Combinatorial and physical content of Kirchhoff polynomials

Karen Yeats

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Spanning trees

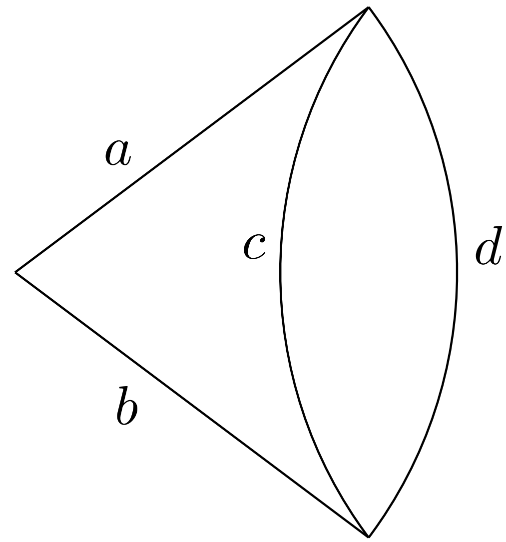
Let G be a connected graph, potentially with multiple edges and loops in the sense of a graph theorist. Perhaps a Feynman graph.

A **spanning forest** of G is a not-necessarily connected subgraph which

- is a disjoint union of trees and
- contains every vertex of G.

A **spanning tree** of G is a connected spanning forest of G.

Example



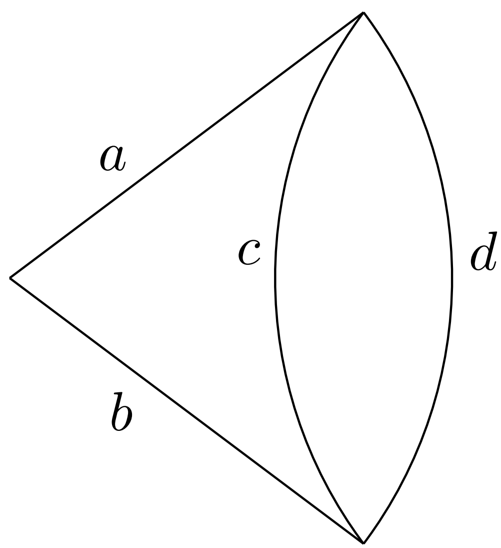
The edges a and c form a spanning tree of

The Kirchhoff polynomial

Associate a variable a_e to each edge e of G. The **Kirchhoff polynomial** of G is

$$\Psi_G = \sum_{\substack{T \text{ spanning} \\ \text{tree of } G}} \prod_{e \notin T} a_e$$

For example

$G =$ 

spanning trees $= \{ac, ad, bc, bd, ab\}$

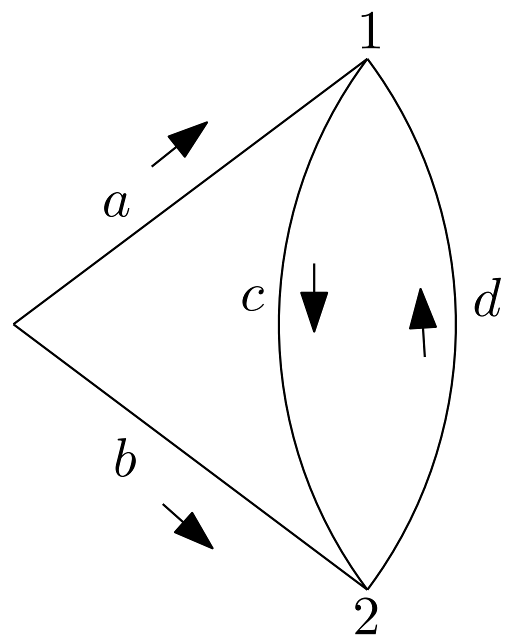
$$\Psi_G = bd + bc + ad + ac + cd = (c+d)(a+b) + cd$$

The matrix-tree theorem

Orient the edges of G and let E be the incidence matrix of G. The Laplacian of G is $L = EE^T$. Then the number of spanning trees of G is the absolute value of any cofactor of L.

Continuing the example

$G =$



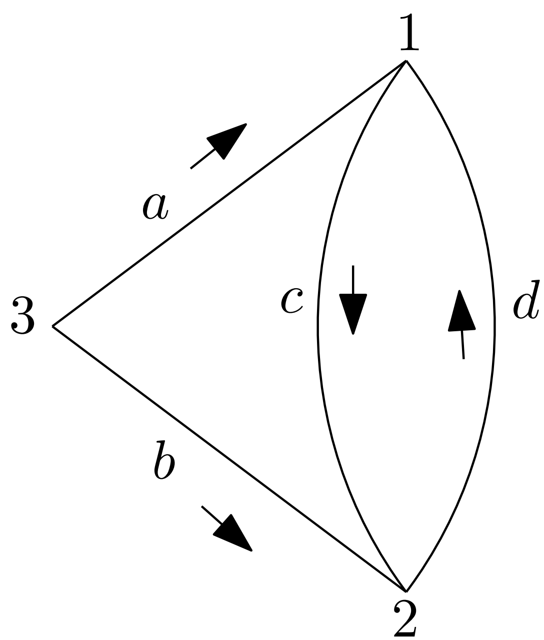
$$E = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ -1 & -1 & 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 3 & -2 & -1 \\ -2 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

G has $\left| \det \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \right| = 5$ spanning trees.

The dual Kirchhoff from a matrix

By putting the variables into the Laplacian we build a monomial for each spanning tree rather than simply counting which gives the dual Kirchhoff polynomial.

$G =$ 

$$L = \begin{bmatrix} a+c+d & -c-d & -a \\ -c-d & b+c+d & -b \\ -a & -b & a+b \end{bmatrix}$$

$$\det \begin{bmatrix} b+c+d & -b \\ -b & a+b \end{bmatrix} = ab + ac + ad + bc + bd = abcd\Psi_G(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d})$$

A variant

Suppose G has n vertices and m edges. Let $\widehat{E}$ be the incidence matrix with one column removed. Build the matrix

$$M = (-1)^{n+1} \left[\begin{array}{ccc|c} a_1 & & & \\ & \ddots & & \widehat{E} \\ & & a_m & \\ \hline & \widehat{E}^T & & 0 \end{array}\right]$$

Then

$$\Psi_G = \det(M)$$

Seeing the Kirchhoff polynomial as a determinant makes it unsurprising that it appears after an appropriate change of variable in integrals – in Feynman integrals.

Feynman graphs

Feynman graphs are graphs (up to isomorphism) formed out of half edges, with multiple edges and self loops permitted.

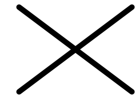
Pairs of adjacent half edges are called internal edges, the remaining half edges are called external edges.

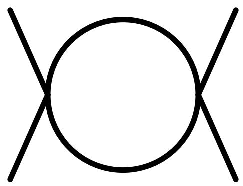
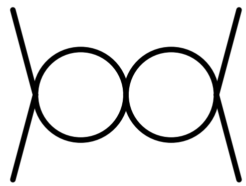
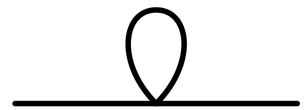
Graphs are said to be 1-particle irreducible (1PI) if they are 2-edge connected.

The physical theory gives permissible types of edges and vertices.

The edges represent particles; the vertices, their interactions. Different theories discuss different particles and interactions.

Feynman graphs in ϕ^4

In ϕ^4 there is one edge type, , and one vertex type,  .
Divergent 1PI Feynman graphs:

 ,  ,  , ...

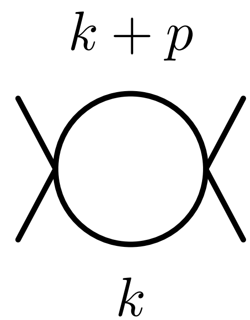
Feynman integrals

Feynman rules associate to a graph in a theory a formal integral. Each internal edge and vertex is associated to a factor of the integrand. Each independent cycle contributes an integration variable.

Associate to each internal and external edge a variable (the momentum) subject to momentum conservation at each vertex.

In massless ϕ^4, the vertices contribute nothing to the integrand; an edge with momentum k contributes a factor of $\frac{1}{k^2}$

For example



is associated to

$$\int d^4k \frac{1}{(k^2)(p+k)^2}.$$

Feynman periods

Use (Schwinger parameters)

$$\int_0^\infty da\, e^{-ak^2} = \frac{1}{k^2}$$

on each factor of the integrand. Integrate the Gaussian part. This leaves

$$\int_{e_i \geq 0} \frac{\prod de_i}{\Phi^2}$$

where Φ is the Kirchhoff polynomial of the graph.

Now this feels more like something an algebraist (or a combinatorialist or an algebraic geometer) would like.

Spanning forest calculus

Joint work with Francis Brown.

Let G be a connected graph. Colour some of its vertices. Let C be the colouring.

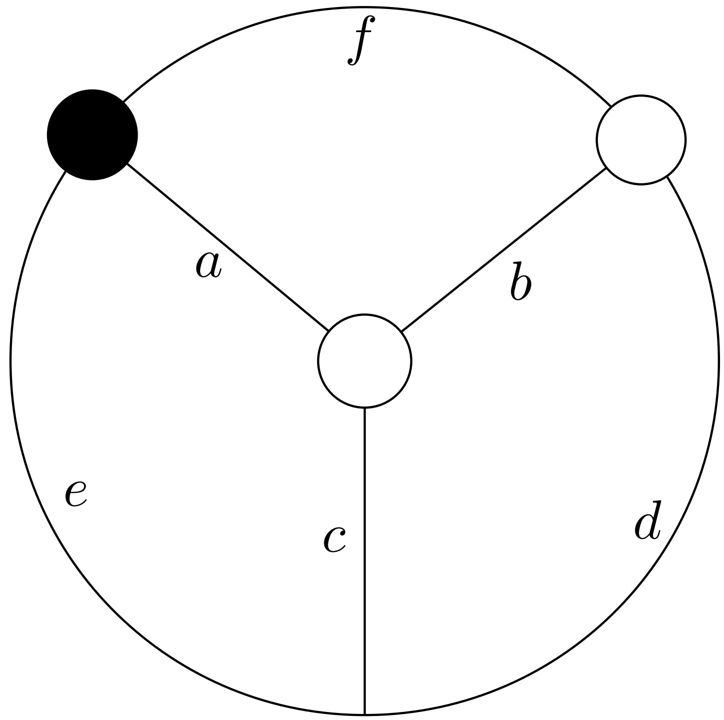
Rather than spanning trees we are now interested in spanning forests with the properties that

- there are exactly as many trees as colours and
- vertices which are the same colour are in the same tree.

Let

$$\Psi_G^C = \sum_{F \text{ as above}} \prod_{e \notin F} a_e$$

Example

G with $C =$ 

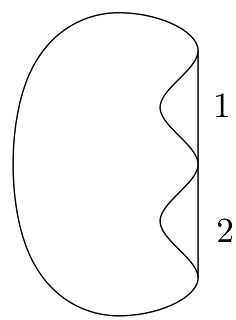
allowable spanning forests $= \{be, bc, bd, cd\}$ $\Psi_G^C = af(cd+ed+ec+be)$

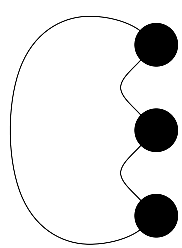
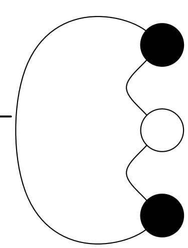
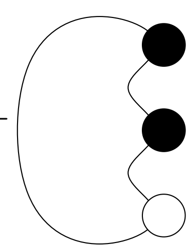
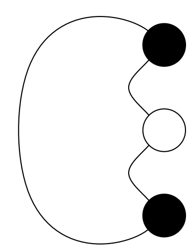
Minors

Recall

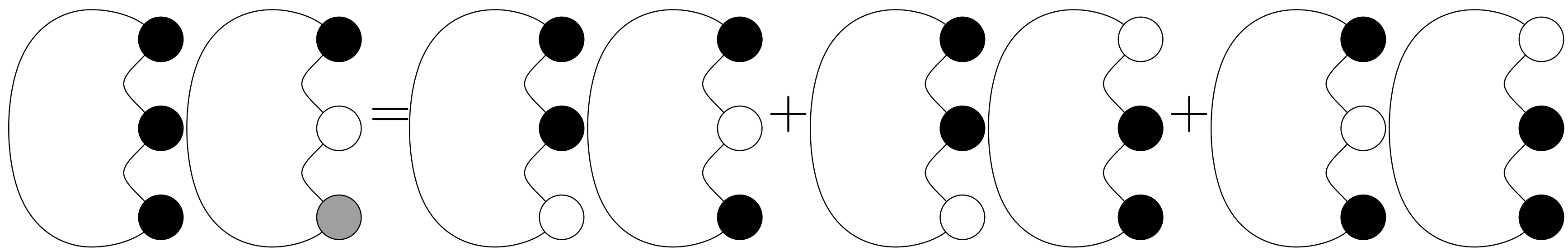
$$M = (-1)^{n+1}\left[\begin{array}{ccc|c} a_1 & & & \\ & \ddots & & \widehat{E} \\ & & a_m & \\ \hline & \widehat{E}^T & & 0 \end{array}\right] \qquad \text{and} \qquad \Psi_G = \det(M)$$

Minors of M can be interpreted in terms of spanning forests. For example let

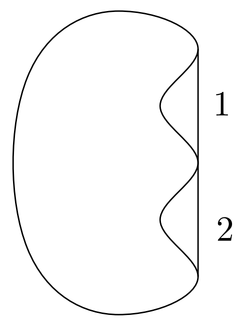
$G =$ 

$M(1,1) \rightsquigarrow a_2$  $+$  $+$  $\qquad M(1,2) \rightsquigarrow$ 

Edge-transferal lemma



Proof via M and its minors with

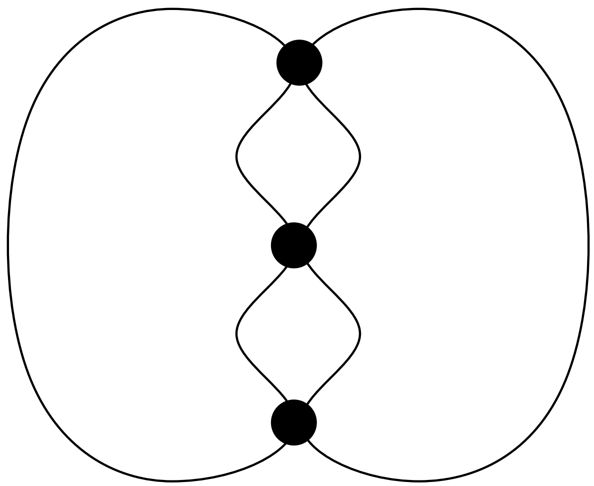
$G =$ 

and the so-called Dodgson identity

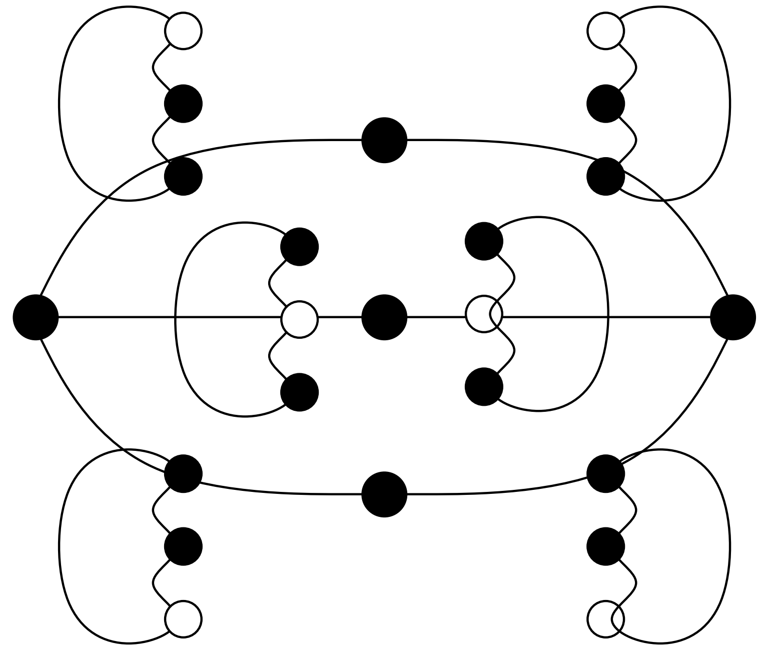
$$\det(M)\det(M(12,12)) = \det(M(1,1))\det(M(2,2)) - \det(M(1,2)M(2,1))$$

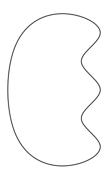
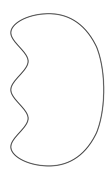
Is there a bijective proof?

3-join theorem



$=$



Where the underlying variables have been scaled by appropriate powers of  and .

Proof by previous lemma.

Tying things together

Suppose we want to integrate Feynman integrals

$$\int_{e_i \geq 0} \frac{\prod de_i}{\Phi^2}$$

one edge variable at a time (see work of Francis Brown).

So long as there is always a variable e so that the numerator is a product of two linear polynomials in e,

$$(Ae + B)(Ce + D),$$

then we can do the integral, getting increasingly complex polylogarithms in the numerator and

$$AD - BC$$

in the denominator.

The key is a purely combinatorial game on factoring polynomials. The polynomials and their identities are captured in a spanning tree calculus.

Next time

Evaluating these polylogarithms give the famous multiple zeta values of Feynman integrals of this conference.

Ideally we'd like to predict the multiple zeta values from combinatorics of the graph. This is too hard.

Perhaps we can predict some useful feature of the multiple zeta values, such as the weight. Still hard, but hopeful. Weight drops will occur when the denominator becomes a square. So far we have found a non-trivial infinite family of graphs where the weight drops.