Recent progress on an arithmetic graph invariant with applications in quantum field theory.

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Numbers in Feynman integrals.

As many of you know, interesting numbers show up in perturbative quantum field theory calculations.

These *interesting numbers* include

multiple zeta values

other values of multiple polylogarithms

elliptic generalizations and onwards

What are these calculations?

- In perturbative quantum field theory you calculate physical things using series indexed by Feynman diagrams – certain graphs describing particle interactions.
- Each Feynman diagram contributes an integral where the integrand is built out of pieces corresponding to edges and vertices in the graph.
- Or you can try to be more clever and index the sum in other ways.

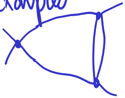
Where do the interesting numbers appear?

These numbers appear almost no matter what you do.

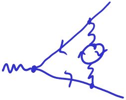
- These numbers appear in Feynman integrals for different graphs.
- They appear in essentially any interesting choice of quantum field theory

two examples

ϕ^4



QED



- They appear in the sum (not just in the individual graphs).
- There are patterns to how they appear, but it is still mysterious.

Why do the numbers appear?

Why indeed?

MZVs should come from mixed Tate motives etc.

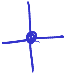
There must be some good mathematical reason.

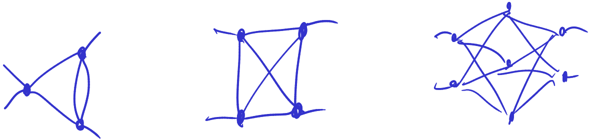
That would be interesting mathematics and it would explain something about Feynman integrals and hence let us do more physics too.

Keep it simple

One example of a quantum field theory is ϕ^4

one edge type

one vertex 

eg 

These will be our graphs today.

The Kirchhoff polynomial

Let K be a connected 4-regular graph
Let $G = K - v$. These are connected ϕ^4 graphs with 4 external edges.

Assign a variable for each edge a_e for edge e

Define

$$\Psi_G = \sum_{\substack{T \\ \text{spanning tree of } G}} \prod_{\substack{e \notin T \\ \text{internal}}} a_e$$

Eg:

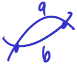
Ψ  $= cd + ad + bd + bc + ac$

The Feynman period

Rather than consider the full Feynman integral and what numbers and functions of the parameters show up, we'll just think about:

$$P_G = \int_{a_e \geq 0} \frac{da_1 \cdots da_{n-1}}{\psi_G^2}\Big|_{a_n=1}$$

Eg:



$$\psi = a+b$$

$$\int_{a \geq 0} \frac{1}{(a+b)^2}$$

Period – geometry – arithmetic

to understand P_G without doing the integral

I want to understand the geometry of $\Psi_G = 0$

Another direction into this geometry, count points on Ψ_G over finite fields

The c_2 invariant

For $f \in \mathbb{Z}[x_1, \ldots, x_n]$ define $[f]_q$ to be the number of $\mathbb{F}_q$-rational points on the variety $f = 0$.

Define

$$c_2^{(p)}(G) = \frac{[\Psi_G]_p}{p^2} \mod p$$

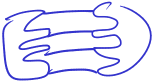
Arithmetic structure

- If $c_2^{(p)}(G)$ is independent of p then P_G should be MZV.
- If $c_2^{(p)}(G) = 0$ then P_G should have less than maximal transcendental weight.
- If $c_2^{(p)}(G)$ is constant in some field extension then P_G should be a multiple polylogarithm evaluated at the roots.
- Some $c_2^{(p)}(G)$ are proven to be coefficient sequences of modular forms.
- In this case P_G should be more exotic.

Known graph-related properties

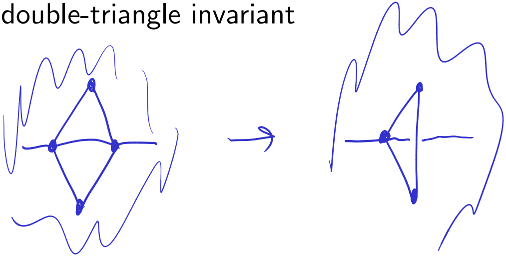
the 4 regular graph

- If K has a 3-separation then $c_2^{(p)}(G) = 0$.
- If K has an internal 4-edge-cut then $c_2^{(p)}(G) = 0$.



K-v

- If G has vertex width 3 then $c_2^{(p)}$ is a constant.
- c_2 is double-triangle invariant



Known and conjectured symmetries

P_G

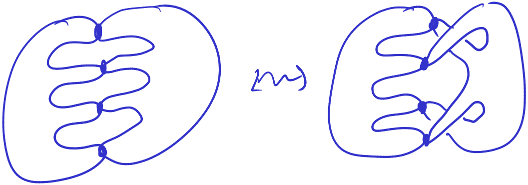
The period is proven to be invariant under

- Completion/decompletion

  $G = K - v$ ← what v doesn't matter

- Planar duality for G
- Schnetz twist

on K



The c_2 invariant should have these symmetries as well.

A result

Brown and Schnetz conjecture that for all p, 4-regular K, $v_1, v_2 \in V(K)$

$$c_2^{(p)}(K - v_1) = c_2^{(p)}(K - v_2)$$

I prove that if K has an odd number of vertices, $v_1, v_2 \in V(K)$, then

$$c_2^{(2)}(K - v_1) = c_2^{(2)}(K - v_2)$$

Expanded Laplacian and more polynomials

Let

$$M_G = \begin{bmatrix} \Lambda & E^T \\ -E & 0 \end{bmatrix}$$

$$E = \begin{array}{cc} & \text{edges} \\ \text{vertices} & \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{array}$$

where $\Lambda = \operatorname{diag}(a_1, a_2, \dots, a_n)$ and E is the signed incidence matrix with one row removed.

Then as another way to view the matrix tree theorem we have

$$\Psi_G = \det M_G$$

$I \cap J$ is deletion

remove rows indexed by I

remove columns indexed by J

We also care about minors

$$\Psi^{I,J}_{G,K} = \det M_G(I, J)|_{a_e=0, e\in K}$$

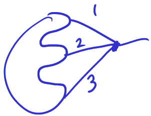
contraction

One known result we need

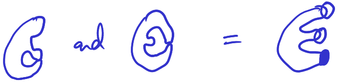
Proposition (Brown and Schnetz)

$$c_2^{(p)}(G) = [\Psi^{1,2}_{G,3}\Psi^{13,23}_G]_p \quad \text{mod } p$$

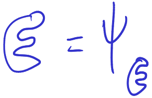
In particular if edges $1, 2, 3$ meet at a 3-valent vertex:



$\psi^{1,2}_{3}$ = edges common to



$\psi^{13,23}$ =



Another known result we need

Proposition (Corollary of Chevalley-Warning)

If f has total degree n in $x_1, x_2, \ldots, x_n$ then

$$[f]_p = \textit{coefficient of } x_1^{p-1} \cdots x_n^{p-1} \textit{ in } f^{p-1} \mod p$$

Reduction to counting certain edge bipartitions

Apply these to our situation.

$$c_2^{(2)}(G) = [\Psi^{1,2}_{G,3}\Psi^{13,23}_{G}]_2 \quad \bmod 2$$

$$= \text{coefficient of } x_1 \cdots x_n \text{ in } \cancel{f} \ \cancel{\bmod 2} \ \Psi^{1,2}_3 \Psi^{13,23} \bmod 2$$

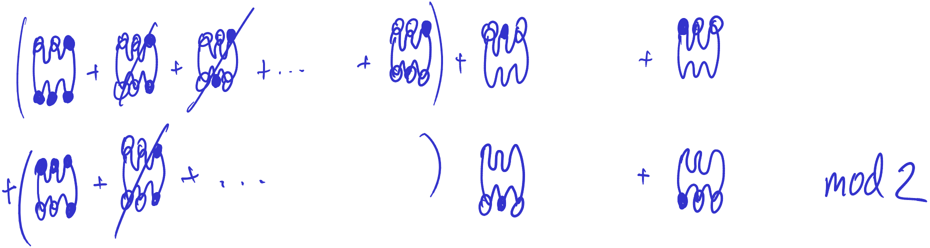
let $G =$ 

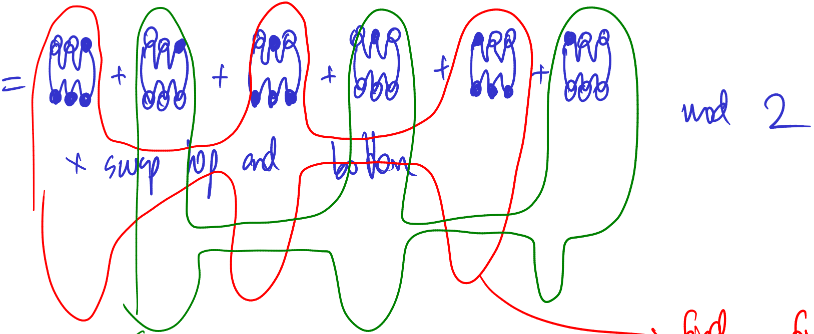
$H =$ 

$=$ number of ways of partitioning the edges of H so that one part is a spanning tree and one is a spanning forest corr. to

Proof sketch

similar but easier if v_1, v_2 have common neighbours





find a fixed point free involution so evenly many

couldn't find something like

instead can swap around cycles

build an auxiliary graph where the vertices are the things I want to count and joined if you can swap between them

This auxiliary graph has all vertices of odd deg (there even # of them)

when K has an odd number of vertices