Algebraic cut structures on Feynman graphs

Karen Yeats

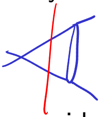
Department of Combinatorics and Optimization, University of Waterloo

KMPB day, September 27, 2021

Cuts in QFT

Cutkosky cuts tell us about the monodromy of Feynman integrals.

Cut edges are edges put on shell.

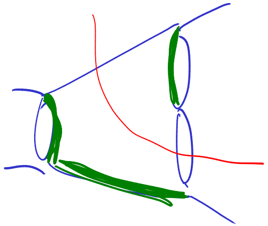


Cuts should separate the graph – don't consider cutting edges that have both ends in the same component after the cut.

But we do care about cutting into more that 2 pieces – anomalous thresholds.

Trees and Cutkosky

To make connections to Vogtmann's Outer Space, we may want to fix a spanning tree in each component.



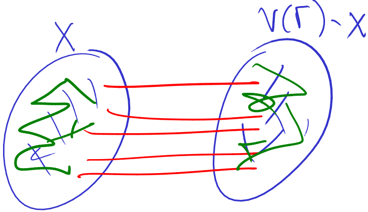
These are a familiar kind of cut

These are familiar cuts from graph theory.

$X \subseteq V(\Gamma)$

Typically a cut is defined by a partition of the vertices. The edges of the cut are the edges with ends in different parts of the partition.

A spanning forest defines a vertex partition and hence a cut.



Cut space

Put an arbitrary orientation on each edge of the graph.

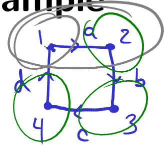
For an ordered bipartition of the vertices form the signed sum of the edges of the cut.

The span of these is the **cut space** of the graph.

Equivalently the cut space is the row space of the signed incidence matrix of the graph.

The cut space is also the dual of the flow space – essentially the space of momentum flows.

Example



$$\mathrm{Span}\{a-d,\ b-d,\ c-d,\ a-c,\ \ldots\}$$

$\Vert$

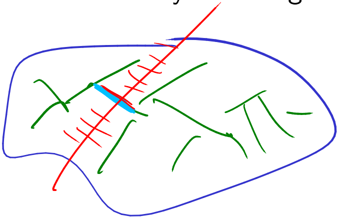
$$\mathrm{Row}\left(\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{array}\right)$$

Nice bases

From the signed incidence matrix perspective we see one basis of the cut space is those cuts that detach single vertices.

(with one vertex removed)

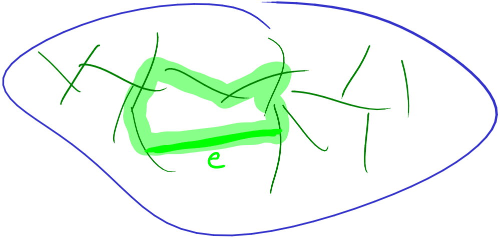
Another nice basis is given by choosing a spanning tree and using the cuts defined by each edge of the spanning tree.



Fundamental cycles

The edges of a spanning tree determine cuts.

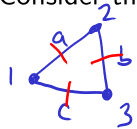
The edges not in the spanning tree determine **fundamental cycles**.



Anomalous thresholds don't live in the cut space

Anomalous thresholds don't live straightforwardly in the cut space.

Consider the triangle graph as an example.



$\mathrm{Span}\{a-b,\ b-c\}$

$\not\ni$

$a \pm b \pm c$

but provided char $\neq 2$ there is

$\alpha a + \beta b + \gamma c$

in cut space $\alpha, \beta, \gamma \neq 0$

Supports are tricky because they take us outside of linear algebra.

So what is it then?

So if it isn't linear algebra, what is it?

We are interested in the containment structure of cuts. Furthermore we may wish to mark some edges as uncuttable, so we are looking at an incidence structure.

Incidence coalgebras

locally finite

Given a partially ordered set S with partial order $\leq$ define

interval

$$\longrightarrow [x, y] = \{z \in S : x \leq z \leq y\}$$

Then the set of intervals of S $\cancel{M}$ has a coproduct

$$\rho([x, y]) = \sum_{x \leq z \leq y} [x, z] \otimes [z, y]$$

and with the counit $[x, y] \mapsto \delta_{x,y}$ this gives the **indicence coalgebra** on S.

Incidence coalgebra in the cut context

Fix a graph and a spanning tree T of the graph.

We are interested in the incidence coalgebra on E_T.

cut complement of uncuttable.

Interpret an interval $[A, B]$ as A defining a cut (into potentially many pieces if A has many elements) and having the edges of $E_T - B$ be uncuttable, or equivalently contracting them.

Cointeracting bialgebras

Perhaps you have heard Loïc Foissy talk recently on cointeracting bialgebras.

Let (A, m_A, Δ) and (B, m_B, δ) be bialgebras with a right coaction $\rho : A \to A \otimes B$ such that the structure maps of A are comodule morphisms, that is

- $\rho(1_A) = 1_A \otimes 1_B$,
- $(\Delta \otimes \mathrm{id}) \circ \rho = m_{1,3,24} \circ (\rho \otimes \rho) \circ \Delta$,
- for $a, b \in A$, $\rho(ab) = \rho(a)\rho(b)$,
- for $a \in A$, $(\epsilon_A \otimes \mathrm{id}) \circ \rho(a) = \epsilon_A(a)1_B$,

Then the two bialgebras are **cointeracting**.

Often $(A, m_A) = (B, m_B)$ and $\rho = \delta$.

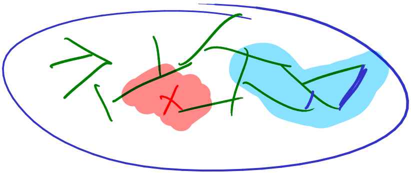
$m_{1,3,24} : A \otimes B \otimes A \otimes B \to A \otimes A \otimes B$

$a_1 \otimes b_1 \otimes a_2 \otimes b_2 \mapsto a_1 \otimes a_2 \otimes m_B(b_1, b_2).$

For QFT

We want to use the ρ defined already. What about Δ and m?

For Δ we want the core Hopf algebra, but with a spanning tree fixed. We can implement this via the fundamental cycles.



Subgraphs that are allowed are unions of fundamental cycles

So core coproduct w tree is a coproduct on $E_\Gamma \setminus E_T$ by subsets.

For QTF continued

We want the product not to be disjoint union but union within the fixed graph. core ✓

intervals → set to 0 if cut or contract info is not consistent.

However, ρ acts on tree edges and Δ on non-tree edges.

Put them together on the space of formal symbols

$$B^{[A_1, A_2]}$$

where $B \subseteq E_\Gamma - E_T$, $[A_1, A_2]$ is an interval in $\mathcal{P}(f(B))$ and $f : E_\Gamma - E_T \to \mathcal{P}(E_T)$ gives the fundamental cycles.

$f(e) = (\text{fundamental cycle of } T, e) \cap E_T$

think: B determines the subgraph.

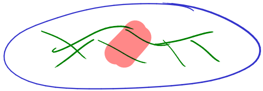
$B \cup f(B)$ is the edges of the subgraph.

A_1 determines cut of subgraph. $f(B) \setminus A_2$ uncuttable.

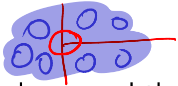
Result

Then (Kreimer, Y, 2021, arXiv:2105.05948) these are cointeracting (wtih Δ restricted to the subspace with intervals $[\emptyset, A]$ and ρ restricted to the subspace where no forbidden tadpoles are contracted.)

When we fix a forest. . .



When we fix a forest, we don't see the cuts on the sub-side of the modified core coproduct.



So the insertion structure works around the cuts rather than building them.

If we don't fix a tree or forest and allow sub-objects across the cut, then we have a different picture

We can build Dyson-Schwinger equation

We can build Dyson-Schwinger equations that interact with the cut non-trivially.

$$X^{+\!\!<} = B_+^{\triangleleft}\left(\frac{(X^{<})^3}{X^{-}}(X^{+})^2\right) + B_+^{\triangleleft}\left(\frac{X^{+\!\!<}\,(X^{<})^2\,X^{+}}{(X^{-})^2}\right) + \dots$$

We need cuts through corollas in the skeletons that we insert into.

The usual story

The usual story holds:

- the lemma on the coproduct of the Green functions
- the importance of invariant charges
- put the ρ coaction on Green functions into a matrix.

Or count the gluings

Or turn it around and ask a different question. In a renormalization Hopf algebra instead of the core Hopf algebra, with higher order primitives, we then have higher order primitives with cuts too.

Instead of starting with the graph and cutting, what about starting with the parts and gluing? This has connections to Vogtmann's assembly maps, but also asks an enumeration question: how many subdivergence free gluings are there of graphs?

Subdivergence free gluings to trees

With Xinle (Clair) Dai and Jordan Long (arXiv:2106.07494), we counted how many subdivergence-free gluings of trees there are in certain special cases: