Perspectives on the moduli space of torsion-free G_2-structures

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The moduli space of torsion-free G_2-structures for a compact 7-manifold forms a non-singular smooth manifold. This was originally proved by Joyce [Joy00]. In this thesis, we present the details of this proof, modifying some of the arguments using techniques in [Kar08] and [DGK23]. Next, we consider the action of gauge transformations of the form e^{tA} where A is a 2-tensor, on the space of torsion-free G_2-structures. This gives us a new framework to study the moduli space.

We show that a G_2-structure $\widetilde{\varphi} = P^*\varphi$ acted upon by a gauge transformation $P = e^{tA}$ is infinitesimal torsion-free condition almost exactly corresponds to $A \diamond \varphi$ being harmonic if A satisfies a "gauge-fixing" condition, where $A \diamond \varphi$ is a 3-form defined using the *diamond operator* $\diamond$ which features in [DGK23]. This may be the first step in giving an alternate proof of the fact that the moduli space forms a manifold in our framework of gauge transformations.

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This thesis is dedicated to Mama, Papa and Didi.

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Chapter 1

Introduction

Our aim in thesis is to study the moduli space of torsion-free G_2-structures and present a new way to explore these moduli spaces through the framework of gauge transformations.

In this chapter, we will set the notation and conventions for the thesis and state some fundamental results which we will need later on. Chapter 2 provides a brief overview of facts and essential results about homogeneous manifolds.

In Chapter 3, we will discuss H-structures, their torsion and their interplay with holonomy groups. The theory of H-structures provides a different way of looking at connections on the underlying manifold and their holonomy groups, and proves to be useful for studying geometrical structures.

We study G_2-structures, which are examples of H-structures, in detail in Chapter 4. We will see how a G_2-structure on a manifold arises from the canonical G_2-structure on $\mathbb{R}^7$ which is isomorphic to the imaginary octonions. Moreover, we will present various identities and properties of differential forms on manifolds with such a structure and look at ways to package the torsion of these G_2-structures.

In Chapter 5, we define the moduli space of G_2-structures on a compact manifold of dimension 7 and prove that it forms a non-singular smooth manifold of dimension b^3. Finally, we present a new framework to study these moduli spaces through gauge transformations in Chapter 6. In particular, we show that infinitesimally, the torsion-free condition under the action of gauge transformations almost exactly corresponds to a particular 3-form, which arises naturally from the G_2-structure and the gauge transformation, being harmonic when we add a "gauge-fixing" condition.

1.1 Notation and conventions

The following standards for notation and conventions are followed throughout Chapter 4, 5 and 6. Unless specified otherwise, M is a 7-dimensional smooth manifold equipped with a Riemannian metric g from a G_2-structure φ (which we define in Chapter 4). We use the metric to identify vector fields with 1-forms and the tensors are expressed with respect to a local frame $\{e_1, \ldots, e_n\}$ that is orthonormal with respect to g. Due to this, all of our indices are subscripts. When an operator like ∇_p appears, unless specified otherwise by parentheses, it **only** acts on the term which follows it immediately. For instance, $\nabla_p \varphi_{ijk} \psi_{qijk}$ means $(\nabla_p \varphi_{ijk})\psi_{qijk}$ and **not** $\nabla_p(\varphi_{ijk}\psi_{qijk})$.

For a fibre bundle E over M, we denote the space of smooth sections of E by $\Gamma(E)$. For some special cases, we use the following notation:

- $\Omega^k = \Gamma(\Lambda^k(T^*M))$ is the space of smooth k-forms on M
- $\mathfrak{X} = \Gamma(TM)$ is the space of smooth vector fields on M
- $\mathcal{T}^k = \Gamma(\otimes^k(T^*M))$ is the space of smooth covariant k-tensors on M
- $\mathcal{S}^k = \Gamma(S^k(T^*M))$ is the space of smooth symmetric k-tensors on M

Next, we set notations for the three Banach spaces we will encounter in this thesis: $L^q(M), C^k(M)$ and $C^{k,\alpha}(M)$. For $q \geq 1$, define $L^q(M)$ to be the Lebesgue space, that is, the set of locally integrable functions f on M for which the norm

$$\|f\|_{L^q} = \left(\int_M |f|^q \operatorname{vol}_g \right)^{1/q}$$

is finite, where vol_g is the volume form associated to the metric g. We denote the space of continuous, bounded functions f on M that have k continuous, bounded derivatives as $C^k(M)$, for integers $k \geq 0$ and define the norm $\|\cdot\|_{C^k}$ as

$$\|f\|_{C^k} = \sum_{j=0}^{k} \sup_M |\nabla^j f|,$$

where ∇ is the Levi-Civita connection on M. If $d(x,y)$ is the distance between $x, y \in M$, for sections v of a vector bundle V over M, $[v]_\alpha$ is given as

$$[v]_\alpha = \sup_{\substack{x \neq y \in M \\ d(x,y) < \delta(g)}} \frac{|v(x) - v(y)|}{d(x,y)^\alpha},$$

where $\delta(g)$ is the injectivity radius of g. For an integer $k \geq 0$ and $\alpha \in (0,1)$, we define the Hölder spaces $C^{k,\alpha}(M)$ as the set of $f \in C^k(M)$ for which the supremum $[\nabla^k f]_\alpha$ exists, working in the vector bundle $\otimes^k T^*M$ with its natural metric and connection. The norm on $C^{k,\alpha}(M)$ is then given as

$$\|f\|_{C^{k,\alpha}} = \|f\|_{C^k} + [\nabla^k f]_\alpha.$$

We view k-forms as totally skew-symmetric k-tensors on M and hence the inner products of tensors, which include k-forms, are inner products as tensors. For k-forms $\alpha = \frac{1}{k!}\alpha_{i_1\cdots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}$, $\beta = \frac{1}{k!}\beta_{j_1\cdots j_k} e_{j_1} \wedge \cdots \wedge e_{j_k}$, the pointwise inner product as tensors is given as

$$\langle \alpha, \beta \rangle = \alpha_{i_1\cdots i_k}\beta_{j_1\cdots j_k}. \tag{1.1}$$

The exterior derivative $d\alpha$ of a k-form α can be written in terms of the covariant derivative as

$$d\alpha = \frac{1}{k!}(\nabla_{i_0}\alpha_{i_1\ldots i_k} - \nabla_{i_1}\alpha_{i_0 i_2\ldots i_k} + \cdots + (-1)^{k-1}\nabla_{i_k}\alpha_{i_0\ldots i_{k-1}}) e_{i_0} \wedge \cdots \wedge e_{i_k}. \tag{1.2}$$

The adjoint $d^* : \Omega^k \to \Omega^{k-1}$ of the exterior derivative is called the **coderivative**, which can be written in terms of the covariant derivative as

$$d^*\alpha = -\frac{1}{(k-1)!}\nabla_m \alpha_{m i_1\ldots i_{k-1}} e_{i_1} \wedge \cdots \wedge e_{i_{k-1}}.$$

The metric and orientation determine the Hodge star operator $\star$ which takes k-forms to $(7-k)$ forms, satisfying the relation

$$\alpha \wedge \beta = g(\alpha, \beta)\,\mathrm{vol}_g,$$

where α, β are k-forms. Furthermore, we have $\star^2 = 1$. Let v be a vector field and α a k-form. Then, we have the following identities between the interior product, wedge product and the Hodge star operator:

$$\star(v \lrcorner \alpha) = (-1)^{k+1}(v^\flat \wedge \star\alpha), \tag{1.3}$$

$$\star(v \lrcorner \star\alpha) = (-1)^k (v^\flat \wedge \alpha). \tag{1.4}$$

Let $A = A_{ij} dx^i \otimes dx^j \in \mathcal{T}^2$. We set $A^t \in \mathcal{T}^2$ as $(A^t)_{ij}$ to be the **transpose** of A and let us define

$$A_{\mathrm{sym}} = \frac{1}{2}(A + A^t) \tag{1.5}$$

and

$$A_{\text{skew}} = \frac{1}{2}(A - A^t). \tag{1.6}$$

Therefore, we have a decomposition $T^*M \otimes T^*M = \mathrm{S}^2(T^*M) \oplus \Lambda^2(T^*M)$ and hence we can write

$$A = A_{\text{sym}} + A_{\text{skew}} \tag{1.7}$$

uniquely where A_{sym} is a symmetric tensor and A_{skew} is a 2-form.

The **trace** of A with respect to g is given as $\operatorname{tr} A = A_{ii} = \operatorname{tr} A_{\text{sym}}$. Hence, we can further decompose

$$A_{\text{sym}} = \frac{1}{n}(\operatorname{tr} A)g + A_0, \tag{1.8}$$

where $A_0 = A_{\text{sym}} - \frac{1}{n}(\operatorname{tr} A)g$ is the **traceless** part of A_{sym} and $n = \dim M$. Therefore, we get a decomposition

$$A = \frac{1}{n}(\operatorname{tr} A)g + A_0 + A_{\text{skew}}, \tag{1.9}$$

which is orthogonal with respect to the inner product given as

$$\langle A, B\rangle = A_{ij}B_{ij} \tag{1.10}$$

for $A, B \in \mathcal{T}^2$. Thus, denoting the traceless symmetric 2-tensors as S_0^2, we get the following pointwise orthogonal splitting

$$\mathcal{T}^2 = \{fg \mid f \in \Omega^0\} \oplus \mathcal{S}_0^2 \oplus \Omega^2 \cong \Omega^0 \oplus \mathcal{S}^0 \oplus \Omega^2. \tag{1.11}$$

Let $E = \otimes^k T^*M$ and take $S \in \Gamma(E)$. If $W \in \mathfrak{X}$, then

$$\begin{aligned}(\mathcal{L}_W S)(X_1, \ldots, X_k) &= W(S(X_1, \ldots, X_k)) - S(\mathcal{L}_W X_1, \ldots, X_k) - \cdots - S(X_1, \ldots, \mathcal{L}_W X_k) \\ &= (\nabla_W S)(X_1, \ldots, X_k) + S(\nabla_W X_1, \ldots, X_k) + \cdots + S(X_1, \ldots, \nabla_W X_k) \\ &\quad - S(\mathcal{L}_W X_1, \ldots, X_k) - \cdots - S(X_1, \ldots, \mathcal{L}_W X_k).\end{aligned}$$

From $\nabla_W X - \mathcal{L}_W X = \nabla_W X - (\nabla_W X - \nabla_X W) = \nabla_X W$, we get

$$(\mathcal{L}_W S)(X_1, \ldots, X_k) = (\nabla_W S)(X_1, \ldots, X_k) + S(\nabla_{X_1} W, \ldots, X_k) + \cdots + S(X_1, \ldots, \nabla_{X_k} W).$$

When $k = 2$ or $k = 3$, with respect to a local orthonormal frame, we get

$$\begin{aligned}(\mathcal{L}_W S)_{ij} &= W_p \nabla_p S_{ij} + \nabla_i W_p S_{pj} + \nabla_j W_p S_{ip},\\ (\mathcal{L}_W S)_{ijk} &= W_p \nabla_p S_{ijk} + \nabla_i W_p S_{pjk} + \nabla_j W_p S_{ipk} + \nabla_k W_p S_{ijp}.\end{aligned} \tag{1.12}$$

For vector fields X, we define the **divergence** $\operatorname{div} X$ as the function $\nabla_i X_i$ and it equals $-d^* X$ when we identify X with its metric dual 1-form. In terms of a local orthonormal frame, we have $\operatorname{div} X = \nabla_p X_p$.

Let $S \in \Gamma(E)$, where $E = \otimes^k T^*M$. Then, we have that $\nabla S \in \Gamma(T^*M \otimes E) = \Gamma(\otimes^{k+1} T^*M)$ where

$$(\nabla S)(X, \cdot) = (\nabla_X S)(\cdot) \in \Gamma(E).$$

If $k \geq 1$, we define the **divergence** $\operatorname{div} S$ of S as the element of $\Gamma(\otimes^{k+1} T^*M)$ given by

$$(\operatorname{div} S)_{i_1 \dots i_{k-1}} = \nabla_p S_{p i_1 \dots i_{k-1}}.$$

Note that this generalizes the notion of divergence for vector fields when we identify 1-forms with vector fields since the two definitions agree when $k = 1$.

We will denote the Riemann curvature tensor in terms of a local orthonormal frame as

$$R_{ijkl} = g(\nabla_{e_i}(\nabla_{e_j} e_k) - \nabla_{e_j}(\nabla_{e_i} e_k) - \nabla_{[e_i, e_j]} e_k, e_l). \tag{1.13}$$

The Ricci tensor in this convention is given as $R_{jk} = R_{ljkl}$ and the Ricci identity for a k-tensor is given as

$$\nabla_p \nabla_q S_{i_1 \dots i_k} - \nabla_q \nabla_p S_{i_1 \dots i_k} = -\sum_{l=1}^{k} R_{pqi_l m} S_{i_1 \dots i_{l-1} m i_{l+1} \dots i_k}. \tag{1.14}$$

1.2 Standard results

In this section, we present some standard results without proof which we will use in this thesis. First, let us discuss some basic facts about Hodge theory. For a compact, oriented Riemannian manifold M, we define the vector space of **harmonic** k-forms as

$$\mathcal{H}^k = \ker(\Delta_d),$$

where $\Delta_d : \Omega^k \to \Omega^k$ is the **Hodge Laplacian** given as $\Delta_d = dd^* + d^*d$. It is easy to show that a k-form α lies in $\mathcal{H}^k$ if and only if it is closed ($d\alpha = 0$) and co-closed ($d^*\alpha = 0$). In addition if $\alpha \in \mathcal{H}^k$, then $\star\alpha \in \mathcal{H}^{n-k}$. Next, we mention the Hodge decomposition theorem which is proved in [War83, Theorem 6.8]

Theorem 1.2.1 (**The Hodge Decomposition Theorem**). *Let M be a compact, oriented Riemannian manifold. Let us denote the exterior derivative and coderivative acting on k-forms as d_k and d_k^* respectively. Then, we have*

$$\Omega^k = \mathcal{H}^k \oplus \operatorname{Im}(d_{k-1}) \oplus \operatorname{Im}(d_{k+1}^*).$$

Furthermore,

$$\ker(d_k) = \mathcal{H}^k \oplus \operatorname{Im}(d_{k-1})$$

and

$$\ker(d_k^*) = \mathcal{H}^k \oplus \operatorname{Im}(d_{k+1}^*).$$

Now, since the k-th deRham cohomology is given as $H^k(M,\mathbb{R}) = \ker(d_k)/\operatorname{Im}(d_{k-1})$ and as $\ker(d_k) = \mathcal{H}^k \oplus \operatorname{Im}(d_{k-1})$ from above, we have a canonical isomorphism between $\mathcal{H}^k$ and $H^k(M,\mathbb{R})$ given as:

Theorem 1.2.2 (**The Hodge Theorem**). *Let M be a compact, oriented Riemannian manifold. Then, every deRham cohomology class on M contains a unique harmonic representative and $\mathcal{H}^k \cong H^k(M,\mathbb{R})$.*

Now, we state the Implicit mapping theorem which can be found in [Lan12, Theorem 2.1, pg. 364]

Theorem 1.2.3 (**The Implicit Mapping Theorem**). *Let X, Y and Z be Banach spaces, and U, V open neigbourhoods of 0 in X and Y respectively. If the function*

$$F : U \times V \to Z$$

is C^k for some $k \geq 1$ such that $F(0,0) = 0$ and $dF_{(0,0)}|_Y : Y \to Z$ is an isomorphism of Y and Z as vector and topological spaces, then there exists an open neighbourhood $U' \subset U$ of 0 in X and a unique C^k map

$$G : U' \to V$$

such that $G(0) = 0$ and

$$F(x, G(x)) = 0$$

for all $x \in U'$.

Let V and W be vector bundles, with metrics on the fibres, over a compact Riemannian manifold M. Then, let P be a smooth linear elliptic operator from V to W and let P^* denote its formal adjoint from W to V. The **Fredholm alternative** is an existence result for the equation $Pv = w$. That is, it gives a simple condition $w \perp \ker P^*$ for w to satisfy for there to exist a solution v. A proof can be found in [Joy00, Theorem 1.5.3].

Theorem 1.2.4. *Let V, W, M and P be as above. Let k be the order of P and let $l \geq 0$ be an integer, let $p > 1$ and let $\alpha \in (0, 1)$. Then, the image of the map*

$$P : C^{k+l,\alpha}(V) \to C^{l,\alpha}(W)$$

is a closed linear subspace of $C^{l,\alpha}(W)$. If $w \in C^{l,\alpha}(W)$, then there exists $v \in C^{k+l,\alpha}(V)$ with $Pv = w$ if and only if $w \perp \ker P^*$ and if one requires that $v \perp \ker P$, then v is unique.

Chapter 2

Homogeneous Manifolds

We begin by briefly recalling some of the background material required to understand group actions on manifolds and the spaces which are formed through these actions. The main source for this chapter is [Lee12].

Recall that an action of a group G on a set M is **transitive** if for every $p, q \in M$, there exists $g \in G$ such that $g \cdot p = q$. A smooth manifold M with a smooth transitive action by a Lie group G is called a **homogeneous space** of G (or just **homogeneous manifold** if it is clear what the group action is). Due to the transitive action, a homogeneous manifold can be informally seen as a space which looks the same everywhere. Let us look at some basic examples of homogeneous manifolds.

Example 2.0.1. Consider the action of $\mathrm{O}(n)$ on $\mathbb{S}^{n-1}$. Since the action of $\mathrm{O}(n)$ on $\mathbb{R}^n$ is smooth and $\mathbb{S}^{n-1}$ is an embedded submanifold of $\mathbb{R}^n$, it is a smooth action. For $v, v' \in \mathbb{S}^{n-1}$, we can complete v and v' to orthonormal bases and let A and A' be the orthogonal matrices whose columns are these orthonormal bases. Then $A'A^{-1}$ takes v to v'. Since the action is transitive, $\mathbb{S}^{n-1}$ is a homogeneous space of $\mathrm{O}(n)$.

Example 2.0.2. The above action of $\mathrm{O}(n)$ restricts to a smooth action of $\mathrm{SO}(n)$ on $\mathbb{S}^{n-1}$. For $n = 1$, this action is trivial since $\mathrm{SO}(1)$ is just the trivial group. To show that the action is transitive for $n \geq 2$, it suffices to show that for any $v \in \mathbb{S}^{n-1}$, there exists $A \in \mathrm{SO}(n)$ which takes the first standard basic vector e_1 to v. As $\mathrm{O}(n)$ acts transitively, we have some $A \in \mathrm{O}(n)$ which takes e_1 to v. We know that $\det A = \pm 1$. If $\det A = 1$, then we are done. But if $\det A = -1$, then we can take the matrix obtained by multiplying the second column of A by -1 which is in $\mathrm{SO}(n)$ and takes e_1 to v. Hence, S^{n-1} is a homogeneous space of $\mathrm{SO}(n)$ for $n \geq 2$.

Example 2.0.3. The group $\mathrm{SL}(2,\mathbb{R})$ acts smoothly and transitively on the upper half plane $\mathbb{U} = \{z \in \mathbb{C} : \Im(z) > 0\}$ under the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

The induced diffeomorphisms on $\mathbb{U}$ are called Möbius transformations.

Now, we want to show that by taking the quotients of Lie groups by closed subgroups, we can generate many examples of homogeneous manifolds. Let G be a Lie group, $H \subseteq G$ be a Lie subgroup and G/H denote the left coset space of G modulo H. We then have the following theorem:

Theorem 2.0.4 (Homogeneous Space Construction Theorem). *Let G be a Lie group and let H be a closed subgroup of G. The left coset space G/H is a topological manifold with dimension $\dim G - \dim H$ and a unique smooth structure such that the quotient map $\pi : G \to G/H$ is a smooth submersion. The left action of G on G/H given as*

$$g_1 \cdot (g_2 H) = (g_1 g_2)H$$

exhibits G/H as a homogeneous space of G.

Recall that a continuous left action of a Lie group G on a manifold M is said to be a **proper action** if the map $G \times M \to M \times M$ given by $(g, p) \mapsto (g \cdot p, p)$ is proper. It is said to be a **free action** if every isotropy group is trivial. To prove Theorem 2.0.4, we will require a few fundamental theorems about Lie groups, quotient manifolds and proper actions. The proofs of the theorems which we do not prove in this chapter can be found in [Lee12].

Theorem 2.0.5 (Closed Subgroup Theorem). *Let G be a Lie group and $H \subseteq G$ is a subgroup which is also a closed subset of G. Then, H is a Lie subgroup which is an embedded submanifold of G. We say that H is an embedded Lie subgroup. Furthermore, every embedded Lie subgroup is properly embedded. That is, the inclusion $H \hookrightarrow G$ is a proper map.*

Theorem 2.0.6 (Quotient Manifold Theorem). *Let G be a Lie group acting smoothly, freely and properly on a smooth manifold M. Then the orbit space M/G is a topological manifold with dimension $\dim M - \dim G$ and has a unique smooth structure such that the quotient map $\pi : M \to M/G$ is a smooth submersion.*

Theorem 2.0.7 (**Sequential Characterization of Proper Actions**)**.** *Let G be a Lie group acting continuously on a manifold M. Then the action is proper iff for sequences (p_i) in M and (g_i) in G such that (p_i) and $(g_i \cdot p_i)$ converge, a subsequence of (g_i) converges.*

Proof of Theorem 2.0.4. Consider the right action of H on G by translation. Note that the orbit space determined by this right action is the same as the left coset space G/H since $g_1, g_2 \in G$ are in the same H-orbit if and only if $g_1 = g_2 h$ for some $h \in H$ which is equivalent to saying that g_1, g_2 are in the same coset of H.

The H-action on G is smooth as it is the restriction of the multiplication on G which is smooth. In addition, as H is a closed subgroup of G, from Theorem 2.0.5, it is a properly embedded subgroup. Furthermore, since $gh = g$ implies that $h = e$, the action is free. To show that the action is proper we will use Theorem 2.0.7. Let (g_i) be a convergent sequence in G and let (h_i) be a sequence in H such that $(g_i h_i)$ converges in G. Since multiplication and inverses are continuous as G is a Lie group, $h_i = g_i^{-1}(g_i h_i)$ converges in G and as H is closed in G with the subspace topology, (h_i) converges in H.

Hence, from Theorem 2.0.6 it follows that G/H has a unique smooth structure such that $\pi : G \to G/H$ is a smooth submersion. As the product of smooth submersions is also a smooth submersion, we get that the map $\mathrm{Id}_G \times \pi : G \times G \to G \times G/H$ is also a smooth submersion. Denoting the group multiplication by m and the action of G on G/H given in the statement of the theorem as θ, we have the following diagram:

$$\begin{array}{ccc} G \times G & \xrightarrow{\;m\;} & G \\ {\scriptstyle \mathrm{Id}_G \times \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times G/H & \xrightarrow[\theta]{} & G/H \end{array}$$

Note that for $(g, [g']) \in G \times G/H$, we have that

$$(\mathrm{Id}_G \times \pi)^{-1}(g, [g']) = \{(g, g'') \mid g'' H = g' H \text{ for } h \in H\}.$$

It follows that $\pi \circ m$ is constant on the fibres of $\mathrm{Id}_G \times \pi$, because for (g, g'') such that $(\mathrm{Id}_G \times \pi)(g, g'') = (g, [g'])$,

$$\pi(m(g, g'')) = \pi(gg'') = gg'' H = gg' H.$$

Thus, passing smoothly to the quotient it follows that θ is a well-defined and smooth group action. Finally, it is transitive since for $g_1 H, g_2 H \in G/H$, we have $g_2 g_1^{-1} \in G$ such that $(g_2 g_1^{-1}) \cdot g_1 H = g_2 H$. $\square$

As we will see in the next theorem, the homogeneous spaces constructed in Theorem 2.0.4 are special since it turns out that every homogeneous space is equivalent to one of this type. To prove it, we need to use the equivariant rank theorem which we will state without proof:

Theorem 2.0.8 (**Equivariant Rank Theorem**). *For smooth manifolds M, N and a Lie group G, if $F : M \to N$ is a smooth map that is equivariant with respect to a transitive smooth G-action on M and any smooth G-action on N, then F has constant rank. Therefore, F is a smooth submersion if it is surjective, a smooth imersion if it is injective and a diffeomorphism if it is bijective.*

Theorem 2.0.9 (**Homogeneous Space Characterization Theorem**). *Let G be a Lie group, M be a homogeneous space of G and let $p \in M$. Then the isotropy group G_p is a closed subgroup of G, and the map $F : G/G_p \to M$ defined by $F(gG_p) = g \cdot p$ is an equivariant diffeomorphism.*

Proof. Let $\theta^{(p)} : G \to M$ denote the orbit map given as $\theta^{(p)}(g) = g \cdot p$ and let $H = G_p = (\theta^{(p)})^{-1}(p)$. Since $\theta^{(p)}$ is continuous, H is closed. Note that F is well-defined since if $g_1 H = g_2 H$, we have $g_2 = g_1 h$ for some $h \in H$ and hence

$$F(g_2 H) = g_2 \cdot p = g_1 h \cdot p = g_1 \cdot p = F(g_1 H).$$

Furthermore, F is equivariant since

$$F(g' g H) = (g' g) \cdot p = g' \cdot F(gH).$$

As F is obtained from the orbit map $\theta^{(p)} : G \to M$ by passing to the quotient, it is smooth.

By Theorem 2.0.8, we know that equivariant smooth bijections are diffeomorphisms. Thus, it suffices to show that F is bijective. Let $q \in M$. Since the action is transitive, there exists $g \in G$ such that $g \cdot p = q$ and thus $F(gH) = q$, which shows surjectivity. For the injectivity, if $F(g_1 H) = F(g_2 H)$, then

$$g_1 \cdot p = g_2 \cdot p \implies g_1^{-1} g_2 \cdot p = p \implies g_1^{-1} g_2 \in H \implies g_1 H = g_2 H. \qquad \square$$

Therefore, understanding homogeneous spaces can be reduced to the algebraic problem of dealing with quotients of Lie groups by closed subgroups. Equipped with this new perspective, let us take another look at the previously discussed examples of homogeneous spaces.

Example 2.0.10. For the action of $\mathrm{O}(n)$ on $\mathbb{S}^{n-1}$, if we choose the base point to be the north pole $N = (0, \dots, 0, 1)$, the isotropy group is $\mathrm{O}(n-1)$ since it consists of the orthogonal transformations fixing the last coordinate. Hence $\mathbb{S}^{n-1}$ is diffeomorphic to the quotient manifold $\mathrm{O}(n)/\mathrm{O}(n-1)$.

Example 2.0.11. Similar to the previous example, we get that the isotropy group for the action of $\mathrm{SO}(n)$ on $\mathbb{S}^{n-1}$ is $\mathrm{SO}(n-1)$ and hence S^{n-1} is diffeomorphic to $\mathrm{SO}(n)/\mathrm{SO}(n-1)$ for $n \geq 2$.

Example 2.0.12. Consider the transitive action of $\mathrm{SL}(2, \mathbb{R})$ on the upper half-plane by Möbius transformations. Computing the isotropy group of $i \in \mathbb{U}$ directly shows that it consists of matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with $a^2 + b^2 = 1$. Note that this is precisely the group $\mathrm{SO}(2) \subseteq \mathrm{SL}(2, \mathbb{R})$. Thus, we have a diffeomorphism $\mathbb{U} \cong \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$.

Let us conclude this section by defining principal G-bundles, which will be objects of focus in Chapter 3.

Definition 2.0.13. If G is a topological group, a **principal G-bundle** is a fibre bundle $\pi : P \to X$ with a continuous right G-action $P \times G \to P$ given by $(p, g) \mapsto p \cdot g$ which preserves the fibres of P (that is, $\pi(p \cdot g) = \pi(p)$) and acts freely and transitively on the fibres.

From the definition above it follows that each fibre $P_x = \pi^{-1}(x)$ is homeomorphic to G non-canonically through the map $G \to P_x$ which sends g to yg for $x \in X, y \in P_x$. Note that if we take G to be a Lie group and take its action on a manifold M to be smooth, free and proper, then the fibre bundle $\pi : M \to M/G$ with fibre G is a principal G-bundle. For H a subgroup of G, we say that H is **admissible** is the quotient map $G \to G/H$ is a principal H-bundle. Using Theorem 2.0.6, it can be shown that every closed Lie subgroup is admissible.

Chapter 3

H-structures

For this thesis, we will be mainly focusing on G_2-manifolds which are examples of a larger class of objects known as H-structures. Before studying the special properties of G_2-manifolds, we will lay down a theoretical framework for H-structures. This chapter closely follows [FEME23] and [Joy00]. The sources for the Lie algebra part are [KN69] and [CE08].

Let M^n be a connected and orientable smooth n-manifold without boundary where $n > 2$. A **frame** at a point $x \in M$ is given by a linear isomorphism $u : T_xM \to \mathbb{R}^n$. For any $g \in \mathrm{GL}(n, \mathbb{R})$, $g^{-1} \circ u$ is again a frame and for any two frames u, u', there is a unique $g \in \mathrm{GL}(n, \mathbb{R})$ with $u' = g^{-1} \circ u$. That is, $\mathrm{GL}(n, \mathbb{R})$ acts freely and transitively on the set of frames at a point.

Definition 3.0.1. The **frame bundle** of M denoted by $\mathrm{Fr}(M)$ is the principal $\mathrm{GL}(n, \mathbb{R})$-bundle whose fibre over $x \in M$ consists of frames $u : T_xM \to \mathbb{R}^n$ where the right action $\mathrm{GL}(n, \mathbb{R}) \times \mathrm{Fr}(M) \to \mathrm{Fr}(M)$ is given by $(g, u) \mapsto g \cdot u := g^{-1} \circ u$.

Given a principal G-bundle for some group G, one can ask if it "comes from" a subgroup H of G. If it does, then we say that it admits a reduction of structure group to H. More concretely:

Definition 3.0.2. Let H, G be topological groups such that $H < G$. If $\rho : H \to G$ is the inclusion, then a principal G-bundle P_G admits a **reduction of structure group** to H if there is a principal H-bundle P_H and an inclusion $i : P_H \to P_G$ which is H-equivariant, that is, $i(ph) = i(p)h$ for $p \in P_H, h \in H$. The bundle P_H is called an H**-reduction** of P_G.

Let $H \subset \mathrm{GL}(n, \mathbb{R})$ be a Lie subgroup. Then, an H**-structure** on M^n is an H-reduction of $\mathrm{Fr}(M)$. For instance, it can be shown that an $\mathrm{SO}(n)$-structure on M^n is equivalent to a choice of a Riemannian metric g and an orientation. We fix an oriented Riemannian manifold (M^n, g) and denote the associated $\mathrm{SO}(n)$-structure by $\pi_{\mathrm{SO}(n)} : \mathrm{Fr}(M, g) \to M$. Then, if H is a Lie subgroup of $\mathrm{SO}(n)$, we say that Q is a **compatible H-structure** on (M^n, g) if Q is an H-reduction of $\mathrm{Fr}(M, g)$. Now, let us assume that $H \subset \mathrm{SO}(n)$ is closed and connected. From the discussion at the end of Chapter 2, it follows that the quotient map $\pi_H : \mathrm{Fr}(M, g) \to \mathrm{Fr}(M, g)/H$ is a principal H-bundle. Hence, the map $\pi : \mathrm{Fr}(M, g)/H \to M$ which is defined such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Fr}(M,g) & \xrightarrow{\pi_H} & \mathrm{Fr}(M,g)/H \\ {\scriptstyle \pi_{\mathrm{SO}(n)}}\downarrow & \swarrow{\scriptstyle \pi} & \\ M & & \end{array}$$

is a fibre bundle with fibre $\mathrm{SO}(n)/H$.

We then have a one-to-one correspondence between compatible H-structures $Q \subset \mathrm{Fr}(M, g)$ and sections $\sigma \in \Gamma(\mathrm{Fr}(M, g)/H)$. Since σ is a section of a homogeneous fibre bundle $\mathrm{Fr}(M, g)/H \to M$, we call it a **homogeneous section**. Indeed, given $Q \subset \mathrm{Fr}(M, g)$, we can define $\sigma_Q(x) := \pi_H(u)$ where $u \in Q$ with $\pi_{\mathrm{SO}(n)}(u) = x$. This is well defined since for $u, \widetilde{u} \in \pi^{-1}_{\mathrm{SO}(n)}(u) \subset Q$, as Q is an H-bundle, we have $\widetilde{u} = h \cdot u$ for some $h \in H$ which implies that $\pi_H(u) = \pi_H(\widetilde{u})$. Conversely, if $\sigma \in \Gamma(\mathrm{Fr}(M, g)/H)$, we can define a compatible H-structure by $Q_\sigma := \pi_H^{-1}(\sigma(M)) \subset \mathrm{Fr}(M, g)$.

Next, we will briefly discuss some Lie algebra preliminaries relating to Riemannian manifolds which we require in order to define connections associated to H-structures.

3.1 Lie algebra background

We will denote the Lie algebras of G and H by $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Let $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ denote the adjoint representation of G, where

$$\mathrm{Ad}(g)(X) = \mathrm{Ad}_g(X) = gXg^{-1}$$

for $g \in G$ and $X \in \mathfrak{g}$ and let us denote the adjoint representation of $\mathfrak{g}$, which is the derivative of Ad at the identity, by $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ where

$$\mathrm{ad}(X)(Y) = \mathrm{ad}_X(Y) = [X, Y].$$

Definition 3.1.1. If G is a Lie group and H is a closed subgroup, then we say that the homogeneous manifold G/H is **reductive** if for some subspace $\mathfrak{m}$ of $\mathfrak{g}$ we have

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

such that

$$\mathrm{Ad}_h(\mathfrak{m}) \subseteq \mathfrak{m}$$

for all $h \in H$.

We can put certain conditions on G and $\mathfrak{g}$ such that it always admits a reductive decomposition. One such condition is as follows:

Theorem 3.1.2. *Let G/H be a homogeneous space such that G is a connected Lie group. Assume that $\mathfrak{g}$ admits an $\mathrm{Ad}(G)$-invariant inner product $\langle \cdot, \cdot \rangle$ and let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{h}$ with respect to the metric $\langle \cdot, \cdot \rangle$. Then, G/H is reductive with respect to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.*

Proof. As H is closed under conjugation, $\Psi_a(h) = aha^{-1}$ for all $a, h \in H$ and as the adjoint representation is given by $\mathrm{Ad}_h = (d\Psi_h)_e$, it follows that $\mathfrak{h}$ is invariant under Ad_h for all $h \in H$. Furthermore, since each Ad_h is an isomorphism, we get that $\mathrm{Ad}_h(\mathfrak{h}) = \mathfrak{h}$. Now, let $X \in \mathfrak{m}$. As $\mathfrak{m} = \mathfrak{h}^{\perp}$, we have that for all $Y \in \mathfrak{h}$,

$$\langle X, Y \rangle = 0.$$

As the inner product is $\mathrm{Ad}(G)$-invariant, we have that for $h \in H$,

$$\langle \mathrm{Ad}_h(X), \mathrm{Ad}_h(Y) \rangle = 0.$$

As $\mathrm{Ad}_h(\mathfrak{h}) = \mathfrak{h}$, for all $W \in \mathfrak{h}$ we have

$$\langle \mathrm{Ad}_h(X), W \rangle = 0$$

which shows that $\mathrm{Ad}_h(X) \in \mathfrak{h}^{\perp} = \mathfrak{m}$. As X was arbitrary, we have that $\mathrm{Ad}_h(\mathfrak{m}) \subseteq \mathfrak{m}$ and hence, the decomposition is reductive. $\square$

Definition 3.1.3. The **Killing form** B of $\mathfrak{g}$ is the symmetric bilinear form given by

$$B(X, Y) = \mathrm{tr}(\mathrm{ad}\, X \circ \mathrm{ad}\, Y)$$

for $X, Y \in \mathfrak{g}$.

If B is non-degenerate, we say that $\mathfrak{g}$ is **semi-simple**. If the Lie algebra $\mathfrak{g}$ of a Lie group G is semi-simple, then we say that G is semi-simple. From [KN69, Appendix 9], we know that a compact, connected Lie group G is semisimple iff the Killing form B is negative definite. Furthermore, B is $\mathrm{Ad}(G)$-invariant:

Theorem 3.1.4. *The Killing form B of a Lie algebra is Ad_g-invariant for all $g \in G$. That is, for $X, Y \in \mathfrak{g}$,*

$$B(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) = B(X, Y).$$

Proof. Since each Ad_g is a Lie algebra homomorphism, it preserves brackets and hence

$$\mathrm{Ad}_g[X, Y] = [\mathrm{Ad}_g X, \mathrm{Ad}_g Y].$$

Furthermore, since Ad_g is an automorphism, taking $Z = \mathrm{Ad}_g(Y)$, we have that

$$\mathrm{Ad}_g[X, \mathrm{Ad}_g^{-1} Z] = [\mathrm{Ad}_g X, Z].$$

Since $\mathrm{ad}(V)(W) = [V, W]$ for $V, W \in \mathfrak{g}$, we have

$$\mathrm{Ad}_g \circ \mathrm{ad}(X) \circ \mathrm{Ad}_g^{-1} = \mathrm{ad}(\mathrm{Ad}_g(X)).$$

This gives us

$$\begin{aligned} B(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) &= \mathrm{tr}(\mathrm{ad}(\mathrm{Ad}_g X) \circ (\mathrm{ad}\,\mathrm{Ad}_g Y)) \\ &= \mathrm{tr}(\mathrm{Ad}_g \mathrm{ad}(X) \circ \mathrm{ad}(Y) \mathrm{Ad}_g^{-1}) \\ &= \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) \\ &= B(X, Y). \end{aligned}$$

□

Note that in the above proof, we only used the fact that Ad_g is a Lie algebra automorphism. Hence, this proof works for any Lie algebra automorphism ρ. That is, the Killing form B is ρ-invariant for any automorphism ρ of $\mathfrak{g}$.

Therefore, for any Lie group G which is compact, connected and semisimple, the bilinear form $-cB$ for any constant $c > 0$ gives us an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$. We say that a homogeneous space G/H is **normal** if G is a connected, compact and semisimple Lie group. Thus, we obtain the following corollary of Theorem 3.1.2

Corollary 3.1.5. *Let G/H be a normal homogeneous space. That is, G is a connected, compact and semisimple Lie group. Then under the $\mathrm{Ad}(G)$-invariant inner product $\langle\cdot,\cdot\rangle$ given by $-cB$ on $\mathfrak{g}$ where $c > 0$, G/H is reductive with respect to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}$ with respect to the metric $\langle\cdot,\cdot\rangle$.*

We will conclude this section by defining adjoint bundles. Earlier in this chapter, using frame bundles we were able to go from vector bundles to principal bundles. The following is a way to go in the opposite direction.

Definition 3.1.6. Let M be a manifold, G a Lie group and P a principal bundle over M with fibre G. If ρ is a representation of G on a vector space V, then there is an action of G on the product space $P \times V$ where the action on the first factor is the principal bundle action and on the second factor it acts by ρ. We define the quotient of $P \times V$ by this G-action to be

$$\rho(P) = (P \times V)/G.$$

Since $P/G = M$, the natural projection map from $(P \times V)/G$ to P/G gives us a projection from $\rho(P)$ to M. As G acts freely on P, this projection has fibre V and thus $\rho(P)$ is a vector bundle over M with fibre V.

Take $\pi : P \times V \to \rho(P)$ to be the natural projection. Let us consider $P \times V$ as the trivial vector bundle over P with fibre V. Then, if $e \in \Gamma(\rho(P))$ is a smooth section of $\rho(P)$ over M, then the pullback $\pi^*(e)$ is a smooth section of $P \times V$ over P. Furthermore, $\pi^*(e)$ is invariant under the action of G on $P \times V$, which gives us a 1-1 correspondence between sections of $\rho(P)$ over M and G-invariant sections of $P \times V$ over P. Due to this, we often write $\rho(P) = P \times_G V$ if the choice of the representation ρ is clear.

If P is a principal G-bundle over a smooth manifold, then the **adjoint bundle** $\mathrm{Ad}(P)$ of P is the bundle associated to the representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. That is, the elements of the adjoint bundle are equivalence classes $[p, X]$ for $p \in P, X \in \mathfrak{g}$ such that for all $g \in G$,

$$[p \cdot g, X] = [p, \mathrm{Ad}_g(X)].$$

Due to the above 1-1 correspondence, we can also represent $\mathrm{Ad}(P)$ as $P \times_G \mathfrak{g}$.

3.2 H-connections and intrinsic torsion

Consider the quotient $\mathrm{SO}(n)/H$ equipped with the metric on $\mathrm{SO}(n)$ given by $\langle A, B\rangle = -\operatorname{tr}(AB)$. Then, as H is closed and connected, and as $\mathrm{SO}(n)$ is connected, semisimple (for $n > 2$) and compact, $\mathrm{SO}(n)/H$ is a normal, homogeneous Riemannian manifold. From Corollary 3.1.5 we get that there is a reductive decomposition

$$\mathfrak{so}(n) = \mathfrak{h} \oplus \mathfrak{m} \tag{3.1}$$

where $\mathfrak{m} = \mathfrak{h}^\perp$ is the orthogonal complement of $\mathfrak{h}$ with respect to the metric $\langle \cdot, \cdot \rangle$. That is, $\mathrm{Ad}_H(\mathfrak{m}) \subseteq \mathfrak{m}$. If (M^n, g) admits a compatible H-structure $Q \subset \mathrm{Fr}(M, g)$, then the above reductive decomposition induces an orthogonal H-module decomposition

$$\mathfrak{so}(TM) = \mathfrak{h}_Q \oplus \mathfrak{m}_Q \tag{3.2}$$

where the adjoint bundle $\mathfrak{so}(TM) := \mathrm{Fr}(M, g) \times_{\mathrm{SO}(n)} \mathfrak{so}(n)$ is the subbundle of skew-symmetric endomorphisms in $\mathrm{End}(TM) = T^*M \otimes TM$ and $\mathfrak{h}_Q := Q \times_H \mathfrak{h}, \mathfrak{m}_Q := Q \times_H \mathfrak{m}$.

We say that a connection $\widetilde{\nabla}$ on TM is an H**-connection** if for the associated connection 1-form $\widetilde{\omega} \in \Omega^1(\mathrm{Fr}(M), \mathfrak{gl}(n, \mathbb{R}))$, we have that $\iota_Q^* \widetilde{\omega} \in \Omega^1(Q, \mathfrak{h})$ is a connection 1-form on Q where $\iota_Q : Q \hookrightarrow \mathrm{Fr}(M)$ is the H-subbundle inclusion. If $\widetilde{\nabla}$ is an H-connection and ∇ is the Levi-Civita connection on (M^n, g). Then, define

$$\widetilde{T}_X := \widetilde{\nabla}_X - \nabla_X. \tag{3.3}$$

Since Q is compatible with g, the H-connection $\widetilde{\nabla}$ on TM is compatible with g. Thus, we have

$$X(g(Y, Z)) = g(\widetilde{\nabla}_X Y, Z) + g(Y, \widetilde{\nabla}_X Z)$$

and since ∇ is a Levi-Civita connection,

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Combining the two gives us

$$\langle \widetilde{\nabla}_X Y - \nabla_X Y, Z \rangle = -\langle Y, \widetilde{\nabla}_X Z - \nabla_X Z \rangle \implies \langle \widetilde{T}_X Y, Z \rangle = -\langle Y, \widetilde{T}_X Z \rangle.$$

Therefore, for fixed $X \in \mathfrak{X}(M)$, the map $Y \mapsto T_X Y$ is a skew-adjoint endomorphism. This means that $\widetilde{T}_X$ defines a skew-symmetric endomorphism $\widetilde{T}_X \in \Gamma(\mathfrak{so}(TM))$ for all $X \in \mathfrak{X}(M)$. As ∇ is torsion-free, $\widetilde{T}$ is essentially the torsion of $\widetilde{\nabla}$ as

$$\begin{aligned} \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X, Y] &= \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X, Y] - (\nabla_X Y - \nabla_Y X - [X, Y]) \\ &= \widetilde{T}_X Y - \widetilde{T}_Y X \end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$. As $\widetilde{T}_X = \pi_{\mathfrak{h}}(\widetilde{T}_X) + \pi_{\mathfrak{m}}(\widetilde{T}_X)$ where $\pi_{\mathfrak{h}}, \pi_{\mathfrak{m}}$ denote the projections for the decomposition in (3.2), we can define the H-connection

$$\nabla^H_X := \widetilde{\nabla}_X - \pi_{\mathfrak{h}}(\widetilde{T}_X).$$

Note that since the difference between any two H-connections lies in $\Gamma(\mathfrak{h}_Q)$, it follows that ∇^H only depends on the H-structure and not on the choice of the H-connection $\widetilde{\nabla}$. That is, ∇^H is the unique H-connection on M such that its torsion T of ∇^H satisfies

$$T_X = \nabla^H_X - \nabla_X \in \Gamma(\mathfrak{m}_Q).$$

This tensor $T \in \Omega^1(M, \mathfrak{m}_Q)$ is called the **intrinsic torsion** of the H-structure Q and Q is said to be **torsion-free** when $T = 0$. That is, the Levi-Civita connection is an H-connection which as we will see in Section 3.3 shows that its holonomy is a subgroup of H.

3.3 Compatible H-structures and holonomy groups

In this section, we will explore the relations between H-structures and holonomy groups. First, let us briefly recall some facts about holonomy groups on vector bundles.

Given a manifold M, a vector bundle E over M and a connection ∇ on E, for a piecewise-smooth curve $\gamma : [0,1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$ for $x, y \in M$, we have that for each $e \in E_x$ there exists a unique smooth section s of the pullback bundle $\gamma^*(E)$ such that $\nabla_{\gamma'(t)} s(t) = 0$ for $t \in [0,1]$, with $s(0) = e$. We then define the **parallel transport map** as $P_\gamma(e) = s(1)$ where $P_\gamma : E_x \to E_y$ is a well-defined linear map.

We call $\gamma : [0,1] \to M$ a loop based at $x \in M$ if it is piecewise-smooth with $\gamma(0) = \gamma(1) = x$. The parallel transport map for a loop $P_\gamma : E_x \to E_x$ is an invertible linear map, which means that $P_\gamma \in \mathrm{GL}(E_x)$. We define the **holonomy group** of ∇ based at x as

$$\mathrm{Hol}_x(\nabla) = \{P_\gamma : \gamma \text{ is a loop based at } x\} \subset \mathrm{GL}(E_x).$$

It can be easily confirmed that $\mathrm{Hol}_x(\nabla)$ is indeed a a subgroup of $\mathrm{GL}(E_x)$. Furthermore, we can drop the basepoint from the notation and write the holonomy group of ∇ as $\mathrm{Hol}(\nabla)$ since for $x, y \in M$, $\mathrm{Hol}_x(\nabla)$ and $\mathrm{Hol}_y(\nabla)$ determine the same subgroup of $\mathrm{GL}(k, \mathbb{R})$ for some k, up to conjugation.

Next, let us briefly summarize the notion of a connection on a principal bundle. Let P be a principal bundle over a manifold M with fibre G and denote the projection by $\pi : P \to M$. Let $p \in P$ and fix $m = \pi(p)$. Then we define C_p to be the subspace of T_pP given by $C_p = \ker(d\pi_p)$ where $d\pi_p : T_pP \to T_mM$ is the derivative of π. We can then form a vector subbundle C of the tangent bundle TP consisting of the subspaces C_p which we call the **vertical subbundle**. Furthermore, as π is a submersion from the implicit function theorem, we have $C_p = T_p(\pi^{-1}(m))$. But since the fibres of π are the orbits of the free G-action on P, we have a natural isomorphism

$$C_p \cong \mathfrak{g} \tag{3.4}$$

where $\mathfrak{g}$ is the Lie algebra of G. A **connection** on P then is a vector subbundle D of TP called the **horizontal subbundle** which is invariant under the G-action on P satisfying

$$T_pP = C_p \oplus D_p \tag{3.5}$$

for each $p \in P$. As $C_p = \ker d\pi_p$, we have the isomorphism

$$D_p \cong \pi^*(T_{\pi(p)}M). \tag{3.6}$$

Now, let $\gamma : [0,1] \to P$ be a piecewise-smooth curve in P, where P is as above and D is a connection on P. Then we say that γ is a **horizontal curve** if $\gamma'(t) \in D_{\gamma(t)}$ for each t in the open, dense subset of $[0,1]$ where $\gamma'(t)$ is well-defined. Furthermore, if $\gamma : [0,1] \to M$ is piecewise-smooth with $\gamma(0) = m$ and $\pi(p) = m$ for some $p \in P$, then from existence results for ordinary differential equations, it follows that there exists a unique horizontal, piecewise-smooth map $\widetilde{\gamma} : [0,1] \to P$ such that $\widetilde{\gamma}(0) = p$ and $\pi \circ \widetilde{\gamma} = \gamma$. We call $\widetilde{\gamma}$ the **horizontal lift** of γ. We are now ready to define a holonomy group for a principal bundle.

Definition 3.3.1. Let M be a manifold, P a principal bundle over M with fibre G and D a connection on P. Let $p, q \in P$. Then we write $p \sim q$ if there exists a piecewise-smooth horizontal curve in P joining p and q. This is clearly an equivalence relation. Fix $p \in P$ and define the **holonomy group** of (P, D) based at p to be

$$\mathrm{Hol}_p(P, D) = \{g \in G : p \sim g \cdot p\}.$$

Like in the case of vector bundles, it is easy to see that $\mathrm{Hol}_p(P, D)$ is a subgroup of G and that $\mathrm{Hol}_p(P, D)$ depends on the base point $p \in P$ only up to conjugation, which means that the holonomy groups can be regarded as an equivalence class of subgroups of G under conjugation and thus we can write $\mathrm{Hol}(P, D)$.

The next theorem gives us a way to get H-reductions through holonomy groups of a principal G-bundle P for certain closed Lie subgroups H of G.

Theorem 3.3.2 (**Reduction Theorem**). *Let M be a manifold, P a principal bundle over M with fibre G, and D a connection on P. Fix $p \in P$ and let $H = \mathrm{Hol}_p(P, D)$. Suppose that H is a closed Lie subgroup of G. Define $Q = \{q \in P : p \sim q\}$. Then, Q is a principal subbundle of P with fibre H and the connection D on P restricts to a connection D' on Q. That is, P reduces to Q and the connection D on P reduces to D' on Q.*

Proof. Clearly, Q is preserved by the action of H on P, and hence it acts freely on Q. In addition, π restricts to Q which gives us a projection $\pi : Q \to M$, with the fibres of $\pi : Q \to M$ being the orbits of H. Furthermore, as H is a closed subgroup of G, it is a Lie group and thus Q is a submanifold of P. Therefore, Q is a principal subbundle of P with fibre H.

Now, let C' be the vertical subbundle of Q. By definition of Q, a point q lies in Q if it can be joined to p by a horizontal curve. Thus, as a horizontal curve starting in Q must stay in Q, T_qQ must contain all horizontal vectors at q and therefore, $D_q \subset T_qQ$. Then, we have $T_qP = C_q \oplus D_q$, $D_q \subset T_qQ$ and $C'_q = C_q \cap T_qQ$ which combine to give us $T_qQ = C'_q \oplus D_q$. Hence, the restriction D' of the connection D to Q is indeed a connection on Q. □

The two definitions of holonomy groups turn out to be equivalent. But to see that, we need to first see how connections on vector and principal bundles relate. Let P, M and G be as above and let ρ be a representation of G on a vector space V. Then let $E \to M$ be the vector bundle $\rho(P)$ over M as defined in Section 3.1. Given a connection D on P, we want to construct a unique connection ∇ on E. Let $e \in \Gamma(E)$ be a smooth section. Then, if $\pi : P \times V \to \rho(P)$ is the natural projection, $\pi^*(e)$ is a section of $P \times V$ over P. Thus, considering $\pi^*(e)$ as a function $\pi^*(e) : P \to V$, its exterior derivative is a linear map given as $d\pi^*(e)|_p : T_pP \to V$ for each $p \in P$. This shows that $d\pi^*(e)$ is a smooth section of the vector bundle $V \otimes T^*P$ over P. Now, from the isomorphisms (3.4), (3.5) and (3.6), we obtain the natural splitting

$$V \otimes T^*P \cong (V \otimes \mathfrak{g}^*) \oplus (V \otimes \pi^*(TM)). \tag{3.7}$$

Let $\pi_D(d\pi^*(e))$ represent the component of $d\pi^*(e)$ in $\Gamma(V \otimes \pi^*(T^*M))$ with respect to the above splitting. Since both $\pi^*(e)$ and the splitting above are G-invariant, we have that $\pi_D(d\pi^*(e))$ is G-invariant. But from Section 3.1, we know that there is a 1-1 correspondence between G-invariant sections of $V \otimes \pi^*(T^*M)$ over P and sections of the corresponding vector bundle $E \otimes T^*M$ over M. Therefore, $\pi_D(d\pi^*(e))$ is the pullback of a unique element in $\Gamma(E \otimes T^*M)$. This allows to state the following definition.

Definition 3.3.3. Let M be a manifold, P a principal bundle over M with fiber G and D a connection on P. Let ρ be a representation of G on a vector space V and let E be the vector bundle $\rho(P)$ over M. If $e \in \Gamma(E)$, then $\pi_D(d\pi^*(e))$ is a G-invariant section of $V \otimes \pi^*(T^*M)$ over P. Let us then define $\nabla e \in \Gamma(E \otimes T^*M)$ to be the unique section of $E \otimes T^*M$ with pullback $\pi_D(d\pi^*(e))$ under the natural projection $V \otimes \pi^*(T^*M) \to E \otimes T^*M$. This determines a connection on the vector bundle E over M.

Therefore, we have associated a connection ∇ on the vector bundle $E = \rho(P)$ for every connection D on a principal bundle P. In particular, if we take $G = \mathrm{GL}(k, \mathbb{R})$ and ρ as the standard representation of G on $\mathbb{R}^k$, such that P is the frame bundle of E, we get a 1-1 correspondence between connections on P and E. But in general, the map $D \mapsto \nabla$ for any G and ρ may be neither injective nor surjective.

Now, this allows us to compare the holonomy groups of connections in vector bundles and principal bundles. The following proposition from [Joy00] gives us the correspondence.

Proposition 3.3.4. *Let M be a manifold, P a principal bundle over M with fiber G. Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G on a vector space V and set $E = \rho(P)$. Let D be a connection on P and let ∇ be the induced connection on E. Then, $\mathrm{Hol}(P, D)$ and $\mathrm{Hol}(\nabla)$ are subgroups of G and $\mathrm{GL}(V)$ defined up to conjugation respectively and*

$$\rho(\mathrm{Hol}(P, D)) = \mathrm{Hol}(\nabla).$$

If F^E is the frame bundle of a vector bundle E over a manifold M with fibre $\mathbb{R}^k$, then we know that F^E is a principal bundle with fibre $\mathrm{GL}(k, \mathbb{R})$. If ∇^E is a connection on E and D^E is the corresponding connection on F^E, then $\mathrm{Hol}(\nabla^E)$ and $\mathrm{Hol}(F^E, D^E)$ are both subgroups of $\mathrm{GL}(k, \mathbb{R})$ defined up to conjugation and

$$\mathrm{Hol}(\nabla^E) = \mathrm{Hol}(F^E, D^E).$$

Finally, the next proposition, whose proof is similar to that of Theorem 3.3.2, shows that for a connection ∇ on TM where M is connected, ∇ is an H-connection if and only if $\mathrm{Hol}(\nabla) \subseteq H$.

Proposition 3.3.5. *Suppose M is a connected manifold of dimension n and let us denote its frame bundle by F. Let ∇ be a connection on TM. Fix $f \in F$. Then, for each Lie subgroup $H \subset \mathrm{GL}(n, \mathbb{R})$, there exists a H-structure Q on M which is compatible with ∇ (that is, ∇ is an H-connection) which contains f if and only if $\mathrm{Hol}_f(\nabla) \subseteq H \subseteq \mathrm{GL}(n, \mathbb{R})$. If such a Q exists, then it is unique. In general, there is a 1-1 correspondence between H-stuctures equipped with a H-connection ∇ but not necessarily containing f and the homogeneous space $H \setminus \{a \in \mathrm{GL}(n, \mathbb{R}) : a \,\mathrm{Hol}_f(\nabla) a^{-1} \subseteq H\}$*

Proof. If Q exists, then it must contain f. As it is closed under H, we get that it contains $h \cdot f$ for each $h \in H$. As ∇ is an H-connection, any horizontal curve starting in Q must remain in Q. Thus, if $q \in Q$ and $p \in F$ such that $p \sim q$, then $p \in Q$, where $\sim$ is the equivalence relation in Definition 3.3.1. Therefore, if $p \in F$ and $p \sim h \cdot f$ for any $h \in H$, then $p \in Q$. But as M is connected, there exists a curve γ from $\pi(p)$ to $\pi(f)$ in M, where

π is the projection from the frame bundle. It follows that the horizontal lift $\widetilde{\gamma}$ of γ from p ends at $h \cdot f$ for some $h \in H$. Hence, every $p \in Q$ must satisfy $p \sim h \cdot f$ for some $h \in H$. Thus, if Q exists, Q must be $\{p \in F : p \sim h \cdot f \text{ for some } h \in H\}$. This set is a principal bundle over M and it has the subgroup of $\mathrm{GL}(n, \mathbb{R})$ generated by H and $\mathrm{Hol}_f(\nabla)$ as its fibre. Therefore, Q exists if and only if $\mathrm{Hol}_f(\nabla) \subseteq H$ and if it exists, it is unique.

Now, if $a \in \mathrm{GL}(n, \mathbb{R})$, then we know that $\mathrm{Hol}_{a \cdot f}(\nabla) = a\,\mathrm{Hol}_f(\nabla)a^{-1}$. Thus, we get that there is a unique H-structure Q containing $a \cdot f$ if and only if $a\,\mathrm{Hol}_f(\nabla)a^{-1} \subseteq H$. But H-structures containing $a \cdot f$ must contain $(ha) \cdot f$ for all $h \in H$. Therefore, we get the 1-1 correspondence of the H-structures with the set as claimed above. □

3.4 Stabilized tensors

This section gives a characterisation of H-structures on manifolds in terms of their stabilized tensors. Consider the canonical right-actions on $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$ by $\mathrm{GL}(n, \mathbb{R})$:

$$(g, v) \mapsto g^{-1}v$$

for $g \in \mathrm{GL}(n, \mathbb{R}), v \in \mathbb{R}^n$ and

$$(g, \alpha) \mapsto g^*\alpha = \alpha \circ g$$

for $\alpha \in (\mathbb{R}^n)^*$. Let $\xi_0 \in \mathcal{T}^{p,q}(\mathbb{R}^n) := (\otimes^p \mathbb{R}^n) \otimes (\otimes^q (\mathbb{R}^n)^*)$ be a (p, q)-tensor. Then, denoting the canonical basis on $\mathbb{R}^n$ as $\{e_i\}$ and the dual basis on $(\mathbb{R}^n)^*$ as $\{e^i\}$, we can represent ξ_0 in terms of its components as

$$\xi_0 = \xi^{i_1 \cdots i_p}_{j_1 \cdots j_q} e_{i_1} \otimes \cdots \otimes e_{i_p} \otimes e^{j_1} \otimes \cdots \otimes e^{j_q}$$

where $\xi^{i_1 \cdots i_p}_{j_1 \cdots j_q} = \xi_0(e^{i_1}, \ldots, e^{i_p}, e_{j_1}, \ldots, e_{j_q}) \in \mathbb{R}$. Then, we extend the canonical actions naturally onto tensors by

$$g \cdot \xi_0 = \xi^{i_1 \cdots i_p}_{j_1 \cdots j_q} g^{-1}e_{i_1} \otimes \cdots \otimes g^{-1}e_{i_p} \otimes g^*e^{j_1} \otimes \cdots \otimes g^*e^{j_q}. \tag{3.8}$$

The stabilizer of this action is denoted by

$$\mathrm{Stab}(\xi_0) = \{g \in \mathrm{GL}(n, \mathbb{R}) : g \cdot \xi_0 = \xi_0\}$$

and if we have a finite collection of tensors $(\xi_0)_i$, then letting $\mathrm{GL}(n, \mathbb{R})$ act on $\xi_0 = ((\xi_0)_1, \ldots, (\xi_0)_k)$ component wise, we have

$$\mathrm{Stab}(\xi_0) = \bigcap_i (\mathrm{Stab}(\xi_0)_i).$$

For example, if $g_0 := \delta_{ij} e^i \otimes e^j$ is the standard Euclidean metric and μ_0 is the standard volume form, then they are stabilized by

$$\mathrm{Stab}(g_0, \mu_0) = \mathrm{Stab}(g_0) \cap \mathrm{Stab}(\mu_0) = \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R}) = \mathrm{SO}(n).$$

Definition 3.4.1. Given an H-structure $\sigma \in \Gamma(\mathrm{Fr}(M)/H)$, we say that $\xi \in \Gamma(\mathcal{T}^{p,q}(TM))$ is **stabilized** by H if $H \subset \mathrm{Stab}(u^{-1} \cdot \xi)$ for a frame $u \in Q_\sigma = \pi_H^{-1}(\sigma(M)) \subset \mathrm{Fr}(M)$, where $u_x : T_xM \xrightarrow{\cong} \mathbb{R}^n$ for all $x \in M$ and π_H is the quotient map $\pi_H : \mathrm{Fr}(M) \to \mathrm{Fr}(M)/H$.

We want to study H-structures that are completely characterised by their stabilised tensors. That is, $H \subset \mathrm{SO}(n)$ is the stabilizer of a finite number of tensors on $\mathbb{R}^n$. Hence,

$$H = \mathrm{Stab}(\xi_0)$$

for some $\xi_0 = ((\xi_0)_1, \ldots, (\xi_0)_k)$ in an r-dimensional $\mathrm{GL}(n, \mathbb{R})$ submodule $V \leq \oplus \mathcal{T}^{p,q}(\mathbb{R}^n)$, where $V = V_1 \oplus \cdots \oplus V_k$ with $V_i \leq \mathcal{T}^{p_i,q_i}(\mathbb{R}^n)$. Then, let $\mathcal{F} \leq \oplus \mathcal{T}^{p,q}(TM)$ be a rank r subbundle with fibre $V \cong \mathbb{R}^r$. We then obtain a natural monomorphism of principal bundles $\rho : \mathrm{Fr}(M) \hookrightarrow \mathrm{Fr}(\mathcal{F})$ given on the fibres as

$$\rho(u_x) : \mathcal{F}_x \xrightarrow{\cong} V \tag{3.9}$$

which identifies $u_x \in \mathrm{Fr}(M)_x$ at each point $x \in M$ with a frame on the fibre $\mathcal{F}_x$.

Definition 3.4.2. A section $\xi \in \Gamma(\mathcal{F})$ is a **geometric structure** modelled on a fixed element $\xi_0 \in V \leq \mathcal{T}^{p,q}(\mathbb{R}^n)$, if for each $x \in M$ there exists a frame of T_xM identifying $\xi(x)$ and ξ_0.

Now, if we have $H \subset \mathrm{SO}(n)$ such that $H = \mathrm{Stab}(\xi_0)$, then we can define the **universal section** $\Xi \in \Gamma(\pi^* \mathcal{F})$ as

$$\Xi(y) = y^* \xi_0.$$

The universal section assigns to each H-class of frames $y \in \mathrm{Fr}(M)/H$, the vector in $\mathcal{F}_{\pi(y)}$ whose coordinates are given by the model tensor ξ_0 in the frame $\rho(u_{\pi(y)})$ as described in (3.9). Therefore, it codifies all smooth H-structures.

Furthermore, to each homogeneous section $\sigma \in \Gamma(\mathrm{Fr}(M)/H)$ defining an H-structure, we can associate a geometric structure $\xi \in \Gamma(\mathcal{F})$ modelled on ξ_0 by

$$\xi_\sigma := \sigma^* \Xi = \Xi \circ \sigma. \tag{3.10}$$

And conversely, if $\xi \in \Gamma(\mathcal{F})$ is a geometric structure stabilized by H we can associate with it at each point $x \in M$, the H-class of frames $\sigma(x) \in \pi^{-1}(x)$ such that

$$\xi(x) = \sigma(x)^* \xi_0.$$

This correspondence allows us to talk about geometric structures, H-structures and homogeneous sections interchangeably. Now, let us give a brief overview of three well-known examples of H-structures through this theoretical framework.

Example 3.4.3 ($\mathrm{U}(m)$**-structures).** Let $n = 2m \geq 4$ and take $H = \mathrm{U}(m) \subset \mathrm{SO}(2m)$ to be the unitary group. If $J_0 \in \mathrm{End}(\mathbb{R}^{2m})$ is the standard almost complex structure on $\mathbb{R}^{2m} = \mathbb{R}^m \oplus \mathbb{R}^m$ given by $J_0 = \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}$ then we have

$$U(m) = \mathrm{Stab}(J_0) \cap \mathrm{Stab}(g_0) = \mathrm{Stab}_{\mathrm{SO}(n)}(J_0).$$

A $U(m)$-structure (g, J) on M^{2m} consists of a Riemannian metric g and a $J \in \Gamma(\mathrm{End}(TM))$ such that $J^2 = -\mathrm{Id}_{TM}$ and $g(J\cdot, J\cdot) = g$, that is, an orthogonal almost complex structure. Compatible $\mathrm{U}(m)$-structures are in one-to-one correspondence with sections of the $\mathrm{SO}(2m)/\mathrm{U}(m)$-bundle $\pi : \mathrm{Fr}(M^{2m}, g)/\mathrm{U}(m) \to M$. Using the metric identification $\Lambda^2 \cong \mathfrak{so}(2m)$, we have that the $\mathrm{U}(m)$-irreducible decomposition as in (3.1) is given as

$$\Lambda^2 = \Lambda^2_{\mathfrak{u}(m)} \oplus \Lambda^2_{\mathfrak{m}}$$

where $\Lambda^2_{\mathfrak{u}(m)} \cong \mathfrak{u}(m) = \{A \in \mathfrak{so}(2m) : J_0 A = A J_0\}$ and $\Lambda^2_{\mathfrak{m}} \cong \mathfrak{m} := \mathfrak{u}(m)^\perp = \{A \in \mathfrak{so}(2m) : J_0 A = -A J_0\}$.

Example 3.4.4 (G_2**-structures).** Let $n = 7$ and take $H = \mathrm{G}_2 \subset \mathrm{SO}(7)$ (see Chapter 4 for details). Then denoting the standard basis of $(\mathbb{R}^7)^*$ as $(e^1, \dots, e^7)$, we define the standard G_2 structure $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$ by

$$\varphi_0 = e^{123} + e^1 \wedge (e^{45} - e^{67}) + e^2 \wedge (e^{46} - e^{75}) + e^3 \wedge (e^{47} - e^{56})$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$ and $e^{ij} = e^i \wedge e^j$. Then, $\mathrm{G}_2 = \mathrm{Stab}(\varphi_0) \subset \mathrm{SO}(7)$. Furthermore, φ_0 induces the standard Euclidean metric g_0 and the orientation μ_0 through the non-linear algebraic relation

$$(X \lrcorner \varphi_0) \wedge (Y \lrcorner \varphi_0) \wedge \varphi_0 = -6 g_0(X, Y) \mu_0 \tag{3.11}$$

for $X, Y \in \mathbb{R}^7$. Then a G_2-structure on a smooth 7-manifold M^7 is a 3-form φ which can be identified pointwise with φ_0 through a linear isomorphism. We call such a structure a positive 3-form $\varphi \in \mathscr{P}^3 M$. It then follows from (3.11) that φ induces a metric g and orientation μ_g on M^7 satisfying the condition

$$(X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi = -6g(X, Y)\mu_g.$$

The compatible G_2 structures on (M^7, g) are in one-to-one correspondence with the sections of the fibre bundle $\pi : \mathrm{Fr}(M^7, g)/\mathrm{G}_2 \to M$ with fibre $\mathrm{SO}(7)/\mathrm{G}_2 \cong \mathbb{RP}^7$. Using the metric identification $\Lambda^2 \cong \mathfrak{so}(7)$, we have that the G_2-irreducible decomposition as in (3.1) is given as

$$\Lambda^2 = \Lambda^2_{\mathfrak{g}_2} \oplus \Lambda^2_{\mathfrak{m}}$$

where $\Lambda^2_{\mathfrak{g}_2} = \{\omega : *(\omega \wedge \varphi) = \omega\} = \{\omega : \omega \wedge *\varphi = 0\} \cong \mathfrak{g}_2$ and $\Lambda^2_{\mathfrak{m}} = \{\omega : *(\omega \wedge \varphi) = -2\omega\} = \{u \lrcorner \varphi : u \in \mathbb{R}^7\} \cong \mathfrak{m}$.

We will explore these structures in detail in Chapter 4.

Example 3.4.5 (Spin(7)**-structures).** Let $n = 8$ and take $H = \mathrm{Spin}(7) \subset \mathrm{SO}(8)$. Denoting the standard basis of $(\mathbb{R}^8)^* = (\mathbb{R})^* \oplus (\mathbb{R}^7)^*$ as $(e^0, e^1, \ldots, e^7)$, we define the structure $\Phi_0 \in \Lambda^4(\mathbb{R}^8)^*$ as

$$\Phi_0 = e^0 \wedge \varphi_0 + *_{\mathbb{R}^7}\varphi_0.$$

Then a Spin(7)-structure on (M^8, g) is a 4-form Φ which can be identified pointwise with Φ_0 through a linear isomorphism. The compatible Spin(7) structures on (M^8, g) are in one-to-one correspondence with the sections of the fibre bundle $\pi : \mathrm{Fr}(M^8, g)/\mathrm{Spin}(7) \to M$ with fibre $\mathrm{SO}(8)/\mathrm{Spin}(7) \cong \mathbb{RP}^7$. Using the metric identification $\Lambda^2 \cong \mathfrak{so}(8)$, we have that the Spin(7)-irreducible decomposition as in (3.1) is given as

$$\Lambda^2 = \Lambda^2_{\mathfrak{spin}(7)} \oplus \Lambda^2_{\mathfrak{m}}$$

where $\Lambda^2_{\mathfrak{spin}(7)} = \{\omega : *(\omega \wedge \Phi) = \omega\} \cong \mathfrak{spin}(7) = \mathfrak{so}(7)$ and $\Lambda^2_{\mathfrak{m}} = \{\omega : *(\omega \wedge \Phi) = -3\omega\} \cong \mathfrak{m}$.

Chapter 4

G_2-structures

In Chapter 3, we briefly discussed G_2-structures as examples of H-structures when $H = G_2 \subset SO(7)$. In this chapter, we will first study the structures on the imaginary part of normed division algebras from which the canonical G_2-structure on $\mathbb{R}^7$ arises. Next, we describe how we can equip the tangent spaces of 7-dimensional Riemannian manifolds with that structure. We will describe how the space of k-forms on such manifolds decomposes and finally, we give an alternate formulation of the torsion of a G_2-structure. The main references for this chapter include [Kar08], [Kar20], [DGK23] and [Joy00].

4.1 Structures on normed division algebras

Let us equip $\mathbb{A} := \mathbb{R}^n$ with the standard Euclidean inner product $\langle \cdot, \cdot \rangle$ and we denote the norm induced from $\langle \cdot, \cdot \rangle$ by $\| \cdot \|$.

Definition 4.1.1. If $\mathbb{A}$ is an algebra over $\mathbb{R}$ with multiplicative identity $1 \neq 0$ such that

$$\|ab\| = \|a\|\|b\| \tag{4.1}$$

for all $a, b \in \mathbb{A}$, then we say that $\mathbb{A}$ is a **normed division algebra**.

That is, for a normed division algebra, the inner product and the algebra structure on $\mathbb{A}$ are compatible through equation (4.1). We define the **real part** of $\mathbb{A}$ to be the span of the multiplicative identity $1 \in \mathbb{A}$ over $\mathbb{R}$ and denote it by $\operatorname{Re}\mathbb{A}$ and we define the **imaginary**

part of $\mathbb{A}$ to be the orthogonal complement of $\operatorname{Re}\mathbb{A}$ with respect to $\langle \cdot, \cdot \rangle$, denoting it by $\operatorname{Im}\mathbb{A}$. Thus, we have an orthogonal decomposition

$$\mathbb{A} = \operatorname{Re}\mathbb{A} \oplus \operatorname{Im}\mathbb{A}$$

Notice that $\operatorname{Im}\mathbb{A} = (\operatorname{Re}\mathbb{A})^{\perp} \cong \mathbb{R}^{n-1}$. For $a \in \mathbb{A}$, we define the **conjugate** of a, which we denote by $\overline{a}$ to be

$$\overline{a} = \operatorname{Re} a - \operatorname{Im} a, \tag{4.2}$$

where $\operatorname{Re} a \in \operatorname{Re}\mathbb{A}$ and $\operatorname{Im} a \in \operatorname{Im}\mathbb{A}$.

Normed division algebras were classified by Hurwitz in 1898. Up to isomorphism, there are exactly four possibilities which are given by the following table:

$n = \dim \mathbb{A}$	1	2	4	8
Symbol	$\mathbb{R}$	$\mathbb{C} \cong \mathbb{R}^2$	$\mathbb{H} \cong \mathbb{R}^4$	$\mathbb{O} \cong \mathbb{R}^8$
Name	Real numbers	Complex numbers	Quaternions	Octonions

Table 4.1: Classification of normed division algebras

Note that as the dimension increases, the algebras in the table are subalgebras of the larger algebras. Furthermore, in the progression $\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O}$, some algebraic property is lost at each step. From $\mathbb{R}$ to $\mathbb{C}$, we lose the natural ordering. From $\mathbb{C}$ to $\mathbb{H}$ we lose commutativity. And from $\mathbb{H}$ to $\mathbb{O}$ we lose associativity. For the next section, the object of our focus would be the octonions $\mathbb{O}$, which allows us to define the canonical G_2-structure on $\mathbb{R}^7$.

From the compatibility condition (4.1), the following lemmas and identities (the proofs can be found in [Kar20, Section 3.1]) follow:

Lemma 4.1.2. *Let $a, b, c \in \mathbb{A}$. Then we have*

$$\langle ac, bc \rangle = \langle ca, cb \rangle = \langle a, b \rangle \|c\|^2, \tag{4.3}$$
$$\langle a, bc \rangle = \langle a\bar{c}, b \rangle, \quad \langle a, cb \rangle = \langle \bar{c}a, b \rangle. \tag{4.4}$$

and

$$\overline{ab} = \bar{b}\overline{a} \tag{4.5}$$

Lemma 4.1.3. *Let $a, b \in \mathbb{A}$. Then*

$$\langle a, b\rangle = \operatorname{Re}(a\bar{b}) = \operatorname{Re}(b\overline{a}) = \operatorname{Re}(\bar{b}a) = \operatorname{Re}(\overline{a}b) \tag{4.6}$$

and

$$\|a\|^2 = a\overline{a} = \overline{a}a \tag{4.7}$$

which gives us that $a^2 = aa$ is real if and only if a is either real or imaginary.

Lemma 4.1.4. *Let $a, b \in \mathbb{A}$. Then, we have*

$$\begin{aligned} (ab)\bar{b} = a(b\bar{b}) = \|b\|^2 a = a(\bar{b}b) = (a\bar{b})b, \\ a(\overline{a}b) = (a\overline{a})b = \|a\|^2 b = (\overline{a}a)b = \overline{a}(ab). \end{aligned} \tag{4.8}$$

Now, we will define two $\mathbb{A}$-valued multilinear maps on $\mathbb{A}$ which induce the associative and coassociative forms on $\operatorname{Im}\mathbb{A}$.

Definition 4.1.5. For $a, b \in \mathbb{A}$, we define a bilinear map $[\cdot,\cdot] : \mathbb{A}^2 \to \mathbb{A}$ by

$$[a, b] = ab - ba \tag{4.9}$$

and we call the map $[\cdot,\cdot]$ the **commutator** of $\mathbb{A}$. For $a, b, c \in \mathbb{A}$ we define a trilinear map $[\cdot,\cdot,\cdot] : \mathbb{A}^3 \to \mathbb{A}$ by

$$[a, b, c] = (ab)c - a(bc) \tag{4.10}$$

and we call the map $[\cdot,\cdot,\cdot]$ the **associator** of $\mathbb{A}$.

We now use the above identities and lemmas to prove some characteristic properties of the commutator and the associator.

Proposition 4.1.6. *The commutator and associator are both totally skew-symmetric (alternating) in their arguments.*

Proof. From the definition of the commutator, it is clear that it is skew-symmetric. As $\mathbb{A}$ is an algebra over $\mathbb{R}$, in particular it is a vector space over $\mathbb{R}$ and thus if any of the arguments of the associator are in $\operatorname{Re}\mathbb{A}$, the associator vanishes. Hence, as the associator is trilinear, it is enough to show that it is alternating when all the arguments are imaginary. Let $a, b \in \operatorname{Im}\mathbb{A}$. Then, $\overline{a} = -a$ and $\bar{b} = -b$. From (4.8), we have

$$-[a, a, b] = [a, \overline{a}, b] = (a\overline{a})b - a(\overline{a}b) = 0$$

and

$$-[a,b,b] = [a,\bar{b},b] = (a\bar{b})b - a(\bar{b}b) = 0.$$

We thus have that $[\cdot,\cdot,\cdot]$ is alternating in the first two arguments and the last two arguments. Therefore, we also have $[a,b,a] = -[a,a,b] = 0$ which shows that it is also alternating in the first and last argument. □

The next property that we will prove will show that the commutator and the associator restrict to $\operatorname{Im}\mathbb{A}$-valued maps on $\operatorname{Im}\mathbb{A}$.

Lemma 4.1.7. *Let $a,b,c \in \operatorname{Im}\mathbb{A}$. Then $[a,b] \in \operatorname{Im}\mathbb{A}$ and $[a,b,c] \in \operatorname{Im}\mathbb{A}$.*

Proof. To show that $[a,b],[a,b,c] \in \operatorname{Im}\mathbb{A}$ it suffices to show that they are orthogonal to every element in $\operatorname{Re}\mathbb{A} = \{t1 : t \in \mathbb{R}\}$. That is, we need to show that $[a,b]$ and $[a,b,c]$ are orthogonal to 1. As $\bar{a} = -a$, from the identity (4.4), we have

$$\begin{aligned}\langle [a,b],1\rangle = \langle ab-ba,1\rangle &= \langle b,\bar{a}\rangle - \langle a,\bar{b}\rangle \\ &= -\langle b,a\rangle + \langle a,b\rangle = 0.\end{aligned}$$

Similarly, as $\bar{b} = -b, \bar{c} = -c$, from (4.5) we have $\overline{bc} = \bar{c}\bar{b} = (-c)(-b) = cb$ and thus we get

$$\begin{aligned}\langle [a,b,c],1\rangle = \langle (ab)c - a(bc),1\rangle &= \langle ab,\bar{c}\rangle - \langle bc,\bar{a}\rangle \\ &= -\langle ab,c\rangle + \langle bc,a\rangle \\ &= -\langle a,c\bar{b}\rangle + \langle bc,a\rangle \\ &= \langle a,cb+bc\rangle = \langle a,bc+\overline{bc}\rangle = 2\langle a,\operatorname{Re}(bc)\rangle = 0,\end{aligned}$$

as desired. □

As the next proposition shows, it turns out that we get multilinear alternating forms on $\mathbb{A}$ by combining the commutator and the associator with the inner product.

Lemma 4.1.8. *For $a,b,c,d \in \mathbb{A}$, the expressions $\langle a,[b,c]\rangle$ and $\langle a,[b,c,d]\rangle$ are both totally skew-symmetric in their arguments.*

Proof. From Lemma 4.1.6, we know that the associator and the commutator are totally skew-symmetric. Therefore, it suffices to show that $\langle a,[a,b]\rangle = 0$ and $\langle a,[a,b,c]\rangle = 0$. From (4.3) we have

$$\langle a,[a,b]\rangle = \langle a,ab-ba\rangle = \|a\|^2\langle 1,b\rangle - \|a\|^2\langle 1,b\rangle = 0$$

and for the associator, using (4.3) and (4.4) we have

$$\begin{aligned}\langle a, [a, b, c]\rangle = \langle a, (ab)c - a(bc)\rangle &= \langle a\bar{c}, ab\rangle - \|a\|^2\langle 1, bc\rangle \\ &= \|a\|^2\langle \bar{c}, b\rangle - \|a\|^2\langle \bar{c}, b\rangle = 0\end{aligned}$$

as desired. □

Definition 4.1.9. Define a 3-form φ and 4-form ψ on $\operatorname{Im}\mathbb{A}$ as follows:

$$\varphi(a, b, c) = \frac{1}{2}\langle a, [b, c]\rangle = \frac{1}{2}\langle [a, b], c\rangle, \tag{4.11}$$

$$\psi(a, b, c, d) = \frac{1}{2}\langle a, [b, c, d]\rangle = -\frac{1}{2}\langle [a, b, c], d\rangle, \tag{4.12}$$

for $a, b, c, d \in \operatorname{Im} A$. We call the form $\varphi \in \Lambda^3(\operatorname{Im}\mathbb{A})^*$ the **associative** 3-form and the form $\psi \in \Lambda^4(\operatorname{Im}\mathbb{A})^*$ the **coassociative** 4-form.

We can also define a cross-product on $\operatorname{Im}\mathbb{A}$ which generalizes the cross-product on $\mathbb{R}^3$.

Definition 4.1.10. The **vector cross product** on $\operatorname{Im}\mathbb{A}$ is the bilinear map $\times : \operatorname{Im}\mathbb{A} \times \operatorname{Im}\mathbb{A} \to \operatorname{Im}\mathbb{A}$ given as

$$a \times b = \operatorname{Im}(ab) \tag{4.13}$$

for all $a, b \in \operatorname{Im}\mathbb{A}$.

From the above identities and lemmas it follows easily that this vector cross product satisfies properties similar to the usual cross product on $\mathbb{R}^3$ such as the following:

Lemma 4.1.11. *For $a, b \in \operatorname{Im}\mathbb{A}$ we have*

$$a \times b = -b \times a, \tag{4.14}$$

$$\langle a \times b, a\rangle = 0, \tag{4.15}$$

$$\operatorname{Re}(ab) = -\langle a, b\rangle 1. \tag{4.16}$$

Proof. As $\overline{a} = -a$ and $\overline{b} = -b$ and as $\operatorname{Re} v = \frac{1}{2}(v + \overline{v}), \operatorname{Im} v = \frac{1}{2}(v - \overline{v})$ for $v \in \mathbb{A}$, from (4.5), we get

$$2a \times b = ab - \overline{ab} = ab - ba = [a, b]. \tag{4.17}$$

Hence, $a \times b = -b \times a$. As $a \in \operatorname{Im} \mathbb{A}$, from the definition of the vector product (4.13) we have

$$\langle a \times b, a \rangle = \langle \operatorname{Im}(ab), a \rangle = \langle ab, a \rangle.$$

Then, using (4.3) we have

$$\langle a \times b, a \rangle = \langle ab, a \rangle = \langle ab, a1 \rangle = \|a\|^2 \langle b, 1 \rangle = 0$$

as $b \in \operatorname{Im} \mathbb{A}$ is orthogonal to $1 \in \operatorname{Re} \mathbb{A}$. Finally, as $\bar{b} = -b$, from (4.6) we get

$$\langle a, b \rangle 1 = \operatorname{Re}(a\bar{b}) = -\operatorname{Re}(ab). \qquad \square$$

From (4.16) and (4.13) we have that for $ab \in \mathbb{A}$ the decomposition into real and imaginary parts is given as

$$ab = -\langle a, b \rangle 1 + a \times b. \tag{4.18}$$

Therefore, from (4.17) and (4.11) it follows that for $a, b, c \in \operatorname{Im} \mathbb{A}$

$$\varphi(a, b, c) = \langle a \times b, c \rangle. \tag{4.19}$$

Furthermore as from (4.18) we know that $a \times b - ab$ is real, we have

$$\varphi(a, b, c) = \langle ab, c \rangle. \tag{4.20}$$

Hence, we have an elegant relation between the vector cross product and the associative 3-form φ.

4.2 Canonical G_2-structure on $\mathbb{R}^7$

The previous section defined the associative 3-form, coassociate 4-form and the vector cross product on $\operatorname{Im} \mathbb{A}$. In this section, we will take $\mathbb{A} = \mathbb{O}$ and describe the canonical G_2-structure on $\mathbb{R}^7 \cong \operatorname{Im} \mathbb{O}$. Let us denote the standard Euclidean metric on $\mathbb{R}^7 \cong \operatorname{Im} \mathbb{O}$ by g_0, the associated orthonormal basis by $e_1, \ldots, e_7$, standard volume form associated to g_0 and the standard orientation by $\mathrm{vol}_0 = e^1 \wedge \cdots \wedge e^7$, the associative 3-form by φ_0, the co-associative 4-form by ψ_0 and finally the cross-product by $\times_0$. We call the tuple $(g_0, \mathrm{vol}_0, \varphi_0, \psi_0, \times_0)$ the **standard G_2-package** on $\mathbb{R}^7$.

Let $e^1, \ldots, e^7$ denote the standard dual basis on $(\mathbb{R}^7)^*$, let $e^{ijk} = e^i \wedge e^j \wedge e^k$ and $e^{ijkl} = e^i \wedge e^j \wedge e^k \wedge e^l$. Then, from the octonion multiplication table it follows that

$$\varphi_0 = e^{123} - e^{167} - e^{527} - e^{563} - e^{415} - e^{426} - e^{437}, \tag{4.21}$$

$$\psi_0 = e^{4567} - e^{4523} - e^{4163} - e^{4127} - e^{2637} - e^{1537} - e^{1526}. \tag{4.22}$$

From these representations, it follows that

$$\psi_0 = \star_0 \varphi_0$$

where $\star_0$ is the Hodge star operator induced from (g_0, vol_0). Now, we use the standard G_2-package on $\mathbb{R}^7$ to define the G_2-group.

Definition 4.2.1. We define the group $\mathrm{G}_2 < \mathrm{GL}(7, \mathbb{R})$ as

$$\mathrm{G}_2 = \{A \in \mathrm{GL}(7, \mathbb{R}) : A^* g_0 = g_0, A^* \mathrm{vol}_0 = \mathrm{vol}_0, A^* \varphi_0 = \varphi_0\}. \tag{4.23}$$

That is, G_2 preserves the standard G_2-package on $\mathbb{R}^7$.

Notice that as $A \in \mathrm{G}_2$ preserves the standard metric and orientation on $\mathbb{R}^7$, it follows that G_2 is a subgroup of $\mathrm{SO}(7)$. Also, note that in the above definition, we are missing the cross product $\times_0$ and the coassociative form ψ_0 which are part of the standard G_2-package on $\mathbb{R}^7$. Since g_0 and vol_0 determine $\star_0$ and as $\psi_0 = \star_0 \varphi_0$, it determines ψ_0. Furthermore, we know from (4.19) that g_0 and φ_0 determine $\times_0$. Therefore, any $A \in G_2$ preserves $\times_0$ and ψ_0 as well. In fact, the following theorem from [Bry87] shows that the definition of the G_2 group only requires the 3-form φ_0:

Theorem 4.2.2. *Let $G = \{A \in \mathrm{GL}(7, \mathbb{R}) : A^* \varphi_0 = \varphi_0\}$. Then $\mathrm{G}_2 = G$. That is, if $A \in \mathrm{GL}(7, \mathbb{R})$ preserves φ_0, then it preserves g_0 and vol_0 as well.*

Proof. From the explicit formula for φ_0 in (4.21), from direct computation it follows that

$$(a \lrcorner \varphi_0) \wedge (b \lrcorner \varphi_0) \wedge \varphi_0 = -6 g_0(a, b) \mathrm{vol}_0 \tag{4.24}$$

where $a, b \in \mathbb{R}^7$. Then, if $A^* \varphi_0 = \varphi_0$, applying A^* on both sides of the equation we have

$$\begin{aligned} g_0(a, b)(\det A) \mathrm{vol}_0 &= g_0(a, b) A^* \mathrm{vol}_0 \\ &= -\frac{1}{6} A^*(a \lrcorner \varphi_0) \wedge A^*(b \lrcorner \varphi_0) \wedge A^* \varphi_0 \\ &= -\frac{1}{6} (A^{-1} a \lrcorner A^* \varphi_0) \wedge (A^{-1} b \lrcorner A^* \varphi_0) \wedge A^* \varphi_0 \\ &= -\frac{1}{6} (A^{-1} a \lrcorner \varphi_0) \wedge (A^{-1} b \lrcorner \varphi_0) \wedge \varphi_0 \\ &= g_0(A^{-1} a, A^{-1} b) \mathrm{vol}_0 \\ &= (A^{-1})^* g_0(a, b) \mathrm{vol}_0 \end{aligned} \tag{4.25}$$

which implies that $(\det A) g_0(Aa, Ab) = g_0(a, b)$. In terms of matrices, this means $g_0 = (\det A) A^T g_0 A$. Taking determinants of both sides, we get $\det g_0 = (\det A)^9 \det g_0$. This gives us $\det A = 1$ and $A^* g_0 = g_0$. Furthermore, from (4.25) we have $A^* \operatorname{vol}_0 = \operatorname{vol}_0$. $\square$

4.3 G_2-manifolds

In this section, we want to associate the G_2-package on $\mathbb{R}^7$ to each tangent space of an orientable 7-manifold.

Let M be an oriented 7-manifold. For each point $p \in M$, let us define $\mathscr{P}^3_p M$ to be the subset of 3-forms φ such that there exists an oriented isomorphism between $T_p M$ and $\mathbb{R}^7$ which identifies φ and the associative 3-form φ_0 on $\mathbb{R}^7$. Since from Theorem 4.2.2 we know that φ_0 has symmetry group G_2, we get $\mathscr{P}^3_p M \cong \mathrm{GL}_+(7, \mathbb{R})/\mathrm{G}_2$.

Note that $\dim \mathrm{GL}_+(7, \mathbb{R}) = 49$ and $\dim \mathrm{G}_2 = 14$ so $\dim \mathrm{GL}_+(7, \mathbb{R})/\mathrm{G}_2 = 49 - 14 = 35$ which is the same as $\dim \Lambda^3 T^*_p M = \binom{7}{3} = 35$ and hence $\mathscr{P}^3_p M$ is an open subset of $\Lambda^3 T^*_p M$. Let $\mathscr{P}^3 M$ be the bundle over M with fibre $\mathscr{P}^3_p M$ over each $p \in M$. Then $\mathscr{P}^3 M$ is an open subbundle of $\Lambda^3 T^* M$. We say that a 3-form φ on M is **positive** if $\varphi_p \in \mathscr{P}^3_p M$ for each $p \in M$.

Definition 4.3.1. Let M be an oriented 7-manifold. A G_2**-structure** on M is a positive 3-form φ.

As φ_0 determines g_0 and vol_0, a G_2-structure φ on M induces a Riemannian metric g_φ and an associated Riemannian volume form $\operatorname{vol}_\varphi$, which then induce a Hodge star operator $\star_\varphi$ and dual 4-form $\psi = \star_\varphi \varphi$.

Note that if φ is a G_2-structure, and if we let Q be the subset of the frame bundle $\mathrm{Fr}(M)$ consisting of isomorphisms between $T_p M$ amd $\mathbb{R}^7$ which idenitfy φ and φ_0, then it can be shown that Q is a principal subbundle of $\mathrm{Fr}(M)$ with fibre G_2. That is, Q is a G_2-structure as in Chapter 3. Therefore, the existence of a G_2-structure depends completely on the topology of the manifold. The next theorem whose proof can be found in [LM89] characterizes topologically which manifolds admit a G_2-structure.

Theorem 4.3.2. *An orientable 7-dimensional manifold M admits a G_2-structure if and only if M is spinnable. That is, if and only if its first and second Stiefel-Whitney classes $w_1(M)$ and $w_2(M)$ vanish.*

Let M be a 7-manifold equipped with a G_2-structure φ and let g be the induced metric. If ∇ is the Levi-Civita connection of g, then we call $\nabla\varphi$ the **torsion** of φ and we say that φ is **torsion-free** if $\nabla\varphi = 0$.

Definition 4.3.3. Let M be a 7-manifold. We say that (M, φ) is a G_2**-manifold** if φ is a torsion-free G_2-structure.

Later in this thesis, we will be using some contractions relating φ, ψ and g. We list these identities in the next few lemmas. The proofs of these lemmas can be found in [Kar08].

Lemma 4.3.4 (Contractions of φ with itself). *With respect to a local orthonormal frame on M, the following identities hold*

$$\varphi_{ijk}\varphi_{ijk} = 42, \tag{4.26}$$

$$\varphi_{ijk}\varphi_{ajk} = 6g_{ia}, \tag{4.27}$$

$$\varphi_{ijk}\varphi_{abk} = g_{ia}g_{jb} - g_{ib}g_{ja} - \psi_{ijab}. \tag{4.28}$$

Lemma 4.3.5 (Contractions of φ with ψ). *With respect to a local orthonormal frame on M, the following identities hold*

$$\varphi_{ijk}\psi_{aijk} = 0, \tag{4.29}$$

$$\varphi_{ijk}\psi_{abjk} = -4\varphi_{iab}, \tag{4.30}$$

$$\begin{aligned}\varphi_{ijk}\psi_{abck} = {} & g_{ia}\varphi_{jbc} + g_{ib}\varphi_{ajc} + g_{ic}\varphi_{abj} \\ & - g_{aj}\varphi_{ibc} - g_{bj}\varphi_{aic} - g_{cj}\varphi_{abi}.\end{aligned} \tag{4.31}$$

Lemma 4.3.6 (Contractions of ψ with itself). *With respect to a local orthonormal frame on M, the following identities hold*

$$\psi_{ijkl}\psi_{ijkl} = 168, \tag{4.32}$$

$$\psi_{ijkl}\psi_{ajkl} = 24g_{ia}, \tag{4.33}$$

$$\psi_{ijkl}\psi_{abkl} = 4g_{ia}g_{jb} - 4g_{ib}g_{ja} - 2\psi_{ijab}, \tag{4.34}$$

$$\begin{aligned}\psi_{ijkl}\psi_{abcl} = {} & -\varphi_{ajk}\varphi_{ibc} - \varphi_{iak}\varphi_{jbc} - \varphi_{ija}\varphi_{kbc} \\ & + g_{ia}g_{jb}g_{kc} + g_{ib}g_{jc}g_{ka} + g_{ic}g_{ja}g_{kb} \\ & - g_{ia}g_{jc}g_{kb} - g_{ib}g_{ja}g_{kc} - g_{ic}g_{jb}g_{ka} \\ & - g_{ia}\psi_{jkbc} - g_{ja}\psi_{kibc} - g_{ka}\psi_{ijbc} \\ & + g_{ab}\psi_{ijkc} - g_{ac}\psi_{ijkb}.\end{aligned} \tag{4.35}$$

4.4 Decomposition of the space of forms on a manifold with G_2-structure

Similar to how complex-valued differential forms on an almost complex manifold decompose into forms of type (p, q), if M is a manifold with a G_2-structure φ on a manifold M, where

φ is not necessarily torsion-free, the space of differential forms Ω^k on M decomposes into irreducible representations of G_2. The characterization of these decompositions given in this section will be useful to simplify our computations involving these differential forms in the later chapters.

When $k = 2, 3$ the decomposition is given as

$$\Omega^2 = \Omega^2_7 \oplus \Omega^2_{14}, \tag{4.36}$$
$$\Omega^3 = \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27}, \tag{4.37}$$

where Ω^k_l has pointwise dimension l and the decomposition is orthogonal with respect to the metric g. Furthermore, since the Hodge star $\star$ is an isometry and $\Omega^k_l = \star(\Omega^{7-k}_l)$, we have

$$\Omega^5 = \Omega^5_7 \oplus \Omega^5_{14}, \tag{4.38}$$
$$\Omega^4 = \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27}. \tag{4.39}$$

In the next two subsections, we will describe some characterizations of these subspaces.

4.4.1 The decomposition of Ω^2

Let $\mathrm{P} : \Omega^2 \to \Omega^2$ be the map

$$\mathrm{P}\beta = 2 \star (\varphi \wedge \beta) \tag{4.40}$$

for $\beta \in \Omega^2$. If $\beta = \frac{1}{2}\beta_{ij} e_i \wedge e_j$ in terms of a local orthonormal frame, then using (1.4),

$$\begin{aligned}
\mathrm{P}\beta = \frac{1}{2}(P\beta)_{ij} e_i \wedge e_j &= \beta_{ij} \star (e_i \wedge e_j \wedge \varphi) \\
&= \beta_{ij} e_i \lrcorner \star(e_j \wedge \varphi) = -\beta_{ij} e_i \lrcorner e_j \lrcorner \star\varphi \\
&= -\frac{1}{2}\beta_{ij}\psi_{jiab} e_a \wedge e_b.
\end{aligned}$$

Hence, we have

$$(\mathrm{P}\beta)_{ab} = \beta_{ij}\psi_{ijab} = \psi_{abij}\beta_{ij} \tag{4.41}$$

Direct computation yields

$$\langle \mathrm{P}\beta, \mu \rangle = \psi_{ijab}\beta_{ij}\mu_{ab} = \langle \beta, \mathrm{P}\mu \rangle. \tag{4.42}$$

That is, P is pointwise self-adjoint and hence it is orthogonally diagonalizable with real eigenvalues. Furthermore, we have

$$\begin{aligned}(\mathrm{P}^2\beta)_{ab} &= \psi_{abij}(\mathrm{P}\beta)_{ij} = \psi_{abij}\psi_{ijpq}\beta_{pq} \\ &= \beta_{ij}(4g_{ia}g_{jb} - 4g_{ib}g_{ja} - 2\psi_{ijab}) \\ &= 4\beta_{ab} - 4\beta_{ba} - 2\beta_{ij}\psi_{ijab} \\ &= 8\beta_{ab} - 2(\mathrm{P}\beta)_{ab}.\end{aligned}$$

Therefore,

$$\mathrm{P}^2 = 8\mathrm{I} - 2\mathrm{P}$$

where $\mathrm{I} : \Omega^2 \to \Omega^2$ is the identity operator. It follows that $(\mathrm{P} + 4\mathrm{I})(\mathrm{P} - 2\mathrm{I}) = 0$ and thus the eigenvalues of P are -4 and $+2$. Hence, we get a decomposition of Ω^2 into the two eigenspaces. The following theorem gives an alternate description of the decomposition of Ω^2.

Theorem 4.4.1. *We have*

$$\Omega^2 = \Omega^2_7 \oplus \Omega^2_{14}$$

where

$$\begin{aligned}\Omega^2_7 &= \{\beta \in \Omega^2 \mid \mathrm{P}\beta = -4\beta\} &(4.43)\\ &= \{X \lrcorner \varphi \mid X \in \mathfrak{X}\} &(4.44)\end{aligned}$$

and

$$\begin{aligned}\Omega^2_{14} &= \{\beta \in \Omega^2 \mid \mathrm{P}\beta = 2\beta\} &(4.45)\\ &= \{\beta \in \Omega^2 \mid \beta \wedge \psi = 0\}. &(4.46)\end{aligned}$$

Proof. If $\beta_{ij} = X_k\varphi_{kij} \in \Omega^2$ for some vector field X, then from (4.31) we have

$$(P\beta)_{ab} = \psi_{abij}X_k\varphi_{kij} = -4X_k\varphi_{kab} = -4\beta_{ab}.$$

Conversely, let $\beta_{ij}\psi_{ijkl} = -4\beta_{kl}$. Then, if X is a vector field defined by $X_m = \frac{1}{6}\beta_{kl}\varphi_{mkl}$, from (4.28) we get

$$\begin{aligned}(X \lrcorner \varphi)_{ij} = X_m\varphi_{mij} = \frac{1}{6}\beta_{kl}\varphi_{mkl}\varphi_{mij} &= \frac{1}{6}\beta_{kl}(g_{ik}g_{jl} - g_{il}g_{jk} - \psi_{ijkl}) \\ = \frac{1}{6}\beta_{ij} - \frac{1}{6}\beta_{ji} + \frac{4}{6}\beta_{ij} &= \beta_{ij}.\end{aligned}$$

Now, let $\beta \wedge \psi = 0$. Then, from (1.3) we have

$$0 = \frac{1}{2}\beta_{ij} dx^i \wedge dx^j \wedge \psi = -\frac{1}{2}\beta_{ij} \star (\partial_i \lrcorner \star (dx^j \wedge \psi))$$
$$= -\frac{1}{2}\beta_{ij} \star (\partial_i \lrcorner \partial_j \lrcorner \star\psi) = -\frac{1}{2} \star (\beta_{ij}\partial_i \lrcorner \partial_j \lrcorner \varphi).$$

As the Hodge star operator is an isomorphism, we have that the above condition is equivalent to $\beta_{ij}\varphi_{ijk} = 0$. On the other hand, if $\mathrm{P}\beta = 2\beta$, as the eigenspace decomposition is orthogonal, we have that for each vector field X,

$$0 = \langle \beta, X \lrcorner \varphi \rangle = \beta_{ij} X_k \varphi_{kij}.$$

Hence, $\beta_{ij}\varphi_{ijk} = 0$. Finally, if $\beta_{ij}\varphi_{ijk} = 0$, then

$$\beta_{ij}\psi_{ijab} = \beta_{ij}(g_{ia}g_{jb} - g_{ib}g_{ja} - \varphi_{ijk}\varphi_{abk}) = \beta_{ab} - \beta_{ba} = 2\beta_{ab}.$$ □

Let $\beta = \frac{1}{6} X \lrcorner \varphi$. That is, in a local orthonormal frame, $\beta_{ij} = \frac{1}{6} X_k \varphi_{kij}$. Then it follows from (4.27) that $\beta_{ij}\varphi_{ijp} = X_p$. Therefore,

$$\beta_{ab} = \frac{1}{6} X_l \varphi_{lab} \iff X_k = \beta_{ab}\varphi_{abk}. \tag{4.47}$$

Furthermore, we have

$$\left(\frac{1}{6} X_k \varphi_{kij}\right)\left(\frac{1}{6} Y_l \varphi_{lij}\right) = \frac{1}{6} X_k Y_k \tag{4.48}$$

which can be written invariantly as

$$\langle X \lrcorner \varphi, Y \lrcorner \varphi \rangle = 6\langle X, Y \rangle. \tag{4.49}$$

Now, we define a map $\mathrm{V} : \mathcal{T}^2 \to \Omega^1$ by

$$(\mathrm{V}A)_k = A_{ij}\varphi_{ijk} \tag{4.50}$$

for $A \in \mathcal{T}^2$. From the previous proof we know that

$$\beta \in \Omega^2_{14} \iff \beta_{ij}\varphi_{ijk} = 0. \tag{4.51}$$

It follows that $\ker \mathrm{V} = \mathcal{S}^2 \oplus \Omega^2_{14}$, where $\mathcal{S}^2$ are the symmetric 2-tensors. Thus, only the Ω^2_7 part of A contributes to $\mathrm{V}A$ and it is called the **vector part** of A. Thus we can write (4.47) as

$$A_7 = \frac{1}{6}(\mathrm{V}A) \lrcorner \varphi, \quad \mathrm{V}(X \lrcorner \varphi) = 6X \tag{4.52}$$

and (4.49) becomes

$$\langle \mathrm{V}A, \mathrm{V}B \rangle = 6\langle A_7, B_7 \rangle$$

for $A, B \in \mathcal{T}^2$. Finally, let π_7 and π_{14} denote the orthogonal projections to Ω^2_7 and Ω^2_{14} respectively. Let us write $\beta_7 = \pi_7\beta$ and $\beta_{14} = \pi_{14}\beta$. Then, we have that

$$\mathrm{P}\beta = -4\beta_7 + 2\beta_{14} \tag{4.53}$$

which gives us

$$\beta_7 = \frac{1}{6}(2\beta - \mathrm{P}\beta), \quad \beta_{14} = \frac{1}{6}(4\beta + \mathrm{P}\beta). \tag{4.54}$$

4.4.2 The decomposition of Ω^3 and Ω^4

For $\sigma \in \Omega^k$ and $A = A_{ij}dx^i \otimes dx^j \in \mathcal{T}^2$, we define

$$(A \diamond \sigma)_{i_1 i_2 \cdots i_k} = A_{i_1 p}\sigma_{p i_2 \cdots i_k} + A_{i_2 p}\sigma_{i_1 p i_3 \cdots i_k} + \cdots + A_{i_k p}\sigma_{i_1 i_2 \cdots i_{k-1} p}. \tag{4.55}$$

If we take $A = g$ in the above equation, we get

$$g \diamond \sigma = k\sigma. \tag{4.56}$$

Then from (4.36) and (1.11), we have the decomposition

$$\mathcal{T}^2 \cong \Omega^0 \oplus \mathcal{S}^2_0 \oplus \Omega^2_7 \oplus \Omega^2_{14} \tag{4.57}$$

which allows us to write $A \in \mathcal{T}^2$ as

$$A = \frac{1}{7}(\operatorname{tr} A)g + A_{27} + A_7 + A_{14} \tag{4.58}$$

where A_{27} is the traceless symmetric part of A. Then, we can extend P to be a map on all of $\mathcal{T}^2$ by defining

$$(\mathrm{P}A)_{ab} = A_{ij}\psi_{ijab}. \tag{4.59}$$

Clearly, $\ker \mathrm{P} = \mathcal{S}$ and hence

$$\mathrm{P}A = \mathrm{P}\left(\frac{1}{7}(\operatorname{tr} A)g + A_{27} + A_7 + A_{14}\right) = -4A_7 + 2A_{14}. \tag{4.60}$$

From (4.55), we have linear maps $\mathcal{T}^2 \to \Omega^k$ for $k = 3, 4$ given as

$$A \mapsto A \diamond \varphi,$$
$$A \mapsto A \diamond \psi,$$

where in a local orthonormal frame,

$$(A \diamond \varphi)_{ijk} = A_{ip}\varphi_{pjk} + A_{jp}\varphi_{ipk} + A_{kp}\varphi_{ijp}, \tag{4.61}$$
$$(A \diamond \psi)_{ijkl} = A_{ip}\psi_{pjkl} + A_{jp}\psi_{ipkl} + A_{kp}\psi_{ijpl} + A_{lp}\psi_{ijkp}. \tag{4.62}$$

A direct computation using the identities in this subsection along with the contraction identities, gives us the following proposition.

Proposition 4.4.2. *For $A, B \in \mathcal{T}^2$, with respect to the decomposition (4.58), we have*

$$\langle A \diamond \varphi, B \diamond \varphi \rangle = \frac{54}{7}(\operatorname{tr} A)(\operatorname{tr} B) + 12\langle A_{27}, B_{27}\rangle + 36\langle A_7, B_7\rangle, \tag{4.63}$$
$$\langle A \diamond \psi, B \diamond \psi \rangle = \frac{384}{7}(\operatorname{tr} A)(\operatorname{tr} B) + 48\langle A_{27}, B_{27}\rangle + 144\langle A_7, B_7\rangle. \tag{4.64}$$

We use this proposition to give a characterization of tensors in Ω^2_{14} in terms of the diamond operator.

Corollary 4.4.3. *Let $A \in \mathcal{T}^2$. Then*

$$A \in \Omega^2_{14} \iff A \diamond \varphi = 0 \iff A \diamond \psi = 0. \tag{4.65}$$

Furthermore, the maps $A \mapsto A \diamond \varphi$ and $A \mapsto A \diamond \psi$ when restricted to $\mathcal{S}^2 \oplus \Omega^2_7 \subset \mathcal{T}^2$ which forms the orthogonal complement of Ω^2_{14}, are linear isomorphisms onto Ω^3 and Ω^4 respectively.

Proof. Taking $A = B$ in (4.63) we have

$$|A \diamond \varphi|^2 = \frac{54}{7}(\operatorname{tr} A)^2 + 12|A_{27}|^2 + 36|A_7|^2.$$

Thus, we have $A \diamond \varphi = 0 \iff A = A_{14}$, and as we can use the same argument for $A \diamond \psi$ by taking $A = B$ in (4.64), it proves the first part of the corollary.

If $A_{14} = 0$, then from above, we get that $A \diamond \varphi = 0 \iff A = 0$. Thus, the map $A \mapsto A \diamond \varphi$ is injective on the orthogonal complement of Ω^2_{14}. Counting the dimensions, as both sides are 35-dimensional, the map is a linear isomorphism. Similarly, the map $A \mapsto A \diamond \psi$ is also a linear isomorphism. □

Due to the above corollary, in a local orthonormal frame we have

$$\begin{aligned} A_{ij} \in \Omega^2_{14} &\iff A_{ip}\varphi_{pjk} + A_{jp}\varphi_{ipk} + A_{kp}\varphi_{ijp} = 0 \\ &\iff A_{ip}\psi_{pjkl} + A_{jp}\psi_{ipkl} + A_{kp}\psi_{ijpl} + A_{lp}\psi_{ijkp} = 0. \end{aligned} \tag{4.66}$$

Therefore, we have obtained the decompositions

$$\begin{aligned} \Omega^3 &= \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27}, \\ \Omega^4 &= \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27}. \end{aligned}$$

Moreover, using (4.56), we get the following explicit descriptions

$$\begin{aligned} \Omega^3_1 &= \{f\varphi \mid f \in \Omega^0\}, \quad \Omega^4_1 = \{f\psi \mid f \in \Omega^0\}, \\ \Omega^3_7 &= \{A \diamond \varphi \mid A \in \Omega^2_7\}, \quad \Omega^4_7 = \{A \diamond \psi \mid A \in \Omega^2_7\}, \\ \Omega^3_{27} &= \{A \diamond \varphi \mid A \in \mathcal{S}^2_0\}, \quad \Omega^4_{27} = \{A \diamond \psi \mid A \in \mathcal{S}^2_0\}. \end{aligned} \tag{4.67}$$

The next corollary gives us the inverses of the isomorphisms $\mathcal{S}^2 \oplus \Omega^2_7 \xrightarrow{\cong} \Omega^k$, where $k = 3$ or $k = 4$.

Corollary 4.4.4. *Let $\gamma \in \Omega^3$ and let $\eta \in \Omega^4$. Then, from above, $\gamma = A \diamond \varphi$ and $\eta = B \diamond \psi$ for some $A = \frac{1}{7}(\operatorname{tr} A)g + A_{27} + A_7$ and $B = \frac{1}{7}(\operatorname{tr} B)g + B_{27} + B_7$ in $\mathcal{S}^2 \oplus \Omega^2_7$. We define 2-tensors γ^φ and η^ψ as*

$$\gamma^\varphi_{ia} = \gamma_{ijk}\varphi_{ajk}, \quad \eta^\psi_{ia} = \eta_{ijkl}\psi_{ajkl}.$$

Then, we have

$$\operatorname{tr} A = \frac{1}{18}\operatorname{tr}\gamma^\varphi, \quad A_{27} = \frac{1}{4}\gamma^\varphi_{27}, \quad A_7 = \frac{1}{12}\gamma^\varphi_7, \tag{4.68}$$

and

$$\operatorname{tr} B = \frac{1}{96}\operatorname{tr}\eta^\psi, \quad B_{27} = \frac{1}{12}\eta^\psi_{27}, \quad B_7 = \frac{1}{36}\eta^\psi_7. \tag{4.69}$$

Proof. Let $C = \frac{1}{7}(\operatorname{tr} C)g + C_{27} + C_7 \in \mathcal{S}^2 \oplus \Omega^2_7$. From (4.63), we get

$$\langle A \diamond \varphi, C \diamond \varphi \rangle = \frac{54}{7}(\operatorname{tr} A)(\operatorname{tr} C) + 12\langle A_{27}, C_{27}\rangle + 36\langle A_7, C_7\rangle. \tag{4.70}$$

Then,

$$\begin{aligned}\langle \gamma, C \diamond \varphi \rangle &= \gamma_{ijk}(C_{ip}\varphi_{pjk} + C_{jp}\varphi_{ipk} + C_{kp}\varphi_{ijp}) \\ &= 3\gamma_{ijk}C_{ip}\varphi_{pjk} = 3\gamma^{\varphi}_{ip}C_{ip} = 3\langle \gamma^{\varphi}, C\rangle \\ &= 3\langle \frac{1}{7}(\operatorname{tr}\gamma^{\varphi})g + \gamma^{\varphi}_{27} + \gamma^{\varphi}_{7}), \frac{1}{7}(\operatorname{tr} C)g + C_{27} + C_7\rangle \\ &= \frac{3}{7}(\operatorname{tr}\gamma^{\varphi})(\operatorname{tr} C) + 3\langle \gamma^{\varphi}_{27}, C_{27}\rangle + 3\langle \gamma^{\varphi}_{7}, C_7\rangle.\end{aligned}$$

Comparing the two expressions, from nondegeneracy it follows

$$54 \operatorname{tr} A = 3 \operatorname{tr} \gamma^{\varphi}, \quad 12A_{27} = 3\gamma^{\varphi}_{27}, \quad 36A_7 = 3\gamma^{\varphi}_{7}. \tag{4.71}$$

Similarly, we get (4.69) from (4.64). □

Finally, we present an alternate way of expressing 3-forms involving vector fields and ψ.

Corollary 4.4.5. *For 3-forms $\gamma = X \lrcorner \psi$, where X is a vector field, we can define $A = -\frac{1}{3}X \lrcorner \varphi \in \Omega^2_7$ such that $\gamma = A \diamond \varphi$. This gives us $-3X \lrcorner \psi = (X \lrcorner \varphi) \diamond \varphi$.*

Proof. Since $\gamma_{ijk} = X_m\psi_{mijk}$, we have

$$\gamma^{\varphi}_{ia} = \gamma_{ijk}\varphi_{ajk} = X_m\psi_{mijk}\varphi_{ajk} = -4X_m\varphi_{mia}.$$

Thus, from Theorem 4.4.1, we have that $\gamma^{\varphi}_{ia} \in \Omega^3_7$ and hence from Corollary 4.4.4 it follows that $\gamma = A \diamond \varphi$ for $A = A_7 \in \Omega^2_7$ where

$$(A_7)_{ia} = \frac{1}{12}\gamma^{\varphi}_{ia} = -\frac{1}{3}X_m\varphi_{mia}.$$ □

4.5 Torsion of a G_2-structure

Recall that in section 4.3, we defined the torsion of a G_2-structure on a manifold M to be $\nabla\varphi$ where ∇ is the Levi-Civita connection on M. The next lemma gives us a way to express the torsion in an alternate way.

Lemma 4.5.1. *The 3-form $\nabla_X\varphi$ lies in Ω^3_7 for all vector fields X.*

Proof. From Corollary 4.4.4, it suffices to show that for $\gamma = \nabla_m \varphi$, we have that $\gamma^{\varphi}_{ia} = \gamma_{ijk}\varphi_{ajk}$ is skew-symmetric. Hence, using (4.27) we have

$$\begin{aligned}\gamma^{\varphi}_{ia} = \nabla_m \varphi_{ijk}\varphi_{ajk} &= \nabla_m(\varphi_{ijk}\varphi_{ajk}) - \varphi_{ijk}\nabla_m\varphi_{ajk} \\ &= \nabla_m(6g_{ia}) - \varphi_{ijk}\gamma_{ajk} \\ &= -\gamma^{\varphi}_{ai}.\end{aligned}$$

□

Therefore, from Corollary 4.4.5 we know that there exists a 2-tensor T such that

$$\nabla_m \varphi_{ijk} = T_{mp}\psi_{pijk} \tag{4.72}$$

and from now on, we will call this 2-tensor the **torsion** of the G_2-structure. From the contractions in Lemma 4.3.6, we get the expression

$$T_{pq} = \frac{1}{24}\nabla_p \varphi_{jkl}\psi_{qjkl} \tag{4.73}$$

which shows that $T = 0 \iff \nabla\varphi = 0$. That is, φ is torsion-free if and only if $T = 0$. Differentiating (4.26) and using (4.72) and (4.31), we get the following expression for $\nabla\psi$ in terms of T

$$\nabla_p \psi_{ijkl} = -T_{pi}\varphi_{jkl} + T_{pj}\varphi_{ikl} - T_{pk}\varphi_{ijl} + T_{pl}\varphi_{ijk}. \tag{4.74}$$

As the torsion lies in $\mathcal{T}^2$, from the decomposition (4.57), we get

$$T = T_1 + T_{27} + T_7 + T_{14} \tag{4.75}$$

where $T_1 = \frac{1}{7}(\operatorname{tr} T)g$. Furthermore, from (4.53) we get

$$\mathrm{P}T = -4T_7 + 2T_{14}. \tag{4.76}$$

Using all these new tools, we can present an alternate proof of the classical theorem by Fernández and Gray [FG82].

Proposition 4.5.2. *A G_2-structure φ on M is torsion-free if and only if $d\varphi = 0$ and $d^*\varphi = 0$*

Proof. From (4.72), we have

$$\begin{aligned}(d\varphi)_{ijkl} &= \nabla_i\varphi_{jkl} - \nabla_j\varphi_{ikl} + \nabla_k\varphi_{ijl} - \nabla_l\varphi_{ijk} \\ &= T_{ip}\psi_{pjkl} - T_{jp}\psi_{pikl} + T_{kp}\psi_{pijl} - T_{lp}\psi_{pijk} \\ &= T_{ip}\psi_{pjkl} + T_{jp}\psi_{ipkl} + T_{kp}\psi_{ijpl} + T_{lp}\psi_{ijkp} \\ &= (T \diamond \psi)_{ijkl}.\end{aligned}$$

Using the decomposition (4.75), as $T_{14} \diamond \psi = 0$ from Corollary 4.4.3, we have

$$d\varphi = (T_1 + T_{27} + T_7) \diamond \psi. \tag{4.77}$$

Similarly, using (4.72) again along with (4.76), we get

$$(d^*\varphi)_{jk} = -\nabla_i \varphi_{ijk} = -T_{im}\psi_{imjk} = -(\mathrm{P}T)_{jk} = 4T_7 - 2T_{14}. \tag{4.78}$$

The claim then follows from (4.77) and (4.78). □

Remark 4.5.3. When φ is torsion-free, the decomposition for k-forms $\Omega^k = \oplus_i \Omega^k_i$ induce a splitting of the harmonic k-forms $\mathcal{H}^k = \oplus_i \mathcal{H}^k_i$. That is, the projections π_i commute with the Hodge Laplacian $\Delta_d = d^*d + d^*d$. For more, see [Joy00, Section 3.5].

We conclude this section by giving an alternate way of packaging the torsion of a G_2-structure using the isomorphism $\Omega^1 \cong \Omega^2_7$ which we get through (4.47). That is, we define $\widehat{T} \in \Gamma(T^*M \otimes \Lambda^2_7(T^*M))$ as

$$\widehat{T}_{pij} = T_{pq}\varphi_{qij}, \quad T_{pq} = \frac{1}{6}\widehat{T}_{pij}\varphi_{qij}. \tag{4.79}$$

For fixed p, $\widehat{T}_{pij}$ lies in Ω^2_7 in i, j. Therefore, from (4.44) we have

$$\widehat{T}_{pij}\psi_{ijkl} = -4\widehat{T}_{pkl}. \tag{4.80}$$

Via the pairing $(\widehat{T}(X))_{ij} = X_p \widehat{T}_{pij}$, we can think of $\widehat{T}$ as a 1-form on M with values in $\Lambda^2_7(T^*M)$. The following lemma gives us a way to express $\nabla\varphi$ and $\nabla\psi$ in terms of $\widehat{T}$.

Lemma 4.5.4. *Let us fix $p \in \{1, \ldots, 7\}$. At the point $x \in M$, we can write $\widehat{T}_p = \widehat{T}_{pij} e_i \otimes e_j$ as an element of $\Lambda^2_7(T^*_x M)$. Then, we get*

$$\nabla_p \varphi_{abc} = -\frac{1}{3}(\widehat{T}_p \diamond \varphi)_{abc}, \quad \nabla_p \psi_{abcd} = -\frac{1}{3}(\widehat{T}_p \diamond \psi)_{abcd}. \tag{4.81}$$

Proof. From (4.72) and (4.79), we obtain

$$\begin{aligned}\nabla_p \varphi_{abc} = T_{pq}\psi_{qabc} &= -\frac{1}{6}\widehat{T}_{pij}\varphi_{ijq}\psi_{abcq} \\ &= -\frac{1}{6}\widehat{T}_{pij}(g_{ia}\varphi_{jbc} + g_{ib}\varphi_{ajc} + g_{ic}\varphi_{abj} - g_{ja}\varphi_{ibc} - g_{jb}\varphi_{aic} - g_{jc}\varphi_{abi}).\end{aligned}$$

As $\widehat{T}_{pij}$ is skew in i, j, the above becomes

$$\begin{aligned}\nabla_p \varphi_{abc} &= -\frac{1}{3}\widehat{T}_{pij}(g_{ia}\varphi_{jbc} + g_{ib}\varphi_{ajc} + g_{ic}\varphi_{abj}) \\ &= -\frac{1}{3}(\widehat{T}_{paj}\varphi_{jbc} + \widehat{T}_{pbj}\varphi_{ajc} + \widehat{T}_{pcj}\varphi_{abj}),\end{aligned}$$

which proves the first claim. The second formula can be derived by differentiating the identity in (4.28) and then using the formula we derived for $\nabla_p \varphi_{abc}$ and (4.28) again. □

4.6 The G_2-Bianchi identity

In this section we will prove the G_2-Bianchi identity, which gives us a relation between a G_2-structure φ, the Riemann curvature Rm of g_φ, the torision T of φ and its covariant derivative ∇T.

Proposition 4.6.1. *The G_2-Bianchi identity is given as*

$$\nabla_i T_{jk} - \nabla_j T_{ik} = T_{ip}T_{jq}\varphi_{pqk} + \frac{1}{2}R_{ijpq}\varphi_{pqk}. \tag{4.82}$$

Proof. Taking the covariant derivative of (4.73) and substituting (4.74), we get

$$\begin{aligned}\nabla_m \nabla_p \varphi_{ijk} &= \nabla_m T_{pq}\psi_{qijk} + T_{pq}\nabla_m \psi_{qijk} \\ &= \nabla_m T_{pq}\psi_{qijk} + T_{pq}(-T_{mq}\varphi_{ijk} + T_{mi}\varphi_{qjk} - T_{mj}\varphi_{qik} + T_{mk}\varphi_{qij}).\end{aligned}$$

Let us interchange p and m and take the difference. Then, using the fact that $T_{pq}T_{mq}$ is symmetric in p, m, we obtain

$$\begin{aligned}\nabla_m \nabla_p \varphi_{ijk} - \nabla_p \nabla_m \varphi_{ijk} &= (\nabla_m T_{pq} - \nabla_p T_{mq})\psi_{qijk} \\ &\quad + T_{pq}(T_{mi}\varphi_{qjk} - T_{mj}\varphi_{qik} + T_{mk}\varphi_{qij}) \\ &\quad - T_{mq}(T_{pi}\varphi_{qjk} - T_{pj}\varphi_{qik} + T_{pk}\varphi_{qij}).\end{aligned}$$

Using the Ricci identity above, we have

$$\begin{aligned}-R_{mpiq}\varphi_{qjk} - R_{mpjq}\varphi_{iqk} - R_{mpkq}\varphi_{ijq} &= (\nabla_m T_{pq} - \nabla_p T_{mq})\psi_{qijk} \\ &\quad + T_{pq}(T_{mi}\varphi_{qjk} - T_{mj}\varphi_{qik} + T_{mk}\varphi_{qij}) \\ &\quad - T_{mq}(T_{pi}\varphi_{qjk} - T_{pj}\varphi_{qik} + T_{pk}\varphi_{qij}).\end{aligned}$$

Since the left hand side and each of three terms on the right hand side are totally skew in i, j, k, contracting on both sides by ψ_{lijk} yields

$$-3R_{mpiq}\varphi_{qjk}\psi_{lijk} = (\nabla_m T_{pq} - \nabla_p T_{mq})\psi_{qijk}\psi_{lijk} + 3T_{pq}T_{mi}\varphi_{qjk}\psi_{lijk} - 3T_{mq}T_{pi}\varphi_{qjk}\psi_{lijk}.$$

Using the contraction identities on the above expression, we have

$$12R_{mpiq}\varphi_{qli} = 24(\nabla_m T_{pl} - \nabla_p T_{ml}) - 12T_{pq}T_{mi}\varphi_{qli} + 12T_{mq}T_{pi}\varphi_{qli}. \tag{4.83}$$

Rearranging and reindexing the above expression, we get (4.82). □

Chapter 5

The family of compact torsion-free G_2-structures

In Chapter 4, we got familiar with various properties of G_2-structures. It is then natural to wonder what properties would the moduli space of these structures, that is, the "set of all possible G_2-structures", modulo a reasonable notion of equivalence, satisfy. The aim of this chapter is to present a proof given in [Joy00] which shows that the moduli space of torsion-free G_2-structures for a compact 7-manifold forms a non-singular smooth manifold. We will prove some technical results in the first section using techniques from [Kar08] and [DGK23], before moving on to prove the main theorem in the second section.

5.1 Computations with differential forms on G_2-manifolds

In this section, we will carry out some useful computations with differential forms on a G_2-manifold which we will need in the next section. Let us start with the following lemma.

Lemma 5.1.1. *Let (M, φ) be a compact G_2-manifold and η a 2-form on M. Then we have*

$$\pi_7(d^* d\eta) = 0 \iff d^*\pi_1(d\eta) = d^*\pi_7(d\eta) = 0. \tag{5.1}$$

Proof. ($\Rightarrow$) Let $\pi_7(d^* d\eta) = 0$. From Hodge theory, we know that there exists a unique d^*-exact 2-form ξ on M such that $d\xi = d\eta$. Then, as $d^*\xi = 0$, we get that

$$\pi_7((dd^* + d^*d)\xi) = \pi_7(d^*d\eta) = 0.$$

From Remark 4.5.3, we get

$$(dd^* + d^*d)(\pi_7(\xi)) = 0$$

and hence $\pi_7(\xi) \in \mathcal{H}^2_7$. As ξ is d^*-exact, it is L^2-orthogonal to $\mathcal{H}^2_7$ from the Hodge decomposition theorem (Theorem 1.2.1), which means that $\pi_7(\xi) = 0$. Thus, from (4.46), we know that $\xi \wedge \psi = 0$ and as $d\psi = 0$ since φ is torsion-free, we have that $d\xi \wedge \psi = 0$. Furthermore, using the Hodge isomorphism and the description of Ω^3_1 in (4.67), we have

$$d\xi \wedge \psi \cong \pi_1(d\xi).$$

This along with the fact that $d\xi = d\eta$ shows that $\pi_1(d\eta) = 0$.

From (4.40) and (4.45), it follows that $\xi \wedge \varphi = -\star\xi$ and as ξ is d^*-exact, we have $d\star\xi = 0$. Thus, as $d\varphi = 0$, we get $d\xi \wedge \varphi = 0$. Finally, as $d\eta = d\xi$ and as $d\eta \wedge \varphi \cong \pi_7(d\eta)$ from Corollary 4.4.5, we have $\pi_7(d\eta) = 0$. Therefore, we have $d^*\pi_1(d\eta) = d^*\pi_7(d\eta) = 0$ as desired.

($\Leftarrow$) The converse follows from

$$\pi_7(d^*d\eta) = c_1\pi_7(d^*\pi_1(d\eta)) + c_2\pi_7(d^*\pi_7(d\eta)),$$

which appears in [Joy00] without proof. We will now prove that $\pi_7(d^*d\eta)$ can be expressed as a linear combination of $\pi_7(d^*\pi_1(d\eta))$ and $\pi_7(d^*\pi_7(d\eta))$ and find the explicit values for c_1 and c_2 in the above statement. From (4.67), we know that $\pi_1(d\eta) = f\varphi$ for some $f \in \Omega^0$. Then, from (4.26), we have

$$\langle \pi_1(d\eta), \varphi \rangle = \langle f\varphi, \varphi \rangle = 42f.$$

In addition, from (4.56) and (4.63) we obtain

$$\begin{aligned}\langle \pi_1(d\eta), \varphi \rangle = \langle d\eta, \varphi \rangle = (d\eta)_{ijk}\varphi_{ijk} &= (\nabla_i\eta_{jk} + \nabla_j\eta_{ki} + \nabla_k\eta_{ij})\varphi_{ijk} \\ &= 3\nabla_i\eta_{jk}\varphi_{ijk}.\end{aligned}$$

Thus, we have that

$$f = \frac{1}{14}(\nabla_i\eta_{jk})\varphi_{ijk}.$$

This gives us

$$\begin{aligned}(d^*(\pi_1 d\eta))_{ab} = -\nabla_p(\pi_1(d\eta))_{pab} = -\nabla_p(f\varphi_{pab}) &= -(\nabla_p f)\varphi_{pab} \\ &= -\frac{1}{14}(\nabla_p\nabla_i\eta_{jk})\varphi_{ijk}\varphi_{pab},\end{aligned}$$

which when contracted with φ on two indices and using (4.27) yields

$$(d^*(\pi_1(d\eta)))_{ab}\varphi_{abm} = -\frac{1}{14}(\nabla_p\nabla_i\eta_{jk})\varphi_{ijk}\varphi_{pab}\varphi_{abm} = -\frac{3}{7}(\nabla_m\nabla_i\eta_{jk})\varphi_{ijk}. \tag{5.2}$$

From Corollary 4.4.5, it follows that there exists $Y \in \mathfrak{X}$ such that $\pi_7(d\eta) = Y \lrcorner \psi$ and hence from (4.33),

$$\langle \pi_7(d\eta), e_p \lrcorner \psi \rangle = \langle Y \lrcorner \psi, e_p \lrcorner \psi \rangle = Y_i\psi_{iabc}\psi_{pabc} = 24Y_p.$$

But we also have that

$$\langle \pi_7(d\eta), e_p \lrcorner \psi \rangle = \langle d\eta, e_p \lrcorner \psi \rangle = (\nabla_i\eta_{jk} + \nabla_j\eta_{ki} + \nabla_k\eta_{ij})\psi_{pijk} = 3\nabla_i\eta_{jk}\psi_{pijk}.$$

Therefore,

$$Y_p = \frac{1}{8}\nabla_i\eta_{jk}\psi_{pijk}.$$

Substituting the above into the original equation, we have

$$(\pi_7(d\eta))_{qab} = Y_p\psi_{pqab} = \frac{1}{8}(\nabla_i\eta_{jk})\psi_{pijk}\psi_{pqab},$$

and thus

$$(d^*(\pi_7(d\eta)))_{ab} = -\nabla_q(\pi_7(d\eta))_{qab} = -\frac{1}{8}(\nabla_q\nabla_i\eta_{jk})\psi_{pijk}\psi_{pqab}.$$

Then, we have from (4.30) and (4.31) that

$$\begin{aligned}(d^*(\pi_7(d\eta)))_{ab}\varphi_{abm} &= -\frac{1}{8}(\nabla_q\nabla_i\eta_{jk})\psi_{pijk}\psi_{pqab}\varphi_{abm} \\ &= \frac{1}{2}(\nabla_q\nabla_i\eta_{jk})\varphi_{mqp}\psi_{ijkp} \\ &= \frac{1}{2}\nabla_q\nabla_i\eta_{jk}(g_{mi}\varphi_{qjk} + g_{mj}\varphi_{iqk} + g_{mk}\varphi_{ijq} \\ &\quad - g_{qi}\varphi_{mjk} - g_{qj}\varphi_{imk} - g_{qk}\varphi_{ijm}) \\ &= \frac{1}{2}(\nabla_q\nabla_m\eta_{jk})\varphi_{qjk} - \frac{1}{2}(\nabla_i\nabla_i\eta_{jk})\varphi_{mjk} \\ &\quad + (\nabla_q\nabla_i\eta_{mk})\varphi_{iqk} - (\nabla_j\nabla_i\eta_{jk})\varphi_{imk}.\end{aligned} \tag{5.3}$$

Now, we have

$$(d^*d\eta)_{ab} = -\nabla_p(d\eta)_{pab} = -\nabla_p(\nabla_p\eta_{ab} + \nabla_a\eta_{bp} + \nabla_b\eta_{pa}),$$

which gives us

$$(d^*d\eta)_{ab}\varphi_{abm} = -\nabla_p\nabla_p\eta_{ab}\varphi_{abm} - 2\nabla_p\nabla_a\eta_{bp}\varphi_{abm}. \tag{5.4}$$

To eliminate the Laplacian term, let us compute (5.4) $-$ 2(5.3). We obtain

$$\begin{aligned}(d^*d\eta)_{ab}\varphi_{abm} - 2(d^*(\pi_7(d\eta)))_{ab}\varphi_{abm} &= -\nabla_p\nabla_p\eta_{ab}\varphi_{abm} - 2\nabla_p\nabla_a\eta_{bp}\varphi_{abm} \\ &\quad - \nabla_q\nabla_m\eta_{jk}\varphi_{qjk} + \nabla_i\nabla_i\eta_{jk}\varphi_{mjk} \\ &\quad - 2\nabla_q\nabla_i\eta_{mk}\varphi_{iqk} + 2\nabla_j\nabla_i\eta_{jk}\varphi_{imk} \\ &= -\nabla_q\nabla_m\eta_{jk}\varphi_{qjk} - (\nabla_q\nabla_i - \nabla_i\nabla_q)\eta_{mk}\varphi_{iqk} \\ &= -\nabla_q\nabla_m\eta_{jk}\varphi_{qjk} - (-R_{qiml}\eta_{lk} - R_{qikl}\eta_{ml})\varphi_{iqk} \\ &= -\nabla_q\nabla_m\eta_{jk}\varphi_{qjk} + R_{qiml}\eta_{lk}\varphi_{iqk} = -\nabla_q\nabla_m\eta_{jk}\varphi_{qjk},\end{aligned}$$

where the second to last equality follows from the Bianchi identity as

$$R_{qikl} + R_{ikql} + R_{kqil} = 0 \implies 3R_{qikl}\varphi_{iqk} = 0$$

and the last equality follows from the G_2-Bianchi identity (4.82) as φ is torsion-free. Then, computing (5.4) $-$ 2(5.3) $-\frac{7}{3}$(5.2), we have

$$\begin{aligned}(d^*d\eta)_{ab}\varphi_{abm} - 2(d^*(\pi_7(d\eta)))_{ab} - \frac{7}{3}(d^*(\pi_1(d\eta)))_{ab}\varphi_{abm} &= -\nabla_q\nabla_m\eta_{jk}\varphi_{qjk} + \nabla_m\nabla_q\eta_{jk}\varphi_{qjk} \\ &= -(\nabla_q\nabla_m - \nabla_m\nabla_q)\eta_{jk}\varphi_{qjk} \\ &= -(-R_{qmjl}\eta_{lk} - R_{qmkl}\eta_{jl})\varphi_{qjk}.\end{aligned}$$

But note that from (4.82),

$$\begin{aligned}-(-R_{qmjl}\eta_{lk} - R_{qmkl}\eta_{jl})\varphi_{qjk} = 2R_{qmjl}\eta_{lk}\varphi_{qjk} = 2(-R_{mjql} - R_{jqml})\eta_{lk}\varphi_{qjk} &= -2R_{mjql}\eta_{lk}\varphi_{qjk} \\ &= -2R_{mqjl}\eta_{lk}\varphi_{jqk} \\ &= -2R_{qmjl}\eta_{lk}\varphi_{qjk}.\end{aligned}$$

Therefore, as $2R_{qmjl}\eta_{lk}\varphi_{qjk} = -2R_{qmjl}\eta_{lk}\varphi_{qjk} = 0$, we have from (4.51)

$$\pi_7(d^*d\eta) = 2\pi_7(d^*\pi_7(d\eta)) + \frac{7}{3}\pi_7(d^*\pi_1(d\eta)). \tag{5.5}$$

Thus, if $d^*\pi_1(d\eta) = d^*\pi_7(d\eta) = 0$, we have $\pi_7(d^*d\eta) = 0$. □

For the rest of the results in this section, we will need G_2-structures $\widetilde{\varphi}$ close to a given G_2-structure φ in a specific way, which we will describe in the following definition.

Definition 5.1.2. Let $\epsilon_1 > 0$ be a universal constant such that whenever φ is a G_2-structure on a 7-manifold M, then

(i) If $\widetilde{\varphi} \in \Omega^3(M)$ and $\|\widetilde{\varphi} - \varphi\|_{C^0} \leq \epsilon_1$, then we have $\widetilde{\varphi} \in \Gamma(\mathscr{P}^3 M)$. That is, $\widetilde{\varphi}$ is a G_2-structure on M. Let us denote the corresponding splitting on 5-forms as $\Omega^5 \cong \widetilde{\Omega}^5_7 \oplus \widetilde{\Omega}^5_{14}$. Then, for each $\xi \in \widetilde{\Omega}^5_{14}$, we have $|\pi_7(\xi)| \leq |\pi_{14}(\xi)|$, where π_7, π_{14} and $|\cdot|$ are with respect to φ.

(ii) If $\chi \in \Omega^4(M)$ and $\|\chi - \psi\|_{C^0} \leq \epsilon_1$, then for all 1-forms λ on M, we have $|\pi_{14}(\lambda \wedge \chi)| \leq \frac{1}{4}|\pi_7(\lambda \wedge \chi)|$.

If we take ϵ_1 to be sufficiently small, both the conditions hold. Intuitively, the first condition tells us that when ϵ_1 is small, $\widetilde{\Omega}^2_{14}$ is close to Ω^2_{14}, so if $\xi \in \widetilde{\Omega}^2_{14}$ then $\pi_7(\xi)$ is small compared to $\pi_{14}(\xi)$. And the second condition tells that since χ is close to ψ and as $\lambda \wedge \psi \in \Omega^5_7$ from taking the Hodge star of (4.44), we have that $\lambda \wedge \chi$ is close to Ω^5_7 and hence $\pi_{14}(\lambda \wedge \chi)$ is small compared to $\pi_7(\lambda \wedge \chi)$.

Note that the Hodge star $\star$ depends on the metric g which itself depends on the G_2-structure φ. To emphasize this point, we define a map $\Theta : \mathscr{P}^3 M \to \mathscr{P}^4 M$ by

$$\Theta(\varphi) = \star_\varphi \varphi = \psi. \tag{5.6}$$

Notice that Θ depends solely on M and its orientation and it is a non-linear map. Henceforth, when we write $\widetilde{\psi}$, we mean that it is the 4-form associated to $\widetilde{\varphi}$. That is, $\widetilde{\psi} = \star_{\widetilde{\varphi}} \widetilde{\varphi}$.

The following proposition says if $\widetilde{\varphi}$ is a 3-form close to a closed G_2-structure φ in the sense of Definition 5.1.2, then given that $\widetilde{\varphi}$ satisfies some additional conditions, $\widetilde{\varphi}$ defines a torsion-free G_2-structure on M.

Proposition 5.1.3. *Let $\epsilon_1 > 0$ be as in Definition 5.1.2 and let M be a compact 7-manifold. Let φ be a G_2-structure, f a real function, α a 1-form, $\widetilde{\varphi}$ a 3-form and χ a 4-form on M satisfying $\|\widetilde{\varphi} - \varphi\|_{C^0} \leq \epsilon_1, \|\chi - \psi\|_{C^0} \leq \epsilon_1$ and the equations*

$$d\varphi = d\widetilde{\varphi} = d\chi = 0 \text{ and } d\widetilde{\psi} = df \wedge \chi + d\alpha \wedge \varphi. \tag{5.7}$$

Then, we have $d\psi = 0, df = 0$ and $d\alpha = 0$. Therefore, $\widetilde{\varphi}$ is a torsion-free G_2-structure on M.

Proof. Let us define $x_7, y_7, z_7 \in \Omega^5_7$ and $x_{14}, y_{14}, z_{14} \in \Omega^5_{14}$ as

$$\begin{aligned} x_7 = \pi_7(d\widetilde{\psi}), \quad y_7 = \pi_7(df \wedge \chi), \quad z_7 = \pi_7(d\alpha \wedge \varphi), \\ x_{14} = \pi_{14}(d\widetilde{\psi}), \quad y_{14} = \pi_{14}(df \wedge \chi), \quad z_{14} = \pi_{14}(d\alpha \wedge \varphi). \end{aligned} \tag{5.8}$$

Then, taking π_7 and π_{14} of the second equation in (5.7), we have

$$x_7 = y_7 + z_7 \text{ and } x_{14} = y_{14} + z_{14}. \tag{5.9}$$

As $d\widetilde{\varphi} = 0$, from (4.77) and (4.78), it follows that $d\widetilde{\psi} \in \widetilde{\Omega}^5_{14}$. Thus, from (5.8) and (i) of Definition 5.1.2, we get that $|x_7| \leq |x_{14}|$. In addition, making the substitution $\lambda = df$ in (ii) of Definition 5.1.2, from (5.8), we obtain $|y_{14}| \leq \frac{1}{4}|y_7|$. Squaring these two inequalities and integrating over M yields the following two L^2-norm inequalities

$$\|x_7\|_{L^2} \leq \|x_{14}\|_{L^2} \text{ and } \|y_{14}\|_{L^2} \leq \frac{1}{4}\|y_7\|_{L^2}. \tag{5.10}$$

Since φ is closed, $d\alpha \wedge d\alpha \wedge \varphi$ is exact and thus $\int_M d\alpha \wedge d\alpha \wedge \varphi = 0$ by Stokes' Theorem. Then, from (4.40), (4.43) and (4.45) we get

$$\pi_7(d\alpha) = -\frac{1}{2} \star z_7 \text{ and } \pi_{14}(d\alpha) = \star z_{14}$$

and

$$d\alpha \wedge d\alpha \wedge \varphi = (-2|\pi_7(d\alpha)|^2 + |\pi_{14}(d\alpha)|^2)\, \mathrm{vol}_g,$$

from which we obtain

$$d\alpha \wedge d\alpha \wedge \varphi = \left(-\frac{1}{2}|z_7|^2 + |z_{14}|^2\right) \mathrm{vol}_g .$$

Thus integrating over M gives us $-\frac{1}{2}\|z_7\|^2_{L^2} + \|z_{14}\|^2_{L^2} = 0.$ and hence

$$\|z_7\|_{L^2} = \sqrt{2}\|z_{14}\|_{L^2}. \tag{5.11}$$

Similarly, we have

$$\int_M d\widetilde{\psi} \wedge d\alpha = \int_M (x_7 + x_{14}) \wedge (-\frac{1}{2} \star z_7 + \star z_{14}) = 0$$

which gives us

$$\langle x_7, z_7\rangle_{L^2} = 2\langle x_{14}, z_{14}\rangle_{L^2}. \tag{5.12}$$

Thus, from (5.9), (5.10), (5.11) and (5.12), we have

$$\begin{aligned} \|x_{14} - z_{14}\|_{L^2} = \|y_{14}\|_{L^2} \leq \frac{1}{4}\|y_7\|_{L^2} &\leq \frac{1}{4}(\|x_7\|_{L^2} + \|z_7\|_{L^2}) \\ &\leq \frac{\sqrt{2}}{4}(\|x_{14}\|_{L^2} + \|z_{14}\|_{L^2}). \end{aligned} \tag{5.13}$$

Squaring the above and using Cauchy-Schwarz, we have

$$\begin{aligned} \|x_{14}\|_{L^2}^2 + \|z_{14}\|_{L^2}^2 - 2\langle x_{14}, z_{14}\rangle_{L^2} &\leq \frac{2}{16}(\|x_{14}\|_{L^2}^2 + \|z_{14}\|_{L^2}^2 + 2\|x_{14}\|_{L^2}\|z_{14}\|_{L^2}) \\ \implies 14\|x_{14}\|_{L^2}^2 + 14\|z_{14}\|_{L^2}^2 &\leq 32\langle x_{14}, z_{14}\rangle_{L^2} + 4\|x_{14}\|_{L^2}\|z_{14}\|_{L^2} \\ \implies \|x_{14}\|_{L^2}^2 + \|z_{14}\|_{L^2}^2 &\leq \frac{18}{7}\|x_{14}\|_{L^2}\|z_{14}\|_{L^2}. \end{aligned}$$

Then, since $\|x_{14}\|_{L^2}^2 + \|z_{14}\|_{L^2}^2 \geq 2\|x_{14}\|_{L^2}\|z_{14}\|_{L^2}$, we get

$$\begin{aligned} 2\|x_{14}\|_{L^2}\|z_{14}\|_{L^2} - 2\langle x_{14}, z_{14}\rangle_{L^2} &\leq \frac{2}{16}(\|x_{14}\|_{L^2}^2 + \|z_{14}\|_{L^2}^2 + 2\|x_{14}\|_{L^2}\|z_{14}\|_{L^2}) \\ &\leq \frac{2}{16}\left(\frac{18}{7}\|x_{14}\|_{L^2}\|z_{14}\|_{L^2} + 2\|x_{14}\|_{L^2}\|z_{14}\|_{L^2}\right) \\ &= \frac{4}{7}\|x_{14}\|_{L^2}\|z_{14}\|_{L^2} \\ \implies \langle x_{14}, z_{14}\rangle_{L^2} &\geq \frac{5}{7}\|x_{14}\|_{L^2}\|z_{14}\|_{L^2}. \end{aligned}$$

Hence, (5.10), (5.11) and (5.12) give us

$$\|x_7\|_{L^2}\|z_7\|_{L^2} \geq \langle x_7, z_7\rangle_{L^2} = 2\langle x_{14}, z_{14}\rangle_{L^2} \geq \frac{10}{7}\|x_{14}\|_{L^2}\|z_{14}\|_{L^2} \geq \frac{5\sqrt{2}}{7}\|x_7\|_{L^2}\|z_7\|_{L^2}.$$

As $\frac{5\sqrt{2}}{7} > 1$, we must have $\|x_7\|_{L^2} = 0$ or $\|z_7\|_{L^2} = 0$. Thus, $x_7 = 0$ or $z_7 = 0$.

If $z_7 = 0$ then from (5.11) we have $z_{14} = 0$, which will give us $x_7 = y_7$ and $x_{14} = y_{14}$. But as

$$\|x_7\|_{L^2}^2 \leq \|x_{14}\|_{L^2}^2 = \|y_{14}\|_{L^2}^2 \leq \frac{1}{4}\|y_7\|_{L^2}^2 = \frac{1}{4}\|x_7\|_{L^2}^2,$$

we have $\|x_7\|_{L^2} = 0$ and hence $x_7 = x_{14} = y_7 = y_{14} = 0$.

If $x_7 = 0$, then $y_7 = -z_7$ and thus we have $\|y_{14}\|_{L^2} \leq \frac{1}{4}\|y_7\|_{L^2} = \frac{1}{4}\|z_7\|_{L^2} = \frac{\sqrt{2}}{4}\|z_{14}\|_{L^2}$. If $z_{14} \neq 0$, then we have $\|y_{14}\|_{L^2} < \|z_{14}\|_{L^2}$ so that

$$\langle x_{14}, z_{14}\rangle = \langle y_{14}, z_{14}\rangle + \|z_{14}\|^2 \geq -\|z_{14}\|_{L^2}\|y_{14}\|_{L^2} + \|z_{14}\|_{L^2}^2 > 0$$

since $x_{14} = y_{14} + z_{14}$. But this gives us $\langle x_7, z_7\rangle_{L^2} > 0$, which contradicts $x_7 = 0$. Therefore, we get $z_{14} = 0$ which gives us $z_7 = 0$ as before. Thus, in both cases we have $x_7 = y_7 = z_7 = 0$ and $x_{14} = y_{14} = z_{14} = 0$, which gives $d\widetilde{\psi} = df = d\alpha = 0$ as claimed. □

Now, we give a way to estimate the function Θ in (5.6).

Proposition 5.1.4. *Let ϵ_1 be as in Definition 5.1.2. Suppose M is a 7-manifold and φ is a G_2-structure on M. Let $\chi \in \Omega^3(M)$ with $|\chi| < \epsilon_1$. Then $\varphi + \chi \in \Gamma(\mathscr{P}^3 M)$ and $\Theta(\varphi + \chi)$ is given by*

$$\begin{aligned}\Theta(\varphi + \chi) &= \psi + \frac{4}{3} \star \pi_1(\chi) + \star\pi_7(\chi) - \star\pi_{27}(\chi) - F(\chi) \\ &= \psi + \frac{7}{3} \star \pi_1(\chi) + 2 \star \pi_7(\chi) - \star\chi - F(\chi),\end{aligned} \tag{5.14}$$

where F is a smooth function from the closed ball of radius ϵ_1 in $\Lambda^3 T^*M$ to $\Lambda^4 T^*M$ with $F(0) = 0$.

Proof. Since χ is a small 3-form we can think of computing the Taylor expansion of $\Theta(\varphi+\chi)$ which we can see as expanding $G(t) = \Theta(\varphi + t\eta)$, where $\chi = t\eta$ for some 3-form η, about $t = 0$. Thus,

$$G(t) = G(0) + tG'(0) - F(\chi),$$

where $F(\chi)$ represents the remainder of the terms in the Taylor expansion and hence its principal part is quadratic in χ. Thus, F is a smooth function on the closed ball of radius ϵ_1 in $\Lambda^3 T^*M$ with $F(0) = 0$. Now, note that $G(0) = \Theta(\varphi) = \psi$ and

$$\left.\frac{d}{dt}\right|_{t=0} (\varphi + t\eta) = \eta = \pi_1(\eta) + \pi_7(\eta) + \pi_{27}(\eta).$$

Then, from [Kar08, Remark 3.6], we know that

$$G'(0) = \left.\frac{d}{dt}\right|_{t=0} \Theta(\varphi + t\eta) = \frac{4}{3} \star \pi_1(\eta) + \star\pi_7(\eta) - \star\pi_{27}(\eta).$$

Thus, we have

$$\begin{aligned}\Theta(\varphi+\chi) &= \psi + \frac{4}{3}\star\pi_1(t\eta) + \star\pi_7(t\eta) - \star\pi_{27}(t\eta) - F(\chi) \\ &= \psi + \frac{7}{3}\star\pi_1(\chi) + 2\star\pi_7(\chi) - \star\chi - F(\chi).\end{aligned}$$ □

Now, we will use the technical results above to prove the main theorem of this section. The theorem shows that the torsion-free conditions for $\widetilde{\varphi}$, given as $d\widetilde{\varphi} = d\widetilde{\psi} = 0$, along with the "gauge-fixing" condition $\pi_7(d^*\widetilde{\varphi}) = 0$, are equivalent to the equation $(dd^* + d^*d)\eta = \star d(F(\xi + d\eta))$, which is a nonlinear elliptic PDE upon the 2-form η. We will use this theorem to study the family of torsion-free G_2-structures on a compact 7-manifold in the next section.

Theorem 5.1.5. *Let (M, φ) be a compact G_2-manifold. Then, let $\xi \in \mathcal{H}^3$ and η be a 2-form on M such that $\|\xi + d\eta\|_{C^0} \leq \epsilon_1$. Let $\widetilde{\varphi} = \varphi + \xi + d\eta$. Note that $d\widetilde{\varphi} = 0$. Then, we have*

$$(dd^* + d^*d)\eta = \star d(F(\xi + d\eta)) \iff d^*\eta = \pi_7(d^*\widetilde{\varphi}) = d\widetilde{\psi} = 0. \tag{5.15}$$

Proof. Let $(dd^*+d^*d)\eta = \star d(F(\xi+d\eta))$. As M is compact, from the Hodge decomposition theorem (Theorem 1.2.1) we know that $\operatorname{Im} d$ and $\operatorname{Im} d^*$ are L^2-orthogonal. Therefore, as $d^* = -\star d \star$ on Λ^3T^*M, we have $dd^*\eta = -d^*d\eta - d^*(\star F(\xi + d\eta))$, which means that we must have $dd^*\eta = 0$ and hence $d^*\eta = 0$. Thus, $d^*d\eta = \star d(F(\xi+d\eta))$, from which we obtain

$$d \star d\eta + d(F(\xi + d\eta)) = 0. \tag{5.16}$$

Substituting $\chi = \xi + d\eta$ in (5.14), as $\xi \in \mathcal{H}^3$, we have that $\star\pi_1(\xi), \star\pi_7(\xi)$ and $\star\pi_{27}(\xi)$ are closed, which along with (5.16) and the fact that the projections commute with Δ_d, gives us

$$\begin{aligned}d\widetilde{\psi} &= \frac{7}{3}d\star\pi_1(d\eta) + 2d\star\pi_7(d\eta) - d\star d\eta - d(F(\xi + d\eta)) \\ &= \frac{7}{3}d\star\pi_1(d\eta) + 2d\star\pi_7(d\eta).\end{aligned} \tag{5.17}$$

Taking f and α such that $f\varphi = \frac{7}{3}\pi_1(d\eta)$ and $\alpha \wedge \varphi = *(2\pi_7(d\eta))$, which we know exist from the descriptions in Corollary 4.4.5 and (4.67), in Proposition 5.1.3, we get

$$d\widetilde{\psi} = d\star\pi_1(d\eta) = d\star\pi_7(d\eta) = 0. \tag{5.18}$$

Therefore, $d^*\pi_1(d\eta) = d^*\pi_7(d\eta) = 0$, which implies that $\pi_7(d^*d\eta) = 0$ from Lemma 5.1.1. As $d^*d\eta = d^*\widetilde{\varphi}$ from the statement of the theorem, this finishes the proof of the forward direction.

Conversely, suppose $d^*\eta = \pi_7(d^*\widetilde{\varphi}) = d\widetilde{\psi} = 0$. Then, from Lemma 5.1.1, we have that $d^*\pi_1(d\eta) = d^*\pi_7(d\eta) = 0$ which gives us

$$d\widetilde{\psi} = \frac{7}{3} d \star \pi_1(d\eta) + 2d \star \pi_7(d\eta) = 0$$

and thus from the first line of (5.17) we have that $d \star d\eta + dF(d\eta) = 0$. As $d^*\eta = 0$, this gives us $(dd^* + d^*d)\eta = \star d(F(\xi + d\eta))$ as desired. □

5.2 The moduli space of compact torsion-free G_2-structures

Let M be a compact, oriented 7-manifold. Then, let $\mathcal{X}$ be the set of positive 3-forms on M which correspond to torsion-free G_2-structures. That is,

$$\mathcal{X} = \{\varphi \in \Gamma(\mathscr{P}^3 M) : d\varphi = d\psi = 0\}. \tag{5.19}$$

Let $\mathcal{D}$ be the group of all diffeomorphisms Ψ which are isotopic to the identity. That is, each Ψ is connected to the identity map on M by a continuous path on the space Diff of diffeomorphisms of M. Thus, $\mathcal{D}$ is the connected component of the identity in Diff. Then, we have a natural action of $\mathcal{D}$ on $\Gamma(\mathscr{P}^3 M)$ and $\mathcal{X}$ given by

$$\varphi \overset{\Psi}{\mapsto} \Psi^*(\varphi).$$

The reason why we consider $\mathcal{D}$ instead of the entire space Diff is because it acts trivially on cohomology. To see this, let $[\alpha] \in H^k(M, \mathbb{R})$ and let $\Psi \in \mathcal{D}$. We then claim that $[\Psi^*\alpha] = [\alpha]$. Let Ψ_t be a continuous path in Diff with $\Psi_0 = \mathrm{Id}_M$ and $\Psi_1 = \Psi$ given by the flow of the vector field X_t on M. As α is a closed form, we get

$$\begin{aligned}\Psi^*\alpha - \alpha = \int_0^1 \frac{d}{dt}(\Psi_t^*\alpha) = \int_0^1 \mathcal{L}_{X_t}\alpha = \int_0^1 (d(X_t \lrcorner \alpha) + X_t \lrcorner d\alpha) &= \int_0^1 d(X_t \lrcorner \alpha) \\ &= d\left(\int_0^1 X_t \lrcorner \alpha\right),\end{aligned}$$

and hence $\Psi^*\alpha - \alpha$ is exact, as desired.

We define the **moduli space of torsion-free** G_2**-structures** on M as $\mathcal{M} = \mathcal{X}/\mathcal{D}$. It turns out, as we will show, that $\mathcal{M}$ is a non-singular, smooth manifold with dimension $b^3(M)$. We will prove this fact by constructing a "slice" for the action of $\mathcal{D}$ on $\mathcal{X}$. A **slice** S_φ for $\varphi \in \mathcal{X}$ is a submanifold of $\mathcal{X}$ containing φ that is locally transverse to the orbits of $\mathcal{D}$ near φ. That is, all the nearby orbits of $\mathcal{D}$ each intersect S_φ at exactly one point. If we find such a slice, $\mathcal{M} = \mathcal{X}/\mathcal{D}$ is locally homeomorphic to S_φ in a neighbourhood of $[\varphi]_\mathcal{M} \in \mathcal{M}$. Hence, as $\varphi \in \mathcal{X}$ is arbitrary, we get that $\mathcal{M}$ has the structure of a manifold. For more on slices, see [CK21].

The following theorem from [Joy00] which is based on Ebin's Slice Theorem [Ebi70] for the moduli space of Riemannian metrics gives us a slice for the postive 3-forms on a 7-manifold:

Theorem 5.2.1. *Let* (M, φ) *be a compact* G_2*-manifold. Consider the action of* $\mathcal{D}$ *on* $\Gamma(\mathscr{P}^3 M)$*. Let* I_φ *be the stabilizer subgroup of* φ *in* $\mathcal{D}$ *with respect to this action. Define*

$$L_\varphi = \{\widetilde{\varphi} \in \Gamma(\mathscr{P}^3 M) : \pi_7(d^*\widetilde{\varphi}) = 0\}, \tag{5.20}$$

where π_7 *and* d^* *are with respect to* φ*. Then, there exists an open neighbourhood* S_φ *containing* φ *in* L_φ *which is invariant under* I_φ *such that the natural projection from* S_φ/I_φ *to* $\Gamma(\mathscr{P}^3 M)/\mathcal{D}$ *induces a homeomorphism between* S_φ/I_φ *and a neighbourhood of* $\varphi\mathcal{D}$ *in* $\Gamma(\mathscr{P}^3 M)/\mathcal{D}$*.*

Let us see why it makes sense for the condition $\pi_7(d^*\widetilde{\varphi}) = 0$ to be in (5.20). The natural choice for the slice at φ for the action of $\mathcal{D}$ on $\Gamma(\mathscr{P}^3 M)$ would be the L^2-orthogonal subspace to the orbit $\varphi\mathcal{D}$ of $\mathcal{D}$ at φ as it would ensure local transversality for $\widetilde{\varphi}$ close to φ. Consider the tangent vectors at φ to the orbit $\varphi\mathcal{D}$ of $\mathcal{D}$. They are given by

$$\left.\frac{d}{dt}\right|_{t=0} h_t^*\varphi = \mathcal{L}_X\varphi = d(X \lrcorner \varphi),$$

where h_t is the flow of a smooth vector field X on M. As from (4.44) we know that the tangent space at φ of the orbit $\varphi\mathcal{D}$ is the space $d(\Omega^2_7)$, the natural choice for the slice is

$$L_\varphi = \{\widetilde{\varphi} \in \Gamma(\mathscr{P}^3 M) : \langle \widetilde{\varphi} - \varphi, d(X \lrcorner \varphi)\rangle_{L^2} = 0 \ \ \forall\ X \in \Gamma(TM)\}. \tag{5.21}$$

Since φ is torsion-free, we have $d^*\varphi = 0$ and hence from integration of parts we get that (5.20) is equivalent to (5.21). Now, applying Theorem 5.2.1 to the set of torsion-free G_2-structures $\mathcal{X}$, we have:

Corollary 5.2.2. *Let M be a compact 7-manifold and $\mathcal{M} = \mathcal{X}/\mathcal{D}$ the moduli space of torsion-free* G_2*-structures on M. Let $\varphi\mathcal{D} \in \mathcal{M}$, where φ is a torsion-free* G_2*-structure on M and let I_φ be the stabilizer subgroup of φ in $\mathcal{D}$ with respect to this action. Define*

$$L'_\varphi = \{\widetilde{\varphi} \in \Gamma(\mathscr{P}^3 M) : d\widetilde{\varphi} = d\widetilde{\psi} = 0 \text{ and } \pi_7(d^*\widetilde{\varphi}) = 0\}, \tag{5.22}$$

where π_7 and d^* are with respect to φ. Then, there exists an open neighbourhood S'_φ containing φ in L'_φ which is invariant under I_φ such that the natural projection from S'_φ/I_φ to $\mathcal{M}$ induces a homeomorphism between S'_φ/I_φ and a neighbourhood of $\varphi\mathcal{D}$ in $\mathcal{M}$.

Now, we use Theorem 5.1.5 to find a condition equivalent to the ones in (5.22).

Proposition 5.2.3. *Let (M, φ) be a compact* G_2*-manifold and $\widetilde{\varphi}$ a closed 3-form on M such that $\|\widetilde{\varphi} - \varphi\|_{C_0} \le \epsilon_1$. Then, there exist $\xi \in \mathcal{H}^3$ and a d^*-exact 2-form η such that $\widetilde{\varphi} = \varphi + \xi + d\eta$ uniquely. Furthermore, $\widetilde{\varphi}$ lies in L'_φ from (5.22) if and only if $(dd^* + d^*d)\eta = \star d(F(\xi + d\eta))$.*

Proof. Let $[\varphi], [\widetilde{\varphi}]$ be the deRham cohomology classes of $\varphi, \widetilde{\varphi}$. As $\mathcal{H}^3 \cong H^3(M, \mathbb{R})$, there exists a unique $\xi \in \mathcal{H}^3$ such that $[\xi] = [\widetilde{\varphi}] - [\varphi]$. Thus, as $[\widetilde{\varphi} - \varphi - \xi] = 0$ in $H^3(M, \mathbb{R})$, $\widetilde{\varphi} - \varphi - \xi$ is an exact 3-form. Then, we know by Hodge theory that there exists a unique d^*-exact 2-form η on M such that $d\eta = \widetilde{\varphi} - \varphi - \xi$. Thus, $\widetilde{\varphi} = \varphi + \xi + d\eta$ as claimed.

Now, we have that $d^*\eta = 0$ as η is d^*-exact and $\|\widetilde{\varphi} - \varphi\|_{C_0} \le \epsilon_1$ from our assumption. Thus, from Theorem 5.1.5, we have that $(dd^* + d^*d)\eta = \star d(F(\xi + d\eta))$ if and only if $d\widetilde{\psi} = 0$ and $\pi_7(d^*\widetilde{\varphi}) = 0$. Therefore, from (5.22) we have that $\varphi \in L'_\varphi$ if and only if $(dd^* + d^*d)\eta = \star d(F(\xi + d\eta))$. □

Now, we will present a proof of the main result of this chapter.

Theorem 5.2.4. *Let M be a compact 7-manifold. The moduli space of torsion-free* G_2*-structures $\mathcal{M} = \mathcal{X}/\mathcal{D}$ on M is a smooth manifold of dimension $b^3(M) = \dim H^3(M, \mathbb{R})$. Furthermore, the natural projection $\pi : \mathcal{M} \to H^3(M, \mathbb{R})$ which takes an equivalence class $[\varphi]_\mathcal{M}$ in the quotient space $\mathcal{X}/\mathcal{D}$ to the deRham cohomology class $[\varphi]$ is a local diffeomorphism.*

Proof. Let φ be a torsion-free G_2-structure on M. For $k \ge 0$ and $\alpha \in (0, 1)$, let us define $V^{k,\alpha}$ to be the Banach space of 2-forms given as

$$V^{k,\alpha} = \{\eta \in C^{k,\alpha}(\Lambda^2 T^* M) : \eta \text{ is } L^2\text{-orthogonal to } \mathcal{H}^2\}.$$

Then, define an open set $U^{k+2,\alpha} \subset \mathcal{H}^3 \times V^{k+2,\alpha}$ by

$$U^{k+2,\alpha} = \{(\xi, \eta) \in \mathcal{H}^3 \times V^{k+2,\alpha} : \|\xi + d\eta\|_{C^0} < \epsilon_1\}$$

and a map $\Phi : U^{k+2,\alpha} \to V^{k,\alpha}$ by

$$\Phi(\xi, \eta) = (dd^* + d^*d)\eta - \star d(F(\xi + d\eta)).$$

Since M is compact, and each harmonic form is closed and co-closed, Φ is well-defined as $\Phi(\xi, \eta)$ is L^2-orthogonal to harmonic 2-forms for each $(\xi, \eta) \in U^{k+2,\alpha}$. Furthermore, it is a smooth and non-linear map of Banach spaces.

From Proposition 5.1.4, we know that $F(\xi + d\eta)$ is atleast quadratic in $\xi + d\eta$ and hence the first derivative $d\Phi|_{(0,0)} : \mathcal{H}^3 \times V^{k+2,\alpha} \to V^{k,\alpha}$ is given by

$$d\Phi|_{(0,0)}(\xi, \eta) = (dd^* + d^*d)\eta.$$

Note that $dd^* + d^*d$ is a self-adjoint elliptic operator on 2-forms with kernel and cokernel $\mathcal{H}^2$. Thus, from the Fredholm alternative (Theorem 1.2.4), it follows that $dd^* + d^*d : V^{k+2,\alpha} \to V^{k,\alpha}$ is an isomorphism. Therefore, $d\Phi|_{(0,0)} : \mathcal{H}^3 \times V^{k+2,\alpha} \to V^{k,\alpha}$ is surjective with kernel $\mathcal{H}^3$.

Hence by the Implicit Mapping Theorem (Theorem 1.2.3), we have that $\Phi^{-1}(0)$ is a manifold of dimension $b^3(M)$ in a neighbourhood of $(0, 0)$ and the projection $(\xi, \eta) \mapsto \xi$ induces a diffeomorphism between neighbourhoods of $(0, 0)$ in $\Phi^{-1}(0)$ and 0 in $\mathcal{H}^3$. Note that for small $\|\xi + d\eta\|_{C^0}$, any solution η to the equation $\Phi(\xi, \eta) = 0$ is smooth rather than just $C^{k+2,\alpha}$ by elliptic regularity since $\Phi(\xi, \eta) = 0$ is a non-linear elliptic equation when $\|\xi + d\eta\|_{C^0}$ is small.

Then from Proposition 5.2.3, the slice L'_φ from (5.22) is locally isomorphic to the set

$$\{(\xi, \eta) \in \mathcal{H}^3 \times \Gamma(\Lambda^2 T^*M) : \eta \text{ is } L^2\text{-orthogonal to } \mathcal{H}^2, \\ \|\xi + d\eta\|_{C^0} < \epsilon_1 \text{ and } (dd^* + d^*d)\eta = \star d(F(\xi + d\eta))\}.$$

From above, it follows that this set is a manifold of dimension $b^3(M)$ near $(0, 0)$ and the projection to $\mathcal{H}^3$ is a diffeomorphism. Therefore, L'_φ is a smooth manifold of dimension $b^3(M)$ in a neighbourhood of φ and the projection $L'_\varphi \to H^3(M, \mathbb{R})$ which maps $\widetilde{\varphi}$ to the deRham cohomology class $[\widetilde{\varphi}]$ induces a diffeomorphism between neighbourhoods of $\varphi \in L'_\varphi$ and $[\varphi] \in H^3(M, \mathbb{R})$.

From Corollary 5.2.2, we have that the moduli space $\mathcal{M}$ is homeomorphic near $\varphi\mathcal{D}$ to a neighbourhood of φI_φ in L'_{vp}/I_φ, where I_φ is the stabilizer subgroup of $\mathcal{D}$ which fixes

φ. Since I_φ is a group of diffeomorphisms of M which are isotopic to the identity, we know from the discussion earlier in this section that I_φ acts trivially on the cohomology $H^3(M, \mathbb{R})$. As L'_φ is isomorphic to $H^3(M, \mathbb{R})$ near φ, I_φ acts trivially on L'_φ near φ and thus L'_φ / I_φ is locally isomorphic to L'_φ.

Therefore, as $\mathcal{M}$ is homeomorphic near $\varphi\mathcal{D}$ to a neigbourhood of φ in L'_φ, it follows that $\mathcal{M}$ is a smooth manifold of dimension $b^3(M)$ in a neighbourhood of $\varphi\mathcal{D}$ and the projection $\pi : \mathcal{M} \to H^3(M, \mathbb{R})$ which maps $[\widetilde{\varphi}]_{\mathcal{M}}$ to the deRham cohomology class $[\widetilde{\varphi}]$ induces a diffeomorphism between neighbourhoods of $\varphi\mathcal{D} \in \mathcal{M}$ and $[\varphi] \in H^3(M, \mathbb{R})$. As $\varphi\mathcal{D} \in \mathcal{M}$ is arbitrary, we have proved our desired result. $\square$

Note that the above theorem only tells us about the local structure of $\mathcal{M}$ and gives very little information about the global geometry of $\mathcal{M}$.

Chapter 6

Gauge transformations on the space of torsion-free G_2-structures

In this chapter, we will use the techniques from [Kar20] to explore the action of gauge transformations of the form e^{tA} where $A \in \mathcal{T}^2$, on the space of torsion-free G_2-structures. In particular, we show that infinitesimally, the torsion-free condition almost exactly corresponds to $A \diamond \varphi$ being harmonic (that is, closed and co-closed) when we add a "gauge-fixing" condition. This closely matches with the results in [Joy00] that we presented in Chapter 5 but we use a different approach.

6.1 The difference between connections under a gauge transformation

Before focusing entirely on G_2-manifolds, let us do the following computation on a general Riemannian manifold. From this computation, we will obtain a tensor which will be needed for our G_2 computations.

Let (M, g) be a Riemannian manifold and let P be a gauge transformation on TM. That is, $P : TM \to TM$ is an invertible bundle map. Then, consider another metric

$$\widetilde{g} = P^* g$$

on M. With respect to an orthonormal frame with respect to g, we have

$$\widetilde{g}_{ij} = P_i^k P_j^l g_{kl}.$$

Let $\nabla, \widetilde{\nabla}$ be the the Levi-Civita connections of g and $\widetilde{g}$ respectively. Then, since the difference between connections is tensorial, let us denote the difference between these connections as

$$B(X,Y) := \widetilde{\nabla}_X Y - \nabla_X Y$$

for $X, Y \in \mathfrak{X}(M)$. Furthermore, as $\nabla, \widetilde{\nabla}$ are torsion-free, we have

$$\nabla_X Y - \nabla_Y X = [X,Y], \quad \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X = [X,Y],$$

which gives us

$$B(X,Y) = [X,Y] + \widetilde{\nabla}_Y X - [X,Y] - \nabla_Y X = B(Y,X).$$

Thus, B is symmetric. Then, since the metric $\widetilde{g}$ is compatible with $\widetilde{\nabla}$, for $X, Y, Z \in \mathfrak{X}(M)$

$$\begin{aligned} X(\widetilde{g}(Y,Z)) &= \widetilde{g}(\widetilde{\nabla}_X Y, Z) + \widetilde{g}(Y, \widetilde{\nabla}_X Z) \\ &= g(P(\nabla_X Y + B(X,Y)), PZ) + g(PY, P(\nabla_X Z + B(X,Z)), \end{aligned}$$

and

$$\begin{aligned} X(\widetilde{g}(Y,Z)) &= X(g(PY,PZ)) \\ &= g(\nabla_X(PY), PZ) + g(PY, \nabla_X(PZ)) \\ &= g(P(\nabla_X Y), PZ) + g((\nabla_X P)Y, PZ) + g(PY, P(\nabla_X Z)) + g(PY, (\nabla_X P)Z). \end{aligned}$$

Combining the above two, we get

$$\begin{aligned} g(P(B(X,Y)), PZ) + g(PY, P(B(X,Z))) &= g((\nabla_X P)Y, PZ) + g((\nabla_X P)Z, PY) \\ &= g(PP^{-1}(\nabla_X P)Y, PZ) + g(PP^{-1}(\nabla_X P)Z, PY), \end{aligned}$$

which implies

$$\widetilde{g}(B(X,Y),Z) + \widetilde{g}(B(X,Z),Y) = \widetilde{g}(P^{-1}(\nabla_X P)Y, Z) + \widetilde{g}(P^{-1}(\nabla_X P)Z, Y).$$

Permuting X, Y, Z, we get

$$\widetilde{g}(B(X,Y),Z) + \widetilde{g}(B(X,Z),Y) = \widetilde{g}(P^{-1}(\nabla_X P)Y, Z) + \widetilde{g}(P^{-1}(\nabla_X P)Z, Y), \tag{6.1}$$
$$\widetilde{g}(B(Y,Z),X) + \widetilde{g}(B(Y,X),Z) = \widetilde{g}(P^{-1}(\nabla_Y P)Z, X) + \widetilde{g}(P^{-1}(\nabla_Y P)X, Z), \tag{6.2}$$
$$\widetilde{g}(B(Z,X),Y) + \widetilde{g}(B(Z,Y),X) = \widetilde{g}(P^{-1}(\nabla_Z P)X, Y) + \widetilde{g}(P^{-1}(\nabla_Z P)Y, X). \tag{6.3}$$

Then, as B is symmetric, (6.1) + (6.2) − (6.3) gives us

$$\begin{aligned} 2\widetilde{g}(B(X,Y),Z) = \widetilde{g}(P^{-1}(\nabla_X P)Y,Z) + \widetilde{g}(P^{-1}(\nabla_X P)Z,Y) + \widetilde{g}(P^{-1}(\nabla_Y P)Z,X) \\ + \widetilde{g}(P^{-1}(\nabla_Y P)X,Z) - \widetilde{g}(P^{-1}(\nabla_Z P)X,Y) - \widetilde{g}(P^{-1}(\nabla_Z P)Y,X). \end{aligned} \tag{6.4}$$

Now, we want to get everything in terms of an inner product with Z so that we can get an expression for $B(X,Y)$. So, with respect to a local frame, we start with defining

$$(C(X,Y))^m = (P^{-1}(\nabla_p P)Y)^i X^j \widetilde{g}_{ij}\widetilde{g}^{pm},$$

which gives us the term

$$\begin{aligned} \widetilde{g}(C(X,Y),Z) &= \widetilde{g}_{mn}(C(X,Y))^m Z^n \\ &= \widetilde{g}_{mn}(P^{-1}(\nabla_p P)Y)^i X^j \widetilde{g}_{ij}\widetilde{g}^{pm} Z^n \\ &= \delta^p_n \widetilde{g}_{ij}(P^{-1}(\nabla_p P)Y)^i X^j Z^n \\ &= \widetilde{g}_{ij} Z^p (P^{-1}(\nabla_p P)Y)^i X^j \\ &= \widetilde{g}_{ij}(P^{-1}(\nabla_Z P)Y)^i X^j \\ &= \widetilde{g}(P^{-1}(\nabla_Z P)Y,X). \end{aligned}$$

And similarly we define

$$(C'(X,Y))^m = \widetilde{g}_{ij}(P^{-1})^i_a(\nabla_X P)^a_b \widetilde{g}^{bm} Y^j,$$

which gives us the term

$$\begin{aligned} \widetilde{g}(C'(X,Y),Z) &= \widetilde{g}_{mn}\widetilde{g}_{ij}(P^{-1})^i_a(\nabla_X P)^a_b \widetilde{g}^{bm} Y^j Z^n \\ &= \widetilde{g}_{mn}\widetilde{g}^{bm} Z^n \widetilde{g}_{ij}(P^{-1})^i_a(\nabla_X P)^a_b Y^j \\ &= \widetilde{g}_{ij}\delta^b_n Z^n (P^{-1})^i_a(\nabla_X P)^a_b Y^j \\ &= \widetilde{g}_{ij}(P^{-1})^i_a(\nabla_X P)^a_b Z^b Y^j \\ &= \widetilde{g}(P^{-1}(\nabla_X P)Z,Y). \end{aligned}$$

Therefore, we get

$$\begin{aligned} 2\widetilde{g}(B(X,Y),Z) = \widetilde{g}(P^{-1}(\nabla_X P)Y,Z) + \widetilde{g}(C'(X,Y),Z) + \widetilde{g}(C'(Y,X),Z) \\ + \widetilde{g}(P^{-1}(\nabla_Y P)X,Z) - \widetilde{g}(C(X,Y),Z) - \widetilde{g}(C(Y,X),Z), \end{aligned}$$

and since Z is an arbitrary vector field, we have

$$\begin{aligned} B(X,Y) = \frac{1}{2}(P^{-1}(\nabla_X P)Y + C'(X,Y) + C'(Y,X) \\ + P^{-1}(\nabla_Y P)X - C(X,Y) - C(Y,X)). \end{aligned} \tag{6.5}$$

6.2 Harmonicity of $A \diamond \varphi$

In this section, we will find the necessary and sufficient conditions for a 3-form $\gamma = A \diamond \varphi$ to be harmonic where $A \in \mathcal{T}^2$. We will then show that these conditions correspond to the linearization of the torsion-free condition modulo a gauge-fixing condition in Sections 6.3 and 6.4.

Proposition 6.2.1. *Let (M, φ) be a compact G_2-manifold. Suppose that $\gamma = A \diamond \varphi$ is a 3-form where $A \in \mathcal{T}^2$. Then, γ is harmonic if and only if*

$$\nabla_i A_{ip} \varphi_{pjk} + \nabla_i A_{jp} \varphi_{ipk} + \nabla_i A_{kp} \varphi_{ijp} = 0$$

and

$$\nabla_i A_{pq} \varphi_{pqa} + \nabla_p A_{iq} \varphi_{paq} + \nabla_j A_{kp} \varphi_{jkp} g_{ia} - \nabla_j A_{ka} \varphi_{ijk} - \nabla_p A_{kp} \varphi_{aik} - \nabla_j (\operatorname{tr} A) \varphi_{aji} = 0.$$

Moreover, we have

$$(d^* \gamma)_7 = 0 \iff 2 \operatorname{div} A + \nabla \operatorname{Tr} A - \langle \nabla A, \psi \rangle = 0, \tag{6.6}$$

$$(d\gamma)_1 = 0 \iff \operatorname{div}(\mathrm{V} A) = \nabla_a A_{pq} \varphi_{pqa} = 0, \tag{6.7}$$

$$(d\gamma)_7 = 0 \iff 2 \operatorname{div} A^T - 2\nabla \operatorname{Tr} A + \langle \nabla A, \psi \rangle = 0, \tag{6.8}$$

where $\langle \nabla A, \psi \rangle_m = \nabla_i A_{pq} \psi_{ipqm}$.

Proof. Since M is compact, using integration by parts (see [Joy00, Section 1.1.3]), we know that

$$\gamma \text{ is harmonic} \iff d\gamma = 0 \text{ and } d^*\gamma = 0. \tag{6.9}$$

Looking at the co-closed condition first, we have

$$\begin{aligned} 0 = (d^*\gamma)_{jk} &= -\nabla_i \gamma_{ijk} \\ &= -\nabla_i (A_{ip} \varphi_{pjk} + A_{jp} \varphi_{ipk} + A_{kp} \varphi_{ijp}) \\ &= -\nabla_i A_{ip} \varphi_{pjk} - \nabla_i A_{jp} \varphi_{ipk} - \nabla_i A_{kp} \varphi_{ijp} \\ &= -(\operatorname{div} A)_p \varphi_{pjk} + Q_{jk} - Q_{kj}, \end{aligned}$$

where $Q_{ab} = (\nabla_i A_{ap})\varphi_{ibp}$. From (4.28) and (4.53), it follows that

$$\begin{aligned} Q_{ab}\varphi_{mab} &= -(\nabla_i A_{ap})\varphi_{ipb}\varphi_{mab} \\ &= (\nabla_i A_{ap})(g_{ia}g_{pm} - g_{im}g_{pa} - \psi_{pima}) \\ &= (\operatorname{div} A)_m - \nabla_m(A_{ap}g_{pa}) + \nabla_i A_{ap}\psi_{apim}. \end{aligned}$$

Hence, using (4.27) and (4.51), for the Ω^2_7 part of the co-closed condition, we get

$$\begin{aligned} \pi_7((d^*\gamma)_{jk}) = 0 \iff 0 &= (d^*\gamma)_{jk}\varphi_{mjk} \\ &= -(\operatorname{div} A)_p\varphi_{pjk}\varphi_{mjk} + Q_{jk}\varphi_{mjk} - Q_{kj}\varphi_{mjk} \\ &= -6(\operatorname{div} A)_m + 2Q_{jk}\varphi_{mjk} \\ &= -6(\operatorname{div} A)_m + 2((\operatorname{div} A)_m - \nabla_m(A_{ap}g_{pa}) + \nabla_i A_{ap}\psi_{apim}) \\ &= -4(\operatorname{div} A)_m - 2\nabla_m(\operatorname{Tr} A) + 2\langle \nabla A, \psi\rangle_m. \end{aligned}$$

Therefore, we get

$$(d^*\gamma)_7 = 0 \iff 2\operatorname{div} A + \nabla \operatorname{Tr} A - \langle \nabla A, \psi\rangle = 0.$$

Next, consider the closed condition

$$0 = (d\gamma)_{ijkl} = \nabla_i\gamma_{jkl} - \nabla_j\gamma_{ikl} + \nabla_k\gamma_{ijl} - \nabla_l\gamma_{ijk}. \tag{6.10}$$

Let $\eta_{ijkl} = (d\gamma)_{ijkl}$. Using the same notation from Corollary 4.4.4, we have

$$\eta^{\psi}_{ia} = \eta_{ijkl}\psi_{ajkl} = \nabla_i\gamma_{jkl}\psi_{ajkl} - 3\nabla_j\gamma_{ikl}\psi_{ajkl},$$

as the last three terms of (6.10) are skew in j, k and l. Using the fact that $\gamma = A \diamond \varphi$, (4.30), (4.51) and (4.31), we have

$$\begin{aligned} \nabla_i\gamma_{jkl}\psi_{ajkl} - 3\nabla_j\gamma_{ikl}\psi_{ajkl} &= \nabla_i(A_{jp}\varphi_{pkl} + A_{kp}\varphi_{jpl} + A_{lp}\varphi_{jkp})\psi_{ajkl} \\ &\quad - 3\nabla_j((A_{ip}\varphi_{pkl} + A_{kp}\varphi_{ipl} + A_{lp}\varphi_{ikp})\psi_{ajkl}) \\ &= \nabla_i(-4A_{jp}\varphi_{paj} + 2A_{kp}\varphi_{jpl}\psi_{ajkl}) \\ &\quad - 3\nabla_j(-4A_{ip}\varphi_{paj} + 2A_{kp}\varphi_{ipl}\psi_{ajkl}) \\ &= -4\nabla_i A_{pq}\varphi_{pqa} + 2\nabla_i A_{kp}(-4\varphi_{pak}) + 12\nabla_j A_{ip}\varphi_{paj} \\ &\quad - 6(\nabla_j A_{kp})(g_{ia}\varphi_{pjk} + g_{ij}\varphi_{apk} + g_{ik}\varphi_{ajp} \\ &\quad - g_{pa}\varphi_{ijk} - g_{pj}\varphi_{aik} - g_{pk}\varphi_{aji}) \\ &= -12\nabla_i A_{pq}\varphi_{pqa} - 12\nabla_p A_{iq}\varphi_{paq} - 6\nabla_j A_{kp}\varphi_{jkp}g_{ia} + 6\nabla_i A_{kp}\varphi_{kpa} \\ &\quad - 6\nabla_j A_{ip}\varphi_{ajp} + 6\nabla_j A_{ka}\varphi_{ijk} + 6\nabla_p A_{kp}\varphi_{aik} + 6\nabla_j(\operatorname{tr} A)\varphi_{aji} \\ &= -6\nabla_i A_{pq}\varphi_{pqa} - 6\nabla_p A_{iq}\varphi_{paq} - 6\nabla_j A_{kp}\varphi_{jkp}g_{ia} \\ &\quad + 6\nabla_j A_{ka}\varphi_{ijk} + 6\nabla_p A_{kp}\varphi_{aik} + 6\nabla_j(\operatorname{tr} A)\varphi_{aji}, \end{aligned}$$

which gives us

$$-\frac{1}{6}\eta^{\psi}_{ia} = \nabla_i A_{pq}\varphi_{pqa} + \nabla_p A_{iq}\varphi_{paq} + \nabla_j A_{kp}\varphi_{jkp}g_{ia} \\ - \nabla_j A_{ka}\varphi_{ijk} - \nabla_p A_{kp}\varphi_{aik} - \nabla_j(\operatorname{tr} A)\varphi_{aji}. \tag{6.11}$$

Thus, from Corollary 4.4.4, we obtain

$$(d\gamma)_1 = 0 \iff \operatorname{tr}\eta^{\psi} = 0 \\ \iff \operatorname{div}(\mathrm{V}A) = \nabla_a A_{pq}\varphi_{pqa} = 0.$$

Furthermore, from (4.51), it follows that

$$\pi_7(n^{\psi}_{ia}) = 0 \iff \eta^{\psi}_{ia}\varphi_{iam} = 0,$$

and using (4.27) and (4.28) yields

$$\eta^{\psi}_{ia}\varphi_{ami} = 0 \iff 0 = (\nabla_i A_{pq})(g_{mp}g_{iq} - g_{mq}g_{ip} - \psi_{mipq}) - (\nabla_p A_{iq})(g_{mp}g_{iq} - g_{mq}g_{ip} - \psi_{mipq}) \\ - (\nabla_p A_{qa})(g_{ap}g_{mq} - g_{aq}g_{mp} - \psi_{ampq}) + 6\nabla_p A_{mp} - 6\nabla_m(\operatorname{tr} A) \\ = 3\langle \nabla A, \psi\rangle_m + 6(\operatorname{div} A^T)_m - 6\nabla_m(\operatorname{tr} A)$$

Hence, we have

$$(d\gamma)_7 = 0 \iff 2\operatorname{div} A^T - 2\nabla \operatorname{Tr} A + \langle \nabla A, \psi\rangle,$$

which concludes our proof. □

Remark 6.2.2. Recall that Corollary 4.4.3 tells us that $A_{14} \diamond \varphi = 0$ for any $A \in \mathcal{T}^2$. This agrees with the equations (6.6), (6.7) and (6.8), since they are satisfied for any $A \in \Omega^2_{14}$. Indeed, for $A \in \Omega^2_{14}$, we have $\operatorname{Tr} A = 0$ and from (4.45), we get

$$\langle \nabla A, \psi\rangle_m = \nabla_i A_{pq}\psi_{ipqm} = 2\nabla_i A_{im} = 2(\operatorname{div} A)_m = -2(\operatorname{div} A^T)_m,$$

which shows (6.6) and (6.8). In addition, from (4.51) we get $A_{pq}\varphi_{pqm} = 0$, which shows (6.7).

6.3 Linearization of the torsion-free condition

Now, let us take $P = e^{tA}$ and denote $\widetilde{\varphi} = P^*\varphi = (e^{tA})^*\varphi$ and $\widetilde{g} = P^*g = (e^{tA})^*\varphi$. Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\widetilde{g}$. We want to compute the linearization

$$\left.\frac{d}{dt}\right|_{t=0} \widetilde{\nabla}\widetilde{\varphi}. \tag{6.12}$$

For $X, Y, Z, W \in \mathfrak{X}(M)$, we have

$$(\widetilde{\nabla}_X \widetilde{\varphi})(Y, Z, W) = X(\widetilde{\varphi}(Y, Z, W)) - \widetilde{\varphi}(\widetilde{\nabla}_X Y, Z, W) - \widetilde{\varphi}(Y, \widetilde{\nabla}_X Z, W) - \widetilde{\varphi}(Y, Z, \widetilde{\nabla}_X W),$$

and $\nabla_X \varphi = 0$ gives us

$$X(\varphi(PY, PZ, PW)) = \varphi(\nabla_X(PY), PZ, PW) + \varphi(PY, \nabla_X(PZ), PW) + \varphi(PY, PZ, \nabla_X(PW)).$$

Taking B to be as in Section 6.1, expanding the above yields

$$\begin{aligned}
(\widetilde{\nabla}_X \widetilde{\varphi})(Y, Z, W) &= \varphi(\nabla_X(PY), PZ, PW) + \varphi(PY, \nabla_X(PZ), PW) + \varphi(PY, PZ, \nabla_X(PW)) \\
&\quad - \varphi(P(\nabla_X Y + B(X, Y)), PZ, PW) - \varphi(PY, P(\nabla_X Z + B(X, Z)), PW) \\
&\quad - \varphi(PY, PZ, P(\nabla_X W + B(X, W))) \\
&= \varphi((\nabla_X P)Y, PZ, PW) + \varphi(PY, (\nabla_X P)Z, PW) + \varphi(PY, PZ, (\nabla_X P)W) \\
&\quad - \varphi(P(B(X, Y)), PZ, PW) - \varphi(PY, P(B(X, Z)), PW) \\
&\quad - \varphi(PY, PZ, P(B(X, W))),
\end{aligned}$$

and we can rewrite (6.4) as

$$\begin{aligned}
\widetilde{g}(B(X, Y), Z) = g(P(B(X, Y)), PZ) = \frac{1}{2}(&g((\nabla_X P)Y, PZ) + g((\nabla_X P)Z, PY) + g((\nabla_Y P)Z, PX) \\
&+ g((\nabla_Y P)X, PZ) - g((\nabla_Z P)X, PY) - g((\nabla_Z P)Y, PX)).
\end{aligned}$$

As $\frac{d}{dt}|_{t=0} P = A$ and $P|_{t=0} = I$, we have

$$\nabla_X P|_{t=0} = \nabla_X I = 0, \quad \left.\frac{d}{dt}\right|_{t=0} \nabla_X P = \nabla_X A, \quad B|_{t=0} = 0.$$

Thus, denoting $\dot{B} = \frac{d}{dt}|_{t=0} B$, we get

$$\begin{aligned}
g(\dot{B}(X, Y), Z) = \frac{1}{2}(&g((\nabla_X A)Y, Z) + g((\nabla_X A)Z, Y) + g((\nabla_Y A)Z, X) + g((\nabla_Y A)X, Z) \\
&- g((\nabla_Z A)X, Y) - g((\nabla_Z A)Y, X)).
\end{aligned}$$

Now, taking $\frac{d}{dt}|_{t=0}$ of $(\widetilde{\nabla}_X\widetilde{\varphi})(Y,Z,W)$, from the relation between g and φ in (4.19), we obtain

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0}(\widetilde{\nabla}_X\widetilde{\varphi})(Y,Z,W) &= \varphi((\nabla_X A)Y,Z,W)+\varphi(Y,(\nabla_X A)Z,W)+\varphi(Y,Z,(\nabla_X A)W)\\ &\quad -\varphi(\dot{B}(X,Y),Z,W)-\varphi(Y,\dot{B}(X,Z),W)-\varphi(Y,Z,\dot{B}(X,W))\\ &= g((\nabla_X A)Y,Z\times W)-\frac{1}{2}(g((\nabla_X A)Y,Z\times W)+g((\nabla_X A)(Z\times W),Y)\\ &\quad +g((\nabla_Y A)(Z\times W),X)+g((\nabla_Y A)X,Z\times W)-g((\nabla_{Z\times W}A)X,Y)\\ &\quad -g((\nabla_{Z\times W}A)Y,X))+(\text{cyclic terms } Y\to Z\to W)\\ &= \frac{1}{2}(g((\nabla_X A)Y,Z\times W)-g((\nabla_X A)(Z\times W),Y)-g((\nabla_Y A)(Z\times W),X)\\ &\quad -g((\nabla_Y A)X,Z\times W)+g((\nabla_{Z\times W}A)X,Y)+g((\nabla_{Z\times W}A)Y,X))\\ &\quad +(\text{cyclic terms } Y\to Z\to W).\end{aligned} \tag{6.13}$$

Lemma 6.3.1. *For any vector field X on M, $K_X=\frac{d}{dt}|_{t=0}\widetilde{\nabla}_X\widetilde{\varphi}$ lies in Ω^3_7 with respect to the* G_2*-structure* φ.

Proof. From Lemma 4.5.1, we know that $\widetilde{\nabla}_X\widetilde{\varphi}$ lies in Ω^3_7 with respect to $\widetilde{\varphi}$. To emphasize the fact that the decomposition is with respect to $\widetilde{\varphi}$, we write $\widetilde{\varphi}\in\Omega^3_7(\widetilde{\varphi})$. The decomposition in (4.37) tells us that its inner product with any 3-form in $\Omega^3_1(\widetilde{\varphi})\oplus\Omega^3_{27}(\widetilde{\varphi})$ is zero. Furthermore, from Corollary 4.4.4, every $\omega\in\Omega^3_1(\widetilde{\varphi})\oplus\Omega^3_{27}(\widetilde{\varphi})$ is of the form $\omega=C\diamond\widetilde{\varphi}$ for some $C\in\mathcal{S}^2$. Combining these facts gives

$$\widetilde{g}(\widetilde{\nabla}_X\widetilde{\varphi},C\diamond\widetilde{\varphi})=0$$

for all $C\in\mathcal{S}^2$. Then, note that as

$$(\widetilde{\nabla}_X\widetilde{\varphi})|_{t=0}=\nabla_X\varphi=0,\quad \widetilde{\varphi}|_{t=0}=\varphi,$$

we have

$$\frac{d}{dt}\bigg|_{t=0}\widetilde{g}(\widetilde{\nabla}_X\widetilde{\varphi},C\diamond\widetilde{\varphi})=0\implies g(K_X,C\diamond\varphi)=0\implies K_X\in\Omega^3_7(\varphi).$$ □

From Lemma 6.3.1 and Corollary 4.4.5,

$$K_X=K(X)\lrcorner\psi \tag{6.14}$$

for some unique vector field $K(X)\in\mathfrak{X}$. Now, we will find the necessary and sufficient conditions for the 2-tensor K to vanish.

Proposition 6.3.2. *Let K be the 2-tensor defined by the equation $K_X = K(X) \lrcorner \psi$, where $K_X = \frac{d}{dt}|_{t=0}\widetilde{\nabla}_X\widetilde{\varphi}$.*

$$K_{ia} = 0 \iff -\nabla_i A_{pq}\varphi_{apq} + \nabla_p A_{qi}\varphi_{pqa} - \nabla_p A_{iq}\varphi_{paq} = 0.$$

Moreover, we have

$$K_7 = 0 \iff -2\,\mathrm{div}\,A^T + 2\nabla\,\mathrm{tr}\,A - \langle \nabla A, \psi\rangle = 0 \tag{6.15}$$

and

$$K_1 = 0 \iff \nabla_a A_{pq}\varphi_{apq} = 0. \tag{6.16}$$

Proof. Note that for an orthonormal frame e_a, we have

$$e_k \times e_l = g(e_k \times e_l, e_m)e_m = \varphi_{klm}e_m.$$

Thus, taking $X = e_i, Y = e_j, Z = e_k, W = e_l$ in (6.13) gives us

$$\begin{aligned}K_{ijkl} = \frac{1}{2}(&\varphi_{klm}\nabla_i A_{jm} - \varphi_{klm}\nabla_i A_{mj} - \varphi_{klm}\nabla_j A_{mi} - \varphi_{klm}\nabla_j A_{im} + \varphi_{klm}\nabla_m A_{ij} + \varphi_{klm}\nabla_m A_{ji})\\ &+ (\text{cyclic terms } j \to k \to l).\end{aligned}$$

Since from (6.14) we have that $K_{ijkl} = K_{ip}\psi_{ipkl}$, using the identity (4.33) gives us

$$K^{\psi}_{ia} = K_{ijkl}\psi_{ajkl} = 24K_{ia}.$$

Therefore, using the identity (4.30) yields

$$\begin{aligned}
24K_{ia} &= \frac{3}{2}(\varphi_{klm}\nabla_i A_{jm} - \varphi_{klm}\nabla_i A_{mj} - \varphi_{klm}\nabla_j A_{mi}\\
&\qquad - \varphi_{klm}\nabla_j A_{im} + \varphi_{klm}\nabla_m A_{ij} + \varphi_{klm}\nabla_m A_{ji})\psi_{ajkl}\\
\implies 16K_{ia} &= -4\varphi_{ajm}(\nabla_i A_{jm} - \nabla_i A_{mj} - \nabla_j A_{mi} - \nabla_j A_{im} + \nabla_m A_{ij} + \nabla_m A_{ji})\\
\implies 4K_{ia} &= -2\varphi_{ajm}(\nabla_i A_{jm} - \nabla_j A_{mi} - \nabla_j A_{im})\\
\implies 2K_{ia} &= -\nabla_i A_{pq}\varphi_{apq} + \nabla_p A_{qi}\varphi_{pqa} - \nabla_p A_{iq}\varphi_{paq}.
\end{aligned} \tag{6.17}$$

Contracting (6.17) with φ on two indices and using (4.28) and (4.53), we get

$$\begin{aligned}
2K_{ia}\varphi_{iam} &= -\nabla_i A_{pq}\varphi_{apq}\varphi_{iam} + \nabla_p A_{qi}\varphi_{pqa}\varphi_{iam} - \nabla_p A_{iq}\varphi_{paq}\varphi_{iam} \\
&= \nabla_i A_{pq}\varphi_{pqa}\varphi_{ima} - \nabla_p A_{qi}\varphi_{pqa}\varphi_{ima} - \nabla_p A_{iq}\varphi_{pqa}\varphi_{ima} \\
&= \nabla_i A_{pq}(g_{pi}g_{qm} - g_{pm}g_{qi} - \psi_{pqim}) - \nabla_p A_{qi}(g_{pi}g_{qm} - g_{pm}g_{qi} - \psi_{pqim}) \\
&\quad - \nabla_p A_{iq}(g_{pi}g_{qm} - g_{pm}g_{qi} - \psi_{pqim}) \\
&= \nabla_i A_{im} - \nabla_i A_{mi} - \langle \nabla A, \psi\rangle_m - \nabla_i A_{mi} + \nabla_m(\operatorname{tr} A) \\
&\quad + \langle \nabla A, \psi\rangle_m - \nabla_i A_{im} + \nabla_m(\operatorname{tr} A) - \langle \nabla A, \psi\rangle_m \\
&= -2(\operatorname{div} A^T)_m + 2\nabla_m(\operatorname{tr} A) - \langle \nabla A, \psi\rangle_m.
\end{aligned}$$

Hence, we have

$$K_7 = 0 \iff -2\operatorname{div} A^T + 2\nabla \operatorname{tr} A - \langle \nabla A, \psi\rangle = 0,$$

Finally,

$$\begin{aligned}
2K_{ia}g_{ai} &= -\nabla_i A_{pq}\varphi_{apq}g_{ai} + \nabla_p A_{qi}\varphi_{pqa}g_{ai} - \nabla_p A_{iq}\varphi_{paq}g_{ai} \\
&= -\nabla_a A_{pq}\varphi_{apq} + \nabla_p A_{qa}\varphi_{pqa} - \nabla_p A_{aq}\varphi_{paq} \\
&= -\nabla_a(\mathrm{V}A)_a + \nabla_p(\mathrm{V}A)_p - \nabla_p(\mathrm{V}A)_p \\
&= -\nabla_a(\mathrm{V}A)_a = -\operatorname{div}(\mathrm{V}A),
\end{aligned}$$

and thus

$$K_1 = 0 \iff \nabla_a A_{pq}\varphi_{apq} = 0. \qquad \square$$

Remark 6.3.3. Note that A_{14} does not contribute to K since using (4.51) and (1.5), the equation (6.17) can be rewritten as

$$2K_{ia} = -\nabla_i A^7_{pq}\varphi_{pqa} + 2\nabla_p(A_{\mathrm{sym}})_{iq}\varphi_{pqa}.$$

That is, $K = 0$ is always satisfied for any A of type 14. Therefore, any 14-part of A does not contribute to the torsion at leading order, only at higher order.

6.4 Gauge-fixing and the main theorem

As in Section 5.2, we want tangent directions to our "slice" of torsion-free G_2-structures to be L^2-orthogonal to the infinitesimal diffeomorphisms $\mathcal{L}_W\varphi$. That is, our gauge-fixing condition is given as:

$$\langle A \diamond \varphi, \mathcal{L}_W\varphi\rangle_{L^2} = 0. \tag{6.18}$$

Taking $S = \varphi$ in (1.12), we get

$$(\mathcal{L}_W\varphi)_{ijk} = W_p\nabla_p\varphi_{ijk} + \nabla_i W_p\varphi_{pjk} + \nabla_j W_p\varphi_{ipk} + \nabla_k W_p\varphi_{ijp}. \tag{6.19}$$

Using (4.61) and (4.81), we can rewrite the above as

$$\mathcal{L}_W\varphi = -\frac{1}{3}W_p\widehat{T}_p \diamond \varphi + (\nabla W)\diamond\varphi = \left(\nabla W - \frac{1}{3}\widehat{T}(W)\right)\diamond\varphi. \tag{6.20}$$

As φ is torsion-free, $\widehat{T} = 0$ and thus the gauge-fixing condition is

$$\langle A\diamond\varphi, \nabla W\diamond\varphi\rangle_{L^2} = 0 \tag{6.21}$$

for all $W \in \mathfrak{X}(M)$.

Proposition 6.4.1. *Let $A \in \mathcal{T}^2$ and K be a 2-tensor defined by the equation $K_X = K(X)\lrcorner\psi$, where $K_X = \frac{d}{dt}|_{t=0}\widetilde{\nabla}_X\widetilde{\varphi}$. Then, our gauge-fixing (G.F.) condition is given by the equation*

$$2\operatorname{div} A + \nabla\operatorname{tr} A - \langle\nabla A, \psi\rangle = 0. \tag{6.22}$$

Proof. Using (4.61),(4.27), (4.28) and (4.53), we obtain

$$\begin{aligned}
\langle A\diamond\varphi, \nabla W\diamond\varphi\rangle_{L^2} &= (A_{ip}\varphi_{pjk} + A_{jp}\varphi_{ipk} + A_{kp}\varphi_{ijp})(\nabla W\diamond\varphi)_{ijk}\\
&= A_{ip}\varphi_{pjk}(\nabla W\diamond\varphi)_{ijk} + 2A_{jp}\varphi_{ipk}(\nabla W\diamond\varphi)_{ijk}\\
&= A_{ip}\varphi_{pjk}(\nabla_i W_q\varphi_{qjk} + \nabla_j W_q\varphi_{iqk} + \nabla_k W_q\varphi_{ijq})\\
&\quad + 2A_{jp}\varphi_{ipk}(\nabla_i W_q\varphi_{qjk} + \nabla_j W_q\varphi_{iqk} + \nabla_k W_q\varphi_{ijq})\\
&= 6A_{ip}\nabla_i W_p + 2A_{ip}\nabla_j W_q\varphi_{pjk}\varphi_{iqk} + 2A_{jp}\nabla_i W_q\varphi_{ipk}\varphi_{qjk}\\
&\quad + 12A_{jp}\nabla_j W_p + 2A_{jp}\nabla_k W_q\varphi_{ipk}\varphi_{ijq}\\
&= 18A_{ip}\nabla_i W_p + 6A_{ip}\nabla_j W_q(g_{pi}g_{jq} - g_{pq}g_{ji} - \psi_{pjiq})\\
&= 18A_{ip}\nabla_i W_p + 6(\operatorname{tr} A)\nabla_p W_p - 6A_{ip}\nabla_i W_p - 6\langle\nabla A, \psi\rangle_m W_m\\
&= 12A_{ip}\nabla_i W_p + 6(\operatorname{tr} A)\nabla_p W_p - 6\langle\nabla A, \psi\rangle_m W_m.
\end{aligned}$$

Using integration by parts on the above we obtain that for all $W \in \mathfrak{X}(M)$,

$$\langle A\diamond\varphi, \nabla W\diamond\varphi\rangle_{L^2} = 0 \iff -12\langle\operatorname{div} A, W\rangle_{L^2} - 6\langle\nabla\operatorname{tr} A, W\rangle_{L^2} + 6\langle\langle\nabla A, \psi\rangle, W\rangle = 0.$$

Since it is true for all W, we have that the gauge-fixing condition is given as

$$2\operatorname{div} A + \nabla\operatorname{tr} A - \langle\nabla A, \psi\rangle = 0. \qquad \square$$

Theorem 6.4.2. *Let $A \in \mathcal{T}^2$ and K be a 2-tensor defined by the equation $K_X = K(X) \lrcorner \psi$, where $K_X = \frac{d}{dt}|_{t=0} \widetilde{\nabla}_X \widetilde{\varphi}$. We have that*

$$\begin{aligned} (d\gamma)_1 = 0 &\iff K_1 = 0 \\ (d^*\gamma)_7 = 0 &\iff \textit{G.F. condition} = 0 \\ (d\gamma)_7 = 0 &\iff K_7 = 0 \\ (d\gamma)_{27} = 0 &\iff K_{27} = 0. \end{aligned} \tag{6.23}$$

Proof. Note that the conditions for the vanishing of the 1-parts of K (6.16) and $d\gamma$ (6.7) are identical and the same is true for the 7-part of $d\gamma$ (6.8) and 7-part of K (6.15). In addition, the gauge-fixing condition (6.22) is the same as the vanishing of the 7-part of $d^*\gamma$ (6.6). All that remains to show to prove our claim is that

$$K_{27} = 0 \iff (d\gamma)_{27} = 0. \tag{6.24}$$

The symmetric part of K is given as

$$\begin{aligned} (K_{\text{sym}})_{ia} = \frac{1}{4}(&-\nabla_i A_{pq}\varphi_{apq} - \nabla_a A_{pq}\varphi_{ipq} + \nabla_p A_{qi}\varphi_{pqa} + \nabla_p A_{qa}\varphi_{pqi} \\ &- \nabla_p A_{iq}\varphi_{paq} - \nabla_p A_{aq}\varphi_{piq}). \end{aligned} \tag{6.25}$$

and the symmetric part of η^ψ is given as

$$\begin{aligned} -\frac{1}{6}(\eta^\psi_{\text{sym}})_{ia} = \frac{1}{2}(&\nabla_i A_{pq}\varphi_{pqa} + \nabla_a A_{pq}\varphi_{pqi} + \nabla_p A_{iq}\varphi_{paq} + \nabla_p A_{aq}\varphi_{piq} + 2\nabla_j A_{kp}\varphi_{jkp} g_{ia} \\ &- \nabla_p A_{qa}\varphi_{ipq} - \nabla_p A_{qi}\varphi_{apq}). \end{aligned}$$

As $\nabla_j A_{kp}\varphi_{jkp} g_{ia} \in \Omega^0$, from (6.25) we have that

$$\begin{aligned} \pi_{27}(\eta^\psi_{ia}) = \pi_{27}((\eta^\psi_{\text{sym}})_{ia}) &= 3\pi_{27}(-\nabla_i A_{pq}\varphi_{apq} - \nabla_a A_{pq}\varphi_{ipq} + \nabla_p A_{qi}\varphi_{pqa} + \nabla_p A_{qa}\varphi_{pqi} \\ &\qquad - \nabla_p A_{iq}\varphi_{paq} - \nabla_p A_{aq}\varphi_{piq}) \\ &= 12\pi_{27}(K_{\text{sym}})_{ia} \\ &= 12\pi_{27}(K_{ia}), \end{aligned}$$

which proves (6.24). □

Remark 6.4.3. Note that if in the above theorem we could also show that

$$K_{14} = 0 \iff (d^*\gamma)_{14} = 0,$$

then we would have that

$$A \diamond \varphi \text{ is harmonic } \iff (K = 0 + \text{G.F. condition}),$$

which would mean that infinitesimally the torsion-free condition (modulo gauge-fixing) is equivalent to $A \diamond \varphi$ being harmonic. At the time of writing this document, it is not entirely clear if this holds, and in fact our initial analysis appears to indicate that it is false in general. This equivalence may require a further assumption, which could be related to ambiguity (non-uniqueness) in the definition of A for $\widetilde{\varphi} = (e^{tA})^* \varphi$. This is a question that the author hopes to further study in the future.

6.5 Future questions and extensions

In this chapter, using the framework of gauge transformations e^{tA} acting on the torsion-free G_2-structures, we have shown that infinitesimally, being torsion-free and gauge-fixed (except for the 14 part) is the same as $A \diamond \varphi$ being harmonic. One could explore if this method can be used in the non-infinitesimal case to give an alternate proof of the fact that the moduli space of G_2-structures forms a non-singular smooth manifold.

Furthermore, it could prove fruitful to use this method to prove analogous results for the moduli space formed by structures on manifolds with different holonomy groups such as Spin(7) and $U(m)$. In particular, we know from the fundamental work of Kodaira-Spencer and Kuranishi [Kod05] that in general, there are obstructions to deform a complex structure but in the Kähler case, if a particular deformation of complex structure is unobstructed, then the deformation remains Kähler. Using this method, one could attempt to describe these obstructions through a differential-geometric approach. Furthermore, this point of view could give us a differential geometric explanation for why the Kähler moduli space is not smooth in general.

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