} \eta(JX, JY) &= \eta(J(X^{(1,0)} + X^{(0,1)}), J(Y^{(1,0)} + Y^{(0,1)})) \\ &= \eta(iX^{(1,0)} - iX^{(0,1)}, iY^{(1,0)} - iY^{(0,1)}) \\ &= i^2\eta(X^{(1,0)}, Y^{(1,0)}) + i^2\eta(X^{(0,1)}, Y^{(0,1)}) - i^2\eta(X^{(1,0)}, Y^{(0,1)}) - i^2\eta(X^{(0,1)}, Y^{(1,0)}) \\ &= -\eta(X^{(1,0)}, Y^{(1,0)}) - \eta(X^{(0,1)}, Y^{(0,1)}) + \eta(X^{(1,0)}, Y^{(0,1)}) + \eta(X^{(0,1)}, Y^{(1,0)}) \\ &= -\eta^{(2,0)}(X, Y) - \eta^{(0,2)}(X, Y) + \eta^{(1,1)}(X, Y), \end{aligned} \tag{166}$$

and

$$\begin{aligned} \eta(X, Y) &= \eta((X^{(1,0)} + X^{(0,1)}), (Y^{(1,0)} + Y^{(0,1)})) \\ &= \eta(X^{(0,1)}, Y^{(0,1)}) + \eta(X^{(1,0)}, Y^{(1,0)}) + \eta(X^{(1,0)}, Y^{(0,1)}) + \eta(X^{(0,1)}, Y^{(1,0)}) \\ &= \eta^{(2,0)}(X, Y) + \eta^{(0,2)}(X, Y) + \eta^{(1,1)}(X, Y). \end{aligned} \tag{167}$$

If η is of type $(1,1)$ then $\eta^{(2,0)}(X,Y), \eta^{(0,2)}(X,Y)$ are zero and $\eta(JX,JY) = \eta(X,Y)$. If $\eta(JX,JY) = \eta(X,Y)$ then $2\eta^{(2,0)(X,Y)} + 2\eta^{(0,2)(X,Y)}$ is zero for all X,Y and since $\eta^{(2,0)(X,Y)}$ is orthogonal to $\eta^{(0,2)(X,Y)}$ then they both must be zero and η is of type $(1,1)$. $\square$

Lemma 6.5. *We have $\eta = \eta^+ + \lambda\omega$ iff $\pi^*\eta \in \Omega^2(Z)$ is of type $(1,1)$.*

Proof. Let $X,Y \in \Gamma(TM)$ and X^h, Y^h be their respective horizontal lifts. Note that $(\pi^*\eta)_\omega(X,Y) = \eta_{\pi(\omega)}(\pi_*X, \pi_*Y)$ and we drop the subscripts. Then

$$\begin{aligned}
&(\pi^*\eta)(JX^h, JY^h) = (\pi^*\eta)(X^h, Y^h) \qquad \text{by Lemma 6.4}\\
&\Leftrightarrow \eta(\pi_*JX^h, \pi_*JY^h) = \eta(\pi_*X^h, \pi_*Y^h)\\
&\Leftrightarrow \eta((X \lrcorner\, \omega)^\sharp), (Y \lrcorner\, \omega)^\sharp)) = \eta(X,Y) \qquad \text{by Theorem 4.1}\\
&\Leftrightarrow \langle \eta(\omega(X)), \omega(Y)\rangle = \langle \eta(X), Y\rangle \qquad \text{by Theorem 3.3}\\
&\Leftrightarrow \langle -\omega\eta(\omega(X)), Y\rangle = \langle \eta(X), Y\rangle. \qquad (168)
\end{aligned}$$

Thinking of ω, η as endomorphisms, thus we get that

$$\begin{aligned}
&\Leftrightarrow \langle -\omega\eta(\omega(X)), Y\rangle = \langle \eta(X), Y\rangle\\
&\Leftrightarrow -\omega\eta\omega = \eta\\
&\Leftrightarrow \omega\eta = \eta\omega \qquad \text{since } \omega^2 = -1\\
&\Leftrightarrow \eta = \eta^+ + \lambda\omega \qquad \text{by Lemma 6.3} \quad \square \qquad (169)
\end{aligned}$$

In the next theorem we let B be a vector bundle with connection ∇^B. We let F^{∇^B} be the curvature tensor with respect to the connection ∇^B.

Definition 6.6. An instanton is a special solution to the Yang-Mills equations in 4-dimensions [1]. It is a connection whose curvature is self-dual or anti-self-dual.

An instanton is automatically Yang-Mills where the Yang-Mills functional is given by $\int Tr(F \wedge *F)dvol$. It is often that instantons are (but not always) an absolute minimizer of this functional $\int Tr(F \wedge *F)dvol$. A Yang-Mills connection is a connection which is a critical point of the Yang-Mills functional.

Theorem 6.7. *For (B, ∇^B) a vector bundle with connection over a four-manifold, and $\pi : Z \to M$ the projection map, then $F^{\nabla^B} \in \Omega^2_+(M)$ is self-dual iff $\pi^*(F^{\nabla^B}) \in \Omega^2(Z)$ is of type $(1,1)$.*

Proof. Denote SD as self-dual 2-forms. By Lemma 6.2, Lemma 6.3, Lemma 6.4, Lemma 6.5 $\pi^*(F^{\nabla^B})$ is $(1,1)$ iff $(F^{\nabla^B})_{\pi(\omega)} = SD + \lambda\omega$ for all ω such that $\pi(\omega) = p$ iff $(F^{\nabla^B})_{\pi(\omega)} = SD$ [2]. □

7 The vector bundle $E = \Lambda^2_7(T^*M)$

In this section we will look at the case when $M = \mathbb{R}^8$ with a special kind of 4-form called a Cayley form and we will show a similar construction holds.

7.1 The almost complex structure on $S_2(\Lambda^2_7(T^*M))$

Let $M = \mathbb{R}^8$ and let $\{e_i\}$ be an orthonormal frame on M with the standard orientation and $\{e^i\}$ the dual frame to $\{e_i\}$. Let Φ be the standard Spin(7) structure on M which satisfies $*\Phi = \Phi$ and in local coordinates Φ is given in [?] by

$$\Phi = dx^{0123} - dx^{0167} - dx^{0527} - dx^{0563} + dx^{0415} + dx^{0426} + dx^{0437} + dx^{4567} - dx^{4523} - dx^{4163} - dx^{4127} + dx^{2637} + dx^{1537} + dx^{1526}. \quad (170)$$

where dx^{ijkl} denotes $dx^i \wedge dx^j \wedge dx^k \wedge dx^l$. The 4-form Φ satifies the following identity.

$$\begin{aligned}\Phi_{abij}\Phi_{bckl} = g_{ik}g_{jl}g_{ac} + g_{il}g_{jc}g_{ak} + g_{ic}g_{jk}g_{al} \\ - g_{ik}g_{jc}g_{al} - g_{il}g_{jk}g_{ac} - g_{ic}g_{jl}g_{ak} \\ - g_{ik}\Phi_{jalc} - g_{jk}\Phi_{ailc} - g_{ak}\Phi_{ijlc} \\ - g_{il}\Phi_{jack} - g_{jl}\Phi_{aick} - g_{al}\Phi_{ijck} \\ - g_{ic}\Phi_{jakl} - g_{jc}\Phi_{aikl} - g_{ac}\Phi_{ijkl}\end{aligned} \tag{171}$$

A proof of this identity can be found in [8].

One can show that the map $\omega \mapsto *(\Phi \wedge \omega)$ is self-adjoint hence the map is orthogonally diagonalizeable [8]. We have the following relations from [8]:

$$\begin{aligned}\Lambda^2 &= \Lambda^2_7 \oplus \Lambda^2_{21} \\ \Lambda^2_7 &= \{\omega | *(\Phi \wedge \omega) = -3\omega\} \\ \Lambda^2_{21} &= \{\omega | *(\Phi \wedge \omega) = +\omega\}.\end{aligned} \tag{172}$$

A global frame for $\Lambda^2 T^*\mathbb{R}^8$ is given by $\{dx^i \wedge dx^j \mid i < j\}$. Let π_7 be the projection map from $\Lambda^2 T^*\mathbb{R}^8$ to $\Lambda^2_7 T^*\mathbb{R}^8$. The next lemma 7.1 will be used to define the complex structure on the vertical subbundle of TE by identifying it with $\mathbb{R}^7$ so we can use the standard cross product on $\mathbb{R}^7$.

Lemma 7.1. *There is an isomorphism* $\Lambda^1_7(\mathbb{R}^7) \cong \Lambda^2_7(\mathbb{R}^8)$.

Proof. Let $\{e^0, \ldots, e^7\}$ be the dual orthonormal frame to the orthonormal frame $\{e_0, \ldots, e_7\}$ on $\mathbb{R}^8$. Let $p \in \mathbb{R}^8$. Define a map P by

$$\begin{aligned}P : \Lambda^1_7(\mathbb{R}^7) &\to \Lambda^2_7(\mathbb{R}^8) \\ \alpha &\mapsto \pi_7(e^0 \wedge \alpha).\end{aligned} \tag{173}$$

We will show this map is injective.

If $\pi_7(e^0 \wedge \alpha) = 0$, since no non-zero decomposable 2-form can be of pure type [8] $e^0 \wedge \alpha = 0$. We have $e_0 \lrcorner (e^0 \wedge \alpha) = \alpha = 0$. Thus P is injective.

The dimension of $\Lambda^2_7(\mathbb{R}^8)$ and of $\Lambda^1_7(\mathbb{R}^7)$ is 7 and since P is injective then P is an isomorphism. $\square$

From now on the vector bundle is $E = \Lambda^2_7(T^*\mathbb{R}^8)$ which is a rank seven bundle with Euclidean metric $g = \langle \cdot, \cdot \rangle$ and connection ∇. Let $\{\sigma^i\}$ be a global frame on E with induced global coordinates as $(x^0, \dots, x^7, y_1, \dots y_7)$ as in (1) where $(x^1, \dots, x^8)$ are the standard global coordinates on $\mathbb{R}^8$.

We will view ω as an endomorphism from Theorem 3.3. Let the matrix representation for ω be ω_{ij} and let the components of Φ in the stardard global coordinates on $\mathbb{R}^8$ be Φ_{abij}. We compute $*(\Phi \wedge \omega)$ thus

$$\begin{aligned} *(\Phi \wedge \omega) &= \frac{1}{2} * (\Phi \wedge \omega)_{ab} e^a \wedge e^b \\ &= \frac{1}{2}(*(\Phi \wedge (\omega_{ab} e^a \wedge e^b))) \\ &= \frac{1}{2}\omega_{ab}(e^b \lrcorner e^a \lrcorner \Phi) \text{ , since } e_k \lrcorner *\alpha = (-1)^k * (e_k^\flat \wedge \alpha) \text{ for any } k\text{-form } \alpha \\ &= \frac{1}{4}\omega_{ab}(\Phi_{abij} e^i \wedge e^j) \\ &= \frac{1}{4}(\Phi_{abij}\omega_{ij}) e^a \wedge e^b. \end{aligned} \tag{174}$$

The equations in (172) in local coordinates are

$$\begin{aligned} \omega \in \Lambda^2_7 &\iff \Phi_{abij}\omega_{ij} = -6\omega_{ab}, \\ \omega \in \Lambda^2_{21} &\iff \Phi_{abij}\omega_{ij} = 2\omega_{ab}. \end{aligned} \tag{175}$$

Lemma 7.2. *Let $\omega \in \Lambda^2_7$ viewed as an endomorphism with matrix representation ω_{ab}. Then*

$$\omega_{ab}\omega_{bc} = -\delta_{ac}$$

if $\langle\omega,\omega\rangle = |\omega|^2 = 4$.

Proof. Computing $\omega_{ab}\omega_{bc}$ yields

$$
\begin{aligned}
\omega_{ab}\omega_{bc} &= (-\frac{1}{6}\Phi_{abij}\omega_{ij})(-\frac{1}{6}\Phi_{klbc}\omega_{kl}) \\
&= -\frac{1}{36}\omega_{ij}\omega_{kl}\Phi_{ijab}\Phi_{klcb} \\
&= -\frac{1}{36}\omega_{ij}\omega_{kl}(g_{ik}g_{jl}g_{ac} + g_{il}g_{jc}g_{ak} + g_{ic}g_{jk}g_{al} \\
&\quad - g_{ik}g_{jc}g_{al} - g_{il}g_{jk}g_{ac} - g_{ic}g_{jl}g_{ak} \\
&\quad\quad - g_{ik}\Phi_{jalc} - g_{jk}\Phi_{ailc} - g_{ak}\Phi_{ijlc} \\
&\quad\quad - g_{il}\Phi_{jack} - g_{jl}\Phi_{aick} - g_{al}\Phi_{ijck} \\
&\quad\quad - g_{ic}\Phi_{jakl} - g_{jc}\Phi_{aikl} - g_{ac}\Phi_{ijkl}). \qquad \text{by (171)}
\end{aligned} \tag{176}
$$

After relabelling indices, collecting terms and using $g_{il} = \delta_{il}$ (176) becomes

$$
\begin{aligned}
\omega_{ab}\omega_{bc} = -2\frac{1}{36}\omega_{kj}\omega_{kj}\delta_{ac} - 2\frac{1}{36}\omega_{lc}\omega_{al} - 2\frac{1}{36}\omega_{ck}\omega_{ka} \\
-\frac{1}{36}(-4\omega_{kj}\omega_{kl}\Phi_{jalc} - \omega_{ij}\omega_{kl}\delta_{ac}\Phi_{ijkl} - 2\omega_{cj}\omega_{kl}\Phi_{jakl} - 2\omega_{ij}\omega_{al}\Phi_{ijlc}).
\end{aligned} \tag{177}
$$

Let $P = \omega\circ\omega$. The coordinate representation of P is $P_{ac} = \omega_{al}\omega_{lc}$. Note that P_{ac} is symmetric in the indices a, c. From (172) we get that $\omega_{al} = -\frac{1}{6}\Phi_{alib}\omega_{ib}$ hence $\omega_{al}\omega_{lc} = -\frac{1}{6}\Phi_{alib}\omega_{ib}\omega_{lc}$ and as P is symmetric then $\omega_{kj}\omega_{kl}\Phi_{jalc} = -P_{lj}\Phi_{jalc} = 0$. We also have $2|\omega|^2 = \omega_{ij}\omega_{ij}$. Substituting this into (177) yields

$$
\begin{aligned}
\omega_{ab}\omega_{bc} = P_{ac} = -2\frac{1}{36}(2|\omega|^2)\delta_{ac} - 2\frac{1}{36}P_{ac} - 2\frac{1}{36}P_{ac} \\
-\frac{1}{36}(0 + 6(2|\omega|^2\delta_{ac}) + 2(6P_{ac}) + 2(6P_{ac})).
\end{aligned} \tag{178}
$$

Rearranging (178) we get

$$
64P = -16|\omega|^2 I. \tag{179}
$$

If $\langle \omega, \omega \rangle = 4$ this gives $\omega_{ab}\omega_{bc} = -\delta_{ac}$. □

We define Z^{14} to be $Z^{14} = S_2(\Lambda^2_7(T^*M)) \subset E$. Referring to Theorem 4.1 we can define an almost complex structure on Z^{14}.

Theorem 7.3. *Let $\omega \in Z^{14}_p$, $X \in T_pM$. Define $\widehat{J} : T_pM \to T_pM$ by $\widehat{J} := X \mapsto (X \lrcorner \omega)^\sharp$ where $\sharp$ is the musical isomorphism. Then $\widehat{J} : T_pM \to T_pM$ is a complex structure.*

Proof. By Lemma 7.2 $\widehat{J}^2 = -1$ hence $\widehat{J}$ is a complex structure. □

Given the connection on $E = \Lambda^2_7(T^*M))$ and applying the results of section 2.2 and section 2.1 to E we get a splitting $TE = HE \oplus VE$. Applying Lemma 4.3 to Z^{14} we get

$$V_\upsilon Z^{14} = \{Y^v_\upsilon \in V_\upsilon E \mid \langle Y^v_\upsilon, \xi\rangle_{VE} = 0, \;\; \upsilon \in Z^{14}\}. \tag{180}$$

Definition 7.4. Let $\omega \in Z^{14}_p$ and $X \in T_pM$. On $H_\omega Z^{14}$, $\widehat{J}$ is defined by

$$\widehat{J}(X^h) = ((X \lrcorner \omega)^\sharp)^h. \tag{181}$$

Theorem 7.5. *For $M = \mathbb{R}^8$, let Θ be the restriction of the tautological 2-form on T^*E to Z^{14}. Let X^h be the horizontal lift of $X \in \Gamma(TM)$. Then $\widehat{J}(X^h)$ is given by*

$$\widehat{J}(X^h) := \Theta(X^h, e^h_i)e_i,$$

and satisfies $\widehat{J}^2(X^h) = -X^h$.

Proof. Follow the proof of Theorem 4.5. □

We next define an almost complex structure on VZ^{14}. Given the 4-form on Φ on $\mathbb{R}^8$ we can use it to define a 3-form φ and 4-form $\psi = *\phi$ on $\mathbb{R}^7$. Thus we have the following identities

$$\Phi = e^0 \wedge \varphi + \psi$$
$$\varphi = e_0 \lrcorner \Phi \qquad \text{since } e_0 \lrcorner \psi = 0$$
$$\psi = *\varphi$$
$$\psi = \Phi - e^0 \wedge \varphi. \tag{182}$$

Let $p \in \mathbb{R}^8$ then $T_p\mathbb{R}^8 \cong \mathbb{R}^8$.

Definition 7.6. For $X, Y \in T_p\mathbb{R}^7$, φ the 3-form defined above, the cross product $X \times Y$ is defined by $(Y \lrcorner X \lrcorner \varphi)^\sharp$.

In $\mathbb{R}^7$ the cross product satisfies

$$A \times (B \times C) = \langle A, C\rangle B - \langle A, B\rangle C + (\psi(A, B, C, .))^\# \text{ [13]} \tag{183}$$

for $A, B, C \in \Gamma(T\mathbb{R}^7)$. The fibres of E are given by $\mathbb{R}^7$. The fibres of Z^{14} are given by 6-spheres of radius two which have an induced almost complex structure coming from the cross product on the fibres of E. To put an almost complex structure on VZ^{14} we need a cross product on $\Lambda^2(\mathbb{R}^8)$. To do this we use the isomorphism between $P : \Lambda^1_7(\mathbb{R}^7) \to \Lambda^2(\mathbb{R}^8)$ from Lemma 7.1.

Lemma 7.7. *Let $\omega \in Z^{14}_p$, $Y^v \in \Lambda^2_7(T^*_{\pi(\omega)}\mathbb{R}^8)$ such that $Y^v \perp \omega$, $\xi_\omega \in \Gamma(VE)$ the fundamental vector field. Then*

$$P(\Phi(e_0, P^{-1}(\xi_\omega), P^{-1}(Y^v), \cdot))^\sharp = P(P^{-1}(\xi_\omega) \times P^{-1}(Y^v)) \in V_\omega Z^{14}.$$

Proof. Using Definition 7.6 we get that $Y^v \times \xi_\omega = \xi_\omega \lrcorner Y^v \lrcorner \varphi$. By Definition 7.6 and by (182) and Lemma 7.1

$$\begin{aligned} P(\xi_\omega \times Y^v) &= P(P^{-1}(Y^v) \lrcorner P^{-1}(\xi_\omega) \lrcorner e_0 \lrcorner \Phi)^\sharp \\ &= (\Phi(e_0, P^{-1}(\xi_\omega), P^{-1}(Y^v), \cdot))^\sharp. \quad \square \end{aligned} \tag{184}$$

From now on we drop the P for notational simplicity, and write $\xi_\omega \times Y^v$ to mean $P(P^{-1}(\xi_\omega) \times P^{-1}(Y^v))$.

Theorem 7.8. *Let ω, Y^v and ξ be as in Lemma 7.6. Then $\widehat{J}_\omega(Y^v) = \xi_\omega \times Y^v$ defines an almost complex structure.*

Proof. We have to show that $\widehat{J}_\omega^2 = -1$.
The operator $\widehat{J}_\omega^2$ will be in terms of the iterated cross product given by $\widehat{J}_\omega^2 = \xi_\omega \times (\xi_\omega \times Y^v)$. Computing $\widehat{J}^2 = \xi_\omega \times (\xi_\omega \times Y^v)$ yields

$$\begin{aligned}
(\widehat{J}_\omega)^2(Y^v) &= \xi_\omega \times (\xi_\omega \times Y^v) \\
&= \langle \xi_\omega, Y^v \rangle \xi_\omega - \langle \xi_\omega, \xi_\omega \rangle Y^v + (\Phi(\xi_\omega, \xi_\omega, Y^v, .))^{\#} \qquad \text{by (183)} \\
&= 0 - \langle \xi_\omega, \xi_\omega \rangle Y^v + 0 \\
&= -Y^v.
\end{aligned} \tag{185}$$

Thus $\widehat{J}_\omega$ is a complex structure on $V_\omega Z^{14}$. □

7.2 The flow $\Phi_t^{X^h}$ preserves the cross product on VZ^{14}

For the horizontal lift, $X^h \in \Gamma(HZ^{14})$ of $X \in \Gamma(T\mathbb{R}^8)$, the flow $\Phi_t^{X^h}$ will be as it is in Section 2.3. And we get the following following theorems.

Theorem 7.9. *The map $\phi_t^{X^h} : E \to E$ is an isometry in the vertical direction.*

Proof. By Theorem 2.14. □

Theorem 7.10. *The pushforward $(\phi_t^{X^h})_*$ of $\phi_t^{X^h}$ is an isometry when restricted to the vertical subbundle VZ^{14}.*

Proof. By Theorem 2.15. □

Note that in the 4-dimensional case the cross product on the vertical subbundle is with respect to the volume form which is parallel. When $M = \mathbb{R}^8$ the cross product on the vertical subbundle is with respect to the 4-form Φ which is in general not parallel though here we have used a specific Φ which is parallel.

Lemma 7.11. *The flow $\phi_t^{X^h}$ preserves the cross product on VZ^{14}.*

Proof. Let $\xi_\omega = \xi^d \frac{\partial}{\partial y_d}$ and $Y^v = Y^e \frac{\partial}{\partial y_e}$. The cross product on VZ^{14} is given by

$$(\Phi(e_0, \xi_\omega, Y^v, \cdot))^\sharp.$$

In local coordinates this is

$$\varphi_{def} \xi^d Y^e \frac{\partial}{\partial y_f}.$$

Replace $\varepsilon_{def} U^d V^e \frac{\partial}{\partial y_f}$ with $\varphi_{def} \xi^d Y^e \frac{\partial}{\partial y_f}$ in Lemma 4.8. We can do this since $\nabla^E \Phi = 0$. Then follow the rest of Lemma 4.8. $\square$

Lemma 7.12. *The flow preserves the complex structure $\widehat{J}$ on VZ^{14}. That is $(\phi_t^{X^h})_* \widehat{J}_\omega(Y^v) = \widehat{J}_{\phi_t^{X^h}(\omega)}((\phi_t^{X^h})_* Y^v)$.*

Proof. Using Lemma 7.11 follow Theorem 4.9. $\square$

With the above theorems and lemmas we get the following:

Theorem 7.13. *The Lie derivative $(L_{X^h} \widehat{J})(Y^v)$ is zero.*

Proof. By Theorem 7.9, 7.10, and Lemma 7.11,7.12 and Theorem 4.10. $\square$

7.3 Nijenhuis tensor of two horizontal vectors

By Theorem 7.5 we can apply the results of Section 5.4 when $X^h, Y^h \in \Gamma(HZ^{14})$ and get equation (129). Since $M = \mathbb{R}^8$ we get the following theorem.

Theorem 7.14. *For $X^h, Y^h \in \Gamma(HZ^{14})$ horizontal lifts of $X, Y \in \Gamma(T\mathbb{R}^8)$ then*

$$N(X^h, Y^h) = 0.$$

Proof. Using Theorem 7.5 follow Section 5.4 to get

$$\begin{aligned} N(X^h, Y^h)_\eta = J(R(\eta X, Y)\eta) \\ - R(\eta X, \eta Y)\eta \\ + J(R(\eta Y, X)\eta) \\ + R(X, Y)\eta. \end{aligned} \tag{186}$$

Since $\mathbb{R}^8$ is flat $R(X, Y) = 0$ and $N(X^h, Y^h) = 0$. □

7.4 Nijenhuis tensor of a horizontal and vertical vector

By Theorem 7.5 we can apply the results of Sections 5.1 and 5.2 when $X^h \in \Gamma(HZ^{14})$ and $Y^v \in \Gamma(VZ^{14})$. Thus we get the following theorem.

Theorem 7.15. *Let $X^h \in \Gamma(HZ^{14})$ and $Y^v \in \Gamma(VZ^{14})$ then*

$$N(X^h, Y^v) = 0.$$

Proof. By Theorem 7.13, Theorem 7.5, and Theorem 7.8 we can apply the results of Sections 5.1 and 5.2 to $N(X^h, Y^v)$ when $X^h \in \Gamma(HZ^{14})$ and $Y^v \in \Gamma(VZ^{14})$ thus

$$N(X^h, Y^v) = 0. \quad \square$$

7.5 Nijenhuis tensor of two vertical vectors

By Theorem 7.8 we can apply the results of Section 5.5 to the equation

$$N(X^v, Y^v) = [X^v, Y^v] + J[JX^v, Y^v] + J[X^v, JY^v] - [JX^v, JY^v]. \tag{187}$$

When $X^v, Y^v \in \Gamma(VZ^{14})$ and we get the following theorem.

Theorem 7.16. *For $X^v, Y^v \in \Gamma(VZ^{14})$ then*

$$N(X^v, Y^v) = -4\Phi(\eta, X^v, Y^v, \cdot).$$

Proof. This calculation is analagous to the 4 dimensional version so we can apply the results of Section 5.5 and using (183) to obtain

$$\begin{aligned} N(X^v, Y^v) = {} & \nabla_{X^v} Y^v - \nabla_{Y^v} X^v \\ & - (\langle \xi_\eta, X^v \rangle \nabla_{Y^v} \xi_\eta - \langle \xi_\eta, \nabla_{Y^v} \xi_\eta \rangle X^v - \psi(\eta, X^v, \nabla_{Y^v} \eta, \cdot) \\ & + (\langle \xi_\eta, Y^v \rangle \nabla_{X^v} \xi_\eta - \langle \xi_\eta, \nabla_{X^v} \xi_\eta \rangle Y^v + \psi(\eta, Y^v, \nabla_{X^v} \eta, \cdot) \\ & - (\langle \xi_\eta, \nabla_{Y^v} X^v \rangle \xi_\eta - \langle \xi_\eta, \xi_\eta \rangle \nabla_{Y^v} X^v - \psi(\eta, \nabla_{Y^v} X^v, \eta, \cdot) \\ & + (\langle \xi_\eta, \nabla_{X^v} Y^v \rangle \xi_\eta - \langle \xi_\eta, \xi_\eta \rangle \nabla_{X^v} Y^v + \psi(\eta, , \nabla_{X^v} Y^v, \eta, \cdot) \\ & - (\langle \xi_\eta, Y^v \rangle X^v - \langle X^v, Y^v \rangle \xi_\eta) - \psi(\eta, X^v, Y^v, \cdot) \\ & + (\langle \xi_\eta, X^v \rangle Y^v - \langle X^v, Y^v \rangle \xi_\eta) + \psi(\eta, Y^v, X^v, \cdot). \end{aligned} \tag{188}$$

Similar to the four dimensional case all terms that do not have ψ will vanish or cancel. Hence we get

$$
\begin{aligned}
N(X^v, Y^v) = &-\psi(\eta, X^v, \nabla_{Y^v}\eta, \cdot) \\
&+ \psi(\eta, Y^v, \nabla_{X^v}\eta, \cdot) \\
&- \psi(\eta, \nabla_{Y^v}X^v, \eta, \cdot) \\
&+ \psi(\eta, , \nabla_{X^v}Y^v, \eta, \cdot) \\
&- \psi(\eta, X^v, Y^v, \cdot) \\
&+ \psi(\eta, Y^v, X^v, \cdot) \qquad (189)
\end{aligned}
$$

Noting that $\psi(\eta, , \nabla_{X^v}Y^v, \eta, \cdot) = 0, \psi(\eta, \nabla_{Y^v}X^v, \eta, \cdot) = 0$ and simplifying yields

$$
\begin{aligned}
N(X^v, Y^v) &= -\psi(\eta, X^v, Y^v, \cdot) \\
&\quad + \psi(\eta, Y^v, X^v, \cdot) \\
&\quad - \psi(\eta, X^v, Y^v, \cdot) \\
&\quad + \psi(\eta, Y^v, X^v, \cdot) \\
&= -4\psi(\eta, X^v, Y^v, \cdot). \quad \square \qquad (190)
\end{aligned}
$$

It is not surprising that this almost complex structure on Z^{14} is not integrable because the almost complex structure it induces on the 6-sphere fibres is the canonical one which is well known to be non-integrable. The non-trivial observation is that this is the only part of the Nijenhuis tensor (that is, on two vertical vectors) which is non zero.

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