Recent progress on the c_2 invariant

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The Kirchhoff/first Symanzik polynomial

Recall from Erik Panzer's talk:
Let K be a connected 4-regular graph
Let $G = K - v$. These are connected ϕ^4 graphs with 4 external edges.

Define

$$\Psi_G = \sum_{\substack{T \\ \text{spanning tree}}} \prod_{e \notin T} a_e$$

Eg:

Period – geometry – arithmetic

$$P_G = \int_{a_e \geq 0} \frac{1}{\psi_G^2} \rightsquigarrow \text{geometry of } \psi_G = 0$$

how else can we access this geometry

count points over finite fields

The c_2 invariant

For $f \in \mathbb{Z}[x_1, \ldots, x_n]$ define $[f]_q$ to be the number of $\mathbb{F}_q$-rational points on the variety $f = 0$.

Define

$$c_2^{(q)}(G) = \frac{[\Psi_G]_q}{q^2} \mod q$$

would be the quadratic coeff if $[\Psi_G]_q$ were polynomial in q

$$c_2(G) = \left(c_2^{(2)}(G),\ c_2^{(3)}(G),\ \ldots \right)$$

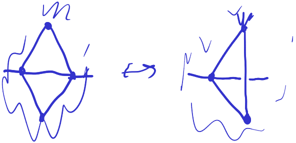
Arithmetic structure

- If $c_2^{(p)}(G) = 0$ then P_G should have less than maximal transcendental weight.
- If P_G is MZV then $c_2^{(p)}(G)$ should be independent of p.
  up to $p=2$
- The same point in field extensions.
- Some $c_2^{(p)}(G)$ are proven to be coefficient sequences of modular forms.
- In this case P_G should be more exotic.

some period of the geometry associated
to the modular form

Known graph-related properties

- If K has a 3-separation then $c_2^{(p)}(G) = 0$.
- If K has an internal 4-edge-cut then $c_2^{(p)}(G) = 0$.
- If G has vertex width 3 then $c_2^{(p)}$ is a constant.
- c_2 is double-triangle invariant



Known and conjectured symmetries

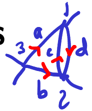
Recall the symmetries Erik discussed:
The period is proven to be invariant under

- Completion/decompletion
- Planar duality for G
- Schnetz twist

The c_2 invariant should have these symmetries as well.

Expanded Laplacian and more polynomials

eg



Let

$$M_G = \begin{bmatrix} \Lambda & E^T \\ -E & 0 \end{bmatrix}$$

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline 1 & -1 & 0 & -1 & 1 \\ 2 & 0 & -1 & 1 & -1 \\ \cancel{3} & \cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} \end{array}$$

where $\Lambda = \text{diag}(a_1, a_2, \dots, a_n)$ and E is the signed incidence matrix with one row removed.

Then as another way to view the matrix tree theorem we have

$$\Psi_G = \det M_G$$

We also care about minors

remove rows I

remove cols indexed by J

$$\Psi^{I,J}_{G,K} = \det M_G(I, J)|_{a_e=0, e\in K}$$

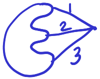
same as contracting

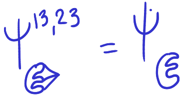
One known result we need

Proposition (Brown and Schnetz)

$$c_2^{(p)}(G) = [\Psi^{1,2}_{G,3} \boxed{\Psi^{13,23}_{G}}]_p \mod p$$

In particular if edges $1, 2, 3$ meet at a 3-valent vertex:





$\Psi^{1,2}_{3} =$  spanning forest polynomials

trees of the forest corr. to different colours of the marked vertices

 $\sim$ common spanning trees in  and 

Another known result we need

Proposition (Corollary of Chevalley-Warning)

If f has total degree n in $x_1, x_2, \ldots, x_n$ then

$$[f]_p = \textit{coefficient of } x_1^{p-1} \cdots x_n^{p-1} \textit{ in } f^{p-1} \mod p$$

The polynomial on the previous page satisfies this as does ψ_G^2

Reduction to counting certain edge bipartitions

Apply these to our situation.

$$c_2^{(2)}(G) = [\Psi^{1,2}_{G,3}\Psi^{13,23}_{G}]_2 \quad \text{mod } 2$$

$$= \text{coefficient of } x_1 \cdots x_n \text{ in } \Psi^{1,2}_{G,3}\Psi^{13,23}_{G} \quad \text{mod } 2$$

= # of ways to partition the edges between $\Psi^{1,2}_{G,3}$ and $\Psi^{13,23}_{G}$ mod 2

= # of ways to partition the edges into a spanning tree and a spanning forest compatible with  mod 2

= parity of # of certain edge bipartitions

A special case of completion

Brown and Schnetz conjecture that for all p, 4-regular K, $v_1, v_2 \in V(K)$

$$c_2^{(p)}(K - v_1) = c_2^{(p)}(K - v_2)$$

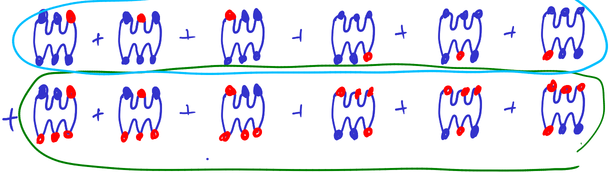
I prove that if K has an odd number of vertices, $v_1, v_2 \in V(K)$, then

$$c_2^{(2)}(K - v_1) = c_2^{(2)}(K - v_2)$$

Proof sketch

requires parity condit on vertices of K

build an aux graph with an even # of vertices



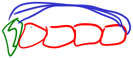
mod 2

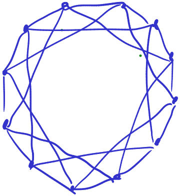
fixed pt free involution

Recursive families

We can fix p but rigorously calculate $c_2^{(p)}(G_n)$ for *recursively constructible families* of graphs. Roughly, graphs with

- an initial piece
- a chain of repeated structures, and
- a cap which may link back to the initial piece.

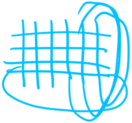




Explicit results

decompletion.

- $c_2^{(2)}(\widetilde{C_n}(1,3)) = n \mod 2$ for $n \geq 7$
- $c_2^{(2)}(\widetilde{C_{2k+2}}(1,k)) = 0 \mod 2$ for $k \geq 3$
- Let G be a (sufficiently large) nonskew toroidal grid. Then $c_2^{(2)}(\widetilde{G}) = 0$ (Chorney, Y.)
- Two other families with Chorney



General recursive family result

- Fix p
- Get started with three edges in the cap and then ~~assgin~~ assign the rest of the cap
- Get sum of products of $2p-2$ spanning forest polynomials; the partitions only use the initial and final pieces.
- There are only finitely many.
- Assigning one piece of chain gives a recurrence. Do so for each product of spanning forest polynomials.
- Calculate initial conditions and solve the system.

This gives a rigorous finite algorithm for any recursively constructible family of graphs with $2|V(G_n)| = |E(G_n)| + 2$ for n sufficiently large (Chorney, Y).

Implementation

The algorithm is very bad in p and complexity of the family.

$C(1,3)$	
p	N
2	29
3	546
5	82703
7	5698505

$C(2,3)$	
p	N
2	248
3	30729

N is the number of products of spanning forest polynomials needed.

How many iterations

$C(1,3)$		
p	c_2 iterations	vector iterations
2	2	4
3	36	59040
5	3720	
7	134064	

$C(2,3)$		
p	c_2 iterations	vector iterations
2	7	56
3	4356	

The ones with vector iterations are proven.

Calculated c_2 invariants

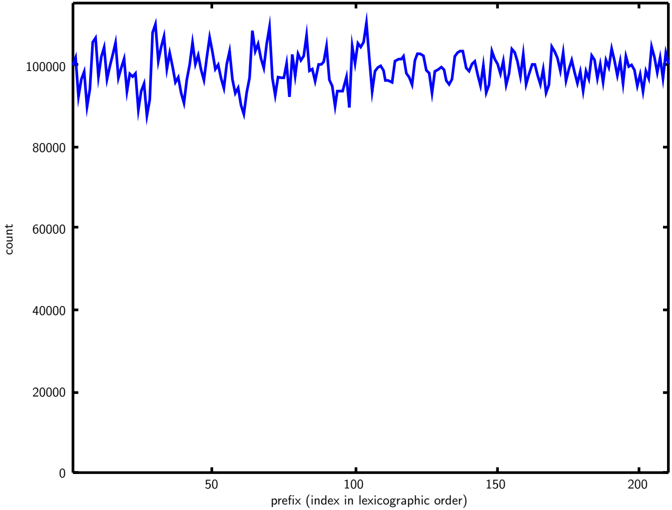
$$c_2^{(2)}(\widetilde{C_n}(1,3)) = (10)^*$$

$$c_2^{(3)}(\widetilde{C_n}(1,3)) = (0000001221222211120102011122221201010)^*$$

$$c_2^{(2)}(\widetilde{C_n}(2,3)) = (1110100)^*$$

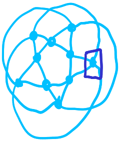
(for the rest see arXiv:1805.11735)

Prefix density



Expanded symmetry (with Crump)

It's inconvenient not to be able to take duals of non-planar graphs.
But we can with matroids.
Let's try with $P_{8,36}$

$P_{8,36} =$ 

Decomplete

Dual of decompleted $P_{8,36}$

Choose a basis for the cycle space and write out the matrix for the dual

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Note the double triangle $(0, 1, 4, 5, 12)$.

Double triangle reduction of it

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & \pm 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

There's a choice, but *any* choice which is invertible mod p is fine.

No double triangles.

Dual again

$$\begin{bmatrix}
0 & \mp 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Take the dual again; now we used the general rule.

Double triangle $(1, 4, 7, 8, 13)$.

Double triangle again

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & -1 & 0 & \pm 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is isomorphic as a matroid to a double triangle reduction of $(P_{7,11} - v_9)^*$. So $P_{8,36}$ and $P_{7,11}$ have the same c_2.

New c_2 identities by dualized double triangles

With Iain Crump
For any prime p

$$c_2^{(p)}(\widetilde{P}_{9,156}) = c_2^{(p)}(\widetilde{P}_{9,159}) = c_2^{(p)}(\widetilde{P}_{7,8})$$

$$c_2^{(p)}(\widetilde{P}_{7,11}) = c_2^{(p)}(\widetilde{P}_{8,36})$$

and we already knew $P_{8,30}$ has the same c_2 as $P_{7,11}$ by usual double triangle.

$$c_2^{(p)}(\widetilde{P}_{9,164}) = c_2^{(p)}(\widetilde{P}_{8,37})$$