Chains of hourglasses and the graph theory of their Feynman integrals

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Yes, it's basically the same talk as at SFU yesterday...

Feynman periods

Let G be a graph, perhaps a scalar Feynman diagram.

Most interesting K 4regular

$G = K - v$

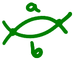
$$\Psi_G = \sum_{\substack{T \\ \text{spanning trees}}} \prod_{e \notin T} a_e$$

associated a_e to edge e

$$P_G = \int_{a_e \geq 0} \frac{da_2\, da_3 \cdots da_{|E(G)|}}{\Psi^2}\Bigg|_{a_1 = 1}$$

$$\int_{a_e \geq 0} \frac{\sum (-1)^i a_i\, da_1 \cdots \widehat{da_i} \cdots da_E}{\Psi^2}$$

Eg



$\Psi = b + a$

$$\int_{a \geq 0} \frac{da}{(1+a)^2}$$

eg $P_{K_4} = 6\zeta(3)$

P_G is a period in the sense of Kontsevich and Zagier

The denominators are important

The denominators are combinatorial or graph theoretic.

The first five integrations sketched look like

$$\int \frac{1}{\Psi_G^2} \longrightarrow \int \frac{1}{\Psi_{G/e_1}\Psi_{G\setminus e_1}}$$

$$\longrightarrow \int \frac{\text{logs}}{\Psi_{G/e_1e_2}\Psi_{G\setminus e_1e_2} - \Psi_{G/e_1\setminus e_2}\Psi_{G/e_2\setminus e_1}} = \int \frac{\text{logs}}{(\Psi^{1,2})^2}$$

sum over certain spanning forests.

$$\longrightarrow \int \frac{\text{logs}}{\sim}$$

← also in terms of spanning forests

$$\longrightarrow \int \frac{\text{dilogs}}{\sim} \longrightarrow \int \frac{\text{trilogs}}{\sim}$$

This is the last step that always works.

Denominator reduction

Francis Brown says

let D_n be the denominator after n integrations

if $D_n = (Aa_{n+1}+B)(Ca_{n+1}+D)$ ← typically also interp. in terms of spanning forests

then $D_{n+1} = AD - BC$ ← in that case contraction deletion

if $D_n = (Aa_{n+1}+B)^2$

then $D_{n+1}=0$ alg. stops.

if D_n doesn't factor alg. stops.

The c_2 invariant

Recall Ψ_G $= \sum_{T} \prod_{e \notin T} a_e$

\# of points on Ψ_G over $\mathbb{F}_p$

$$c_2^{(p)}(G) = \frac{[\Psi_G]_p}{p^2} \mod p$$

Why?

in 90s all P_G which had been calculated were MZVs

maybe all MZV? —— NO

If so should be good math reason

Kontsevich says - good math reason is Ψ_G is mixed Tate

consequently $[\Psi_G]_p$ should be a polynomial in p.

Then $c_2^{(p)}$ is the quad. coeff and in particular $c_2^{(p)}$ indep of p

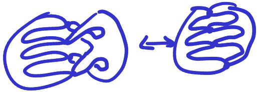
extent to which c_2 not constant becomes a useful measure of how Non-MZV the period is.

Same or compatible graph symmetries?

if $p=2$ true
Simone + me

The c_2 invariant either has or is conjectured to have

- completion invariance (conjectured) $c_2^{(p)}(K-v) = c_2^{(p)}(K-w)$ $\quad P_{K-v} = P_{K-w}$
- duality invariance (proven in planar case and more, conjectured in general) $c_2^{(p)}(G) = c_2^{(p)}(G^*)$ $\quad P_G = P_{G^*}$
- twist invariance (conjectured)
- Fourier split invariance (conjectured)
- 3-join gives 0 (proven)
- Double triangle invariance (proven)



Conjectured: if two graphs have the same period then they have the same c_2 (converse is false)

c_2 is also combinatorial

$c_2^{(p)}$ can be reformulated in terms of counting partitions of $(p-1$ copies of$)$ the edges into $p-1$ spanning trees and $p-1$ appropriate spanning forests.

so c_2 can be calculated by a counting problem

But that's a different talk.

Also D_n from denominator reduction calculates c_2

$$(-1)^n [D_n]_p = c_2^{(p)}$$

Quadratic denominator reduction

Oliver Schnetz says
For $p > 2$, work with the denominators squared, rather than the denominators.

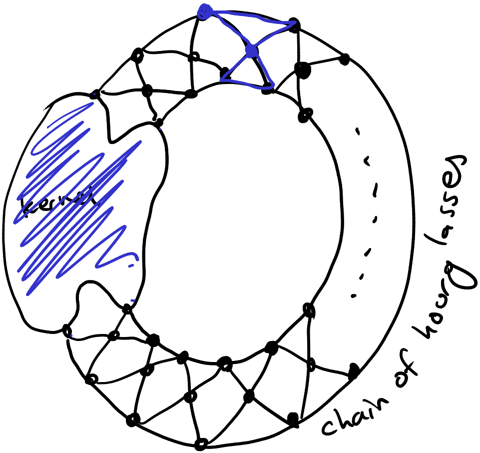
If at the nth step you have	then at the $n+1$ step you have
$(Aa_{n+1}^2 + Ba_{n+1} + C)^2$	$B^2 - 4AC$
$(Aa_{n+1}^2 + Ba_{n+1} + C)(Da_{n+1} + E)^2$	$AE^2 - BDE + CD^2$

Common case is regular denominator reduction.

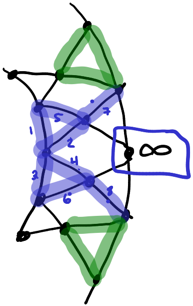
AAA!!!

Hourglass chain graphs

(work with Oliver Schnetz)

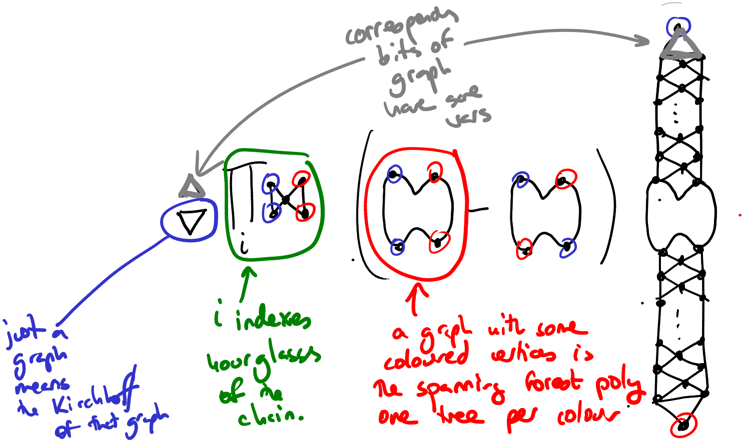


Where to decomplete



Reduce the 8 blue edges. These are not the most obvious ones to start with but they worked well. Then reduce the green edges.

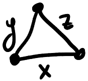
By conventional denominator reduction obtain



Plan: reduce the edges of the two triangles and of the top two hourglasses.

Label as follows

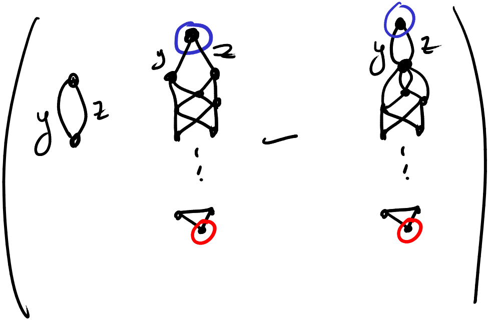
$\left(\text{factors not involving } x, y, z \right)$



Consider x.

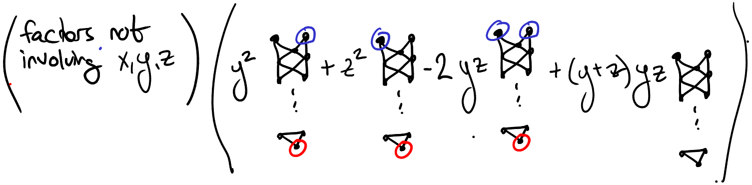
Reduce x

(factors not involving x, y, z)



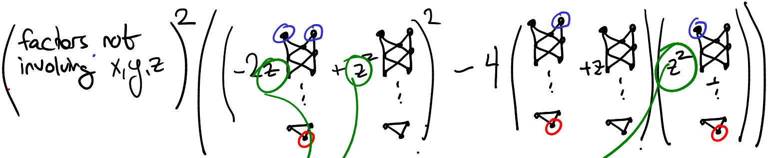
Outcome

Make the ys and zs explicit and collect terms to get



With conventional reduction we are stuck, but not with quadratic reduction. Consider y first.

Reduce y

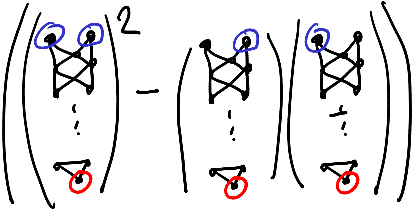


z^2 factors out, so . . .

Reduce z

why $p=2$ doesn't work

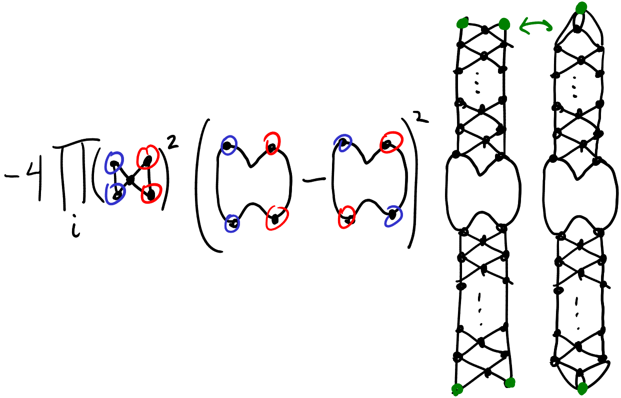
$$\boxed{-4}\left(\text{factors not involving } x, y, z\right)^2$$



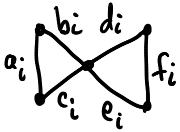
See the -4.

Reduce the bottom triangle

Reduce the bottom triangle to get



Hourglass labelling convention



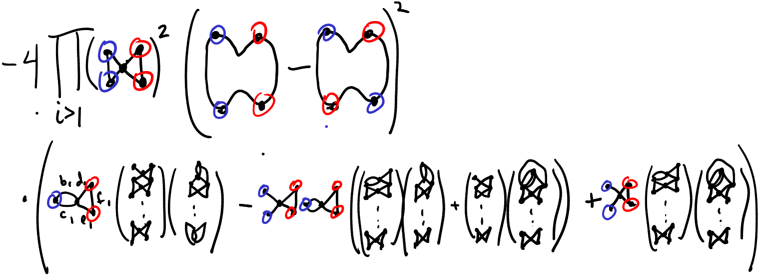
When we're working on things in general for hourglasses, we'll do it without the i subscript. Eg

$$Z = ade + fbc + bcd + bce + bde + cde$$
$$Z_i = a_i d_i e_i + f_i b_i c_i + b_i c_i d_i + b_i c_i e_i + b_i d_i e_i + c_i d_i e_i$$

is the hourglass factor.

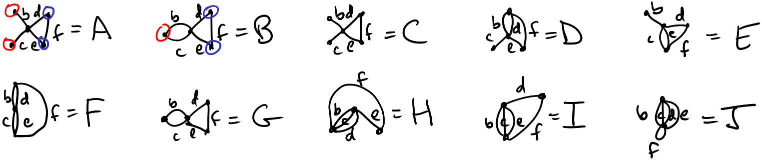
Reduce a_1

Now reduce a_1.



Catalogue of hourglass minors

Calculate the various hourglass minors which appear



On the remaining chains, write $\boxed{\begin{matrix}T\\A\end{matrix}}$ if the top is *together* and the bottom *apart*, etc.

Rewrite

Rewrite to obtain

$$
\begin{aligned}
-4K^2 \prod_{i>1} Z_i^2 \Bigg(& B_1^2 \left(C_1 D_1 \begin{matrix} T\,T \\ A\,T \end{matrix} + E_1 D_1 \begin{matrix} AT \\ AT \end{matrix} + C_1 F_1 \begin{matrix} T\,A \\ A\,T \end{matrix} + E_1 F_1 \begin{matrix} AA \\ AT \end{matrix} \right) \\
& - A_1 B_1 \left(G_1 D_1 \begin{matrix} T\,T \\ A\,T \end{matrix} + I_1 D_1 \begin{matrix} AT \\ AT \end{matrix} + G_1 F_1 \begin{matrix} T\,A \\ A\,T \end{matrix} + I_1 F_1 \begin{matrix} AA \\ AT \end{matrix} \right) \\
& - A_1 B_1 \left(C_1 H_1 \begin{matrix} T\,T \\ A\,T \end{matrix} + E_1 H_1 \begin{matrix} AT \\ AT \end{matrix} + C_1 J_1 \begin{matrix} T\,A \\ A\,T \end{matrix} + E_1 J_1 \begin{matrix} AA \\ AT \end{matrix} \right) \\
& + A_1^2 \left(G_1 H_1 \begin{matrix} T\,T \\ A\,T \end{matrix} + I_1 H_1 \begin{matrix} AT \\ AT \end{matrix} + G_1 J_1 \begin{matrix} T\,A \\ A\,T \end{matrix} + I_1 J_1 \begin{matrix} AA \\ AT \end{matrix} \right) \Bigg)
\end{aligned}
$$

where K^2 is the factor coming from the kernel. This factors.

Factor

$$-4K^2 \prod_{i>1} Z_i^2 \left({}^{T}_{A}(B_1 C_1 - A_1 G_1) + {}^{A}_{A}(B_1 E_1 - A_1 I_1) \right)$$
$$\cdot \left({}^{T}_{T}(B_1 D_1 - A_1 H_1) + {}^{A}_{T}(B_1 F_1 - A_1 J_1) \right)$$

Now work out the polynomials.

Outcome

$$-4K^2 \prod_{i>1} Z_i^2 (d_1 + e_1 + f_1) c_1 X_1 \left(\frac{T}{A} b_1 (d_1 + e_1 + f_1) + \frac{A}{A} Y_1 \right)$$
$$\cdot \left(\frac{T}{T} b_1 (d_1 + e_1 + f_1) + \frac{A}{T} Y_1 \right)$$

where

$$X = bcd + bce + bde + cde + bcf + cdf$$
$$Y = bcd + bce + bde + cde + bcf + bef$$

Reduce a_2

Now do the same thing with a_2. It's the same but bigger.

$$
\begin{aligned}
&-4K^2\prod_{i>2} Z_i^2(d_1+e_1+f_1)c_1X_1\\
&\cdot\Big(\frac{T}{A}((d_1+e_1+f_1)b_1(B_2D_2-A_2H_2)+Y_1(B_2C_2-A_2G_2))\\
&\qquad+\frac{A}{A}((d_1+e_1+f_1)b_1(B_2F_2-A_2J_2)+Y_1(B_2E_2-A_2I_2))\Big)\\
&\cdot\Big(\frac{T}{T}((d_1+e_1+f_1)b_1(B_2D_2-A_2H_2)+Y_1(B_2C_2-A_2G_2))\\
&\qquad+\frac{A}{T}((d_1+e_1+f_1)b_1(B_2F_2-A_2J_2)+Y_1(B_2E_2-A_2I_2))\Big)
\end{aligned}
$$

Subbing in again

But this time when we sub in for the polynomials

$$
\begin{aligned}
&-4K^2\prod_{i>2} Z_i^2(d_1+e_1+f_1)c_1X_1\\
&\boxed{\cdot\Big((d_1+e_1+f_1)b_1X_2+Y_1c_2(d_2+e_2+f_2)\Big)^2}\\
&\cdot\left(\frac{T}{A}(d_2+e_2+f_2)b_2+\frac{A}{A}Y_2\right)\left(\frac{T}{T}(d_2+e_2+f_2)b_2+\frac{A}{T}Y_2\right)
\end{aligned}
$$

a square factor comes out – why is the world so wonderful!

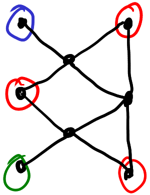
Note that the last line is just what it should be . . .

the bihourglass factor

The square factor

$$\Big((d_1+e_1+f_1)b_1X_2 + Y_1c_2(d_2+e_2+f_2)\Big)^2$$

comes from



Why?

Reduce the top of the bihourglass

We can reduce the top half of the bihourglass, b_1, c_1, d_1 (the last messy one), e_1, f_1.
Finally we get

$$-4K^2 \prod_{i>2} Z_i^2 \left(\frac{T}{A}(d_2+e_2+f_2)b_2 + \frac{A}{A}Y_2\right)$$
$$\cdot \left(\frac{T}{T}(d_2+e_2+f_2)b_2 + \frac{A}{T}Y_2\right) c_2(d_2+e_2+f_2)X_2$$

which is the same as before but with one more hourglass gone.

Induction

Inductively, then, we can consume all the hourglasses above the kernel in this way.

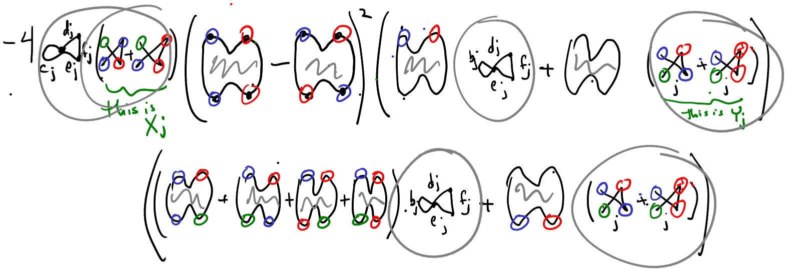
We could have chosen to decomplete so that no hourglasses were left below the kernel.

So all that's left is

- the squared kernel factors that have been carried along,
- the kernel factors from the $\genfrac{}{}{0pt}{}{T}{A}$ applied with no hourglasses left, and
- the remains of the hourglass just above the kernel.

Result

So every graph in the family reduces to



where j is the index of the hourglass just above the kernel.

Conclusion

Now give it to Oliver and he finds a clever scaling to reduce one more variable (needs some more notation to show you).

So what?

- All graphs in each family have the same c_2 at every prime.
- By choosing appropriate kernels, we can get infinite families with various interesting c_2s. Previous families were 0, or -1.
- Quadratic denominator reduction did something cool.
- Blobs were a good way to work with c_2 again.
- Graph theory tells us about Feynman integrals again.