Understanding the log expansions in quantum field theory combinatorially

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Augmented generating functions

Take a combinatorial class $\mathcal{C}$. Build a generating function but keep the objects.

$$C(x) = \sum_{c \in \mathcal{C}} c\, x^{|c|} \quad \in \mathbb{Q}[\mathcal{C}][[x]]$$

- Get the ordinary generating function by evaluating $c \mapsto 1$.
- Count with parameters by evaluating each object as a monomial in the parameters.
- More to today's point if $\mathcal{C}$ is a class of Feynman graphs evaluate by **Feynman rules**.

Two expansion parameters

For us, Feynman rules are an evaluation map ϕ, say

$$\phi : \mathcal{C} \to \mathbb{C}[L]$$

The Green function is ϕ applied to the augmented generating function.

$$G(x, L) = \sum_{c \in \mathcal{C}} \phi(c)\, x^{|c|}$$

The actual physical Feynman rules build an integral from the Feynman graph. L is an energy scale parameter. x is the coupling constant.

log of energy

Which variable to expand in first?

Suppose

$$G(x, L) = 1 + \sum_{i \geq 1} \sum_{j \geq i} a_{i,j} L^i x^j$$

Match the powers of L and x as much as possible

$$G(x, L) = \sum_{k \geq 0} \sum_{n \geq 0} (Lx)^n x^k \; a_{n,n+k}$$

The $k = 0$ part is known as the *leading log expansion*.
The $k = 1$ part is known as the *next-to-leading log expansion*.
The $k = 2$ is known as the *next-to-next-to-leading log expansion*.

What does “leading log” mean?

L is the logarithm of some appropriate energy scale.
x is the coupling constant which is treated as a small parameter.

The leading log expansion captures the maximal powers of x relative to the powers of the energy scale.

The next-to-leading log expansion is next. It is suppressed by one power of x, and so on.

Goal

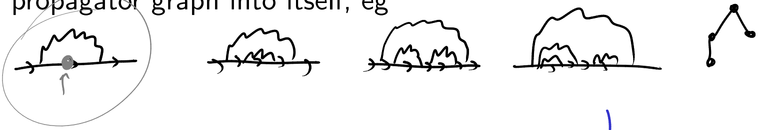
How can we understand the log expansions combinatorially?

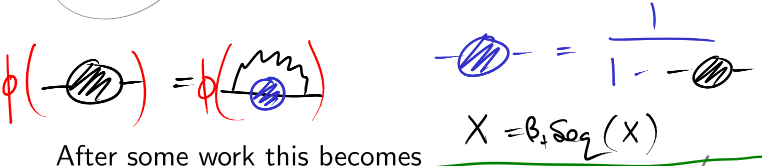
Currently two answers

- Krüger and Kreimer *Filtrations in Dyson-Schwinger equations: next-to^j-leading log expansions systematically*. Annals of Physics, **360**, (2015), 293-340. arXiv:1412.1657
- current work with Julien Courtiel (to be submitted in the next week or two).

A Dyson-Schwinger equation

Consider a Dyson-Schwinger equation for inserting a 1-loop propagator graph into itself, eg





After some work this becomes

$$G(x, L) = 1 - xG(x, \partial_{-\rho})^{-1}(e^{-L\rho} - 1)F(\rho)\Big|_{\rho=0}$$

where

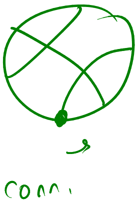
$$F(\rho) = \frac{f_0}{\rho} + f_1 + f_2\rho + f_3\rho^2 + \cdots$$

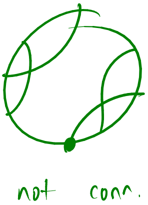
comes from the regularized Feynman integral for the 1-loop graph.

Rooted connected chord diagrams

Can solve this by a chord diagram expansion (with N. Marie, more general case with M. Hihn).

A chord diagram is *rooted* if it has a distinguished vertex.
A chord diagram is *connected* if no set of chords can be separated from the others by a line.
Eg:





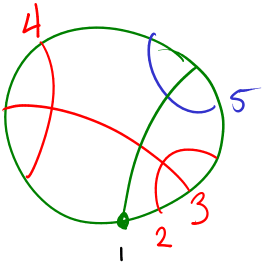
These are really just irreducible matchings of points along a line.

Recursive chord order

Let C be a connected rooted chord diagram. Order the chords recursively:

- c_1 is the root chord
- Order the connected components of $C \setminus c_1$ as they first appear running counterclockwise, $D_1, D_2, \ldots$. Recursively order the chords of D_1, then of D_2, and so on.

Eg:



Terminal chords

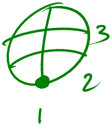
A chord is terminal if it only crosses chords which come before it in the recursive chord order. Let

$$t_1 < t_2 < \cdots < t_\ell$$

be the terminal chords of C. Then

- $b(C) = t_1$ and
- $f_C = f_{t_\ell - t_{\ell-1}} \cdots f_{t_3 - t_2} f_{t_2 - t_1} f_0^{|C|-\ell}$

Eg:



$b(c) = 2$

$f_C = f_{3-2} f_0 = f_0 f_1$

Chord diagram expansion

Theorem

$$G(x,L) = 1 - \sum_{i\geq 1} \frac{(-L)^i}{i!} \sum_{\substack{C \\ b(C)\geq i}} x^{|C|} f_C f_{b(C)-i}$$

solves

$$G(x,L) = 1 - xG(x,\partial_{-\rho})^{-1}(e^{-L\rho} - 1)F(\rho)\Big|_{\rho=0}$$

where

$$F(\rho) = \frac{f_0}{\rho} + f_1 + f_2\rho + f_3\rho^2 + \cdots$$

What is the leading log part?

We had

$$G(x,L) = 1 - \sum_{i\geq 1} \frac{(-L)^i}{i!} \sum_{\substack{C \\ b(C)\geq i}} x^{|C|} f_C f_{b(C)-i}$$

The leading log part is

$$\sum_{b(C)=|C|} \frac{(-Lx)^{|C|}}{|C|!} f_0^{|C|} = \sum_{b(C)=|C|} \frac{(-Lxf_0)^{|C|}}{|C|!}$$

exp. gen fn

What is the next-to-leading log part?

We had

$$G(x, L) = 1 - \sum_{i \geq 1} \frac{(-L)^i}{i!} \sum_{\substack{C \\ b(C) \geq i}} x^{|C|} f_C f_{b(C)-i}$$

$b(C) \geq |C| - 1$

The next-to-leading log part is

$$\sum_{b(C)=|C|} \frac{(-L)^{|C|-1}}{(|C|-1)!} x^{|C|} f_0^{|C|-1} f_1$$

$$+ \sum_{b(C)=|C|-1} \frac{(-L)^{|C|-1}}{(|C|-1)!} x^{|C|} f_1 f_0^{|C|-2} f_0$$

After that

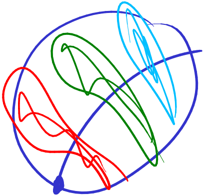
Something different happens for the next-to-next-to leading log part. We had

$$G(x, L) = 1 - \sum_{i \geq 1} \frac{(-L)^i}{i!} \sum_{\substack{C \\ b(C) \geq i}} x^{|C|} f_C f_{b(C)-i}$$

We get

Closed forms

Through the basic decomposition of rooted connected chord diagrams:



We can determine these exponential generating functions (work of Julien Courtiel). Specifically

$$\text{LL:} \qquad 1 - \sqrt{1 - 2Lxf_0}$$

$$\text{NLL:} \qquad xf_1 \left(1 + \frac{1}{\sqrt{1 - 2Lxf_0}} \ln \left(\frac{1}{\sqrt{1 - 2Lxf_0}} \right) \right)$$

and the NNLL is longer and takes more work.

Asymptotics

Theorem (Courtiel)

 # rooted connected chord diagrams, n chords, $b(C) \geq n-k$ $\sim$ *# rooted connected chord diagrams, n chords, last k terminal.*

There's an explicit asymptotic formula.

So, if $F(\rho)$ is not outrageous the next-to^k-leading log expansion is given by

- the exponential generating function for diagrams with $b(C) \geq n-k$
- f_0
- f_1

Nothing else plays a role.

What about Krüger and Kreimer's approach

Krüger and Kreimer approach the problem differently.

- They also start with Dyson-Schwinger equations.
- They use Hopf algebraic properties to map to the shuffle algebra of words.
- Filtering the word Hopf algebra cuts out the different log expansions.

The alphabet for these words corresponds to the primitive Feynman graphs and analogues when graphs are combined.

Similarities and differences

Similarities:

- The master equations.
- An underlying combinatorial perspective.

Differences:

- The basic objects.
- Automaticity.
- Generality.
- Where the new periods come from.

But what about their results?

On the common domain of applicability both groups' results are the same.

They **have to be** because we are both describing the same underlying physics.

So. . . ?

What is going on?

Quite different objects are describing the same physics.

Why? How can we use it?

References

- Nicolas Marie and KY *A chord diagram expansion coming from some Dyson-Schwinger equations*. Communications in Number Theory and Physics, **7**, no 2 (2013), 251-291. arXiv:1210.5457
- Olaf Krüger and Dirk Kreimer *Filtrations in Dyson-Schwinger equations: next-to^j-leading log expansions systematically*. Annals of Physics, **360**, (2015), 293-340. arXiv:1412.1657
- Markus Hihn and KY *Generalized chord diagram expansions of Dyson-Schwinger equations*. arXiv:1602.02550
- Julien Courtiel and KY *Terminal chords in connected chord diagrams*. Ask us (arXiv:16??.?????).