Quantum field theory, algebraic geometry, and graph theory

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Numbers in Feynman integrals.

Interesting numbers show up in perturbative quantum field theory calculations.

These *interesting numbers* include

multiple zeta values
evaluations of multiple polylogarithms
elliptic-polylogarithms
and more

What are these calculations?

- In perturbative quantum field theory you calculate physical things using series indexed by Feynman diagrams – certain graphs describing particle interactions.

- Each Feynman diagram contributes an integral where the integrand is built out of pieces corresponding to edges and vertices in the graph.
- Or you can try to be more clever and index the sum in other ways.

Where do the interesting numbers appear?

These numbers appear almost no matter what you do.

- These numbers appear in Feynman integrals. Which graphs produce which ~~integrals~~ numbers?
- They appear in essentially any interesting choice of quantum field theory

QED

scalar

- They appear in the sum (not just in the individual graphs).
- They appear in other approaches to perturbative quantum field theory.
- There are patterns to how they appear, but it is still mysterious.

Why do the numbers appear?

Why indeed?

There must be some good mathematical reason.

That would be interesting mathematics and it would explain something about Feynman integrals and hence let us do more physics too.

Keep it simple

One example of a quantum field theory is ϕ^4

one edge type

one vertex type all vertices deg 4 (but external edges are allowed)

eg

etc.

These will be our graphs today.

The Kirchhoff polynomial

Let K be a connected 4-regular graph
Let $G = K - v$. These are connected ϕ^4 graphs with 4 external edges.

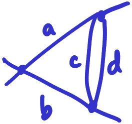
Define

associate a variable to each edge of the graph, a_e for edge e

$$\Psi_G = \sum_T \prod_{e \notin T} a_e$$

, running over spanning trees

Eg:



$$\Psi = bd + bc + ad + ac + cd$$

The Feynman period

Rather than consider the full Feynman integral and what numbers and functions of the parameters show up, we'll just think about:

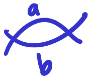
$$P_G = \int_{a_e \geq 0} \frac{\Omega}{\Psi^2}$$ projective form

$$\Omega = \sum_{i=1}^{|E(G)|} (-1)^i da_1 \wedge \cdots \wedge \widehat{da_i} \wedge \cdots$$

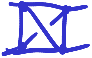
Eg:

or take an affine piece

$$\int_{a_e \geq 0} \frac{da_1 \cdots da_{|E(G)|-1}}{\Psi^2}\Bigg|_{a_{|E(G)|}=1}$$



$\Psi = a+b$ $\quad P = \int_{a=0}^{\infty} \frac{da}{(a+1)^2}$

more exciting  K_4 $\quad P_{K_4} \doteq 6\zeta(3)$

Sketch of Brown's approach to integration

Francis Brown gave an algorithm for how to integrate some of these. There will be multiple polylogarithms in the numerator and polynomials in the denominator.

sketch

$$\int \frac{1}{\Psi_G^2} \to \int \frac{1}{\Psi_{G\setminus e_1}\Psi_{G/e_1}}$$

$$\to \int \frac{\text{logs}}{\text{poly.}}$$ ← turns out by Dodgson to be a square

dilog: $Li_2(z) = \sum_{n\geq 1} \frac{z^n}{n^2}$

$$\to \int \frac{\text{more logs}}{\text{poly}}$$

$Li_k(z) = \sum_{n\geq 1} \frac{z^n}{n^k}$

$$\to \int \frac{\text{dilogs}}{\text{poly}} \to \int \frac{\text{trilogs}}{\text{poly}}$$ $(*)$

multiple polylogs $\sum_{n_1 \geq n_2 \geq \cdots \geq 1} \frac{z^{n_1}}{n_1^{k_1}\cdots n_\ell^{k_\ell}}$

You can't always continue because the polynomial here (*) doesn't necessarily factor

in general $\displaystyle\int \frac{\text{polylog of weight } n}{\text{poly}}$

if poly factors into distinct lin factors

get $\displaystyle\int \frac{\text{polylog of weight } n+1}{\text{disc}}$

if square in denom lower weight

if doesn't factor die.

Period – geometry – arithmetic

$\int \frac{1}{\Psi^2}$ should be controlled by the geometry of $\Psi = 0$

we have a different perspective on this geometry by counting points over finite fields on $\Psi = 0$

The c_2 invariant

For $f \in \mathbb{Z}[x_1, \ldots, x_n]$ define $[f]_q$ to be the number of $\mathbb{F}_q$-rational points on the variety $f = 0$.

Define

$$c_2^{(p)}(G) = \frac{[\Psi_G]_p}{p^2} \mod p$$

note if $[\Psi_G]_p$ were polynomial in p

then $c_2^{(p)}(G)$ would be the quadratic coeff and so independent of p

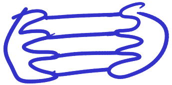
Arithmetic structure

- If $c_2^{(p)}(G)$ is independent of p then P_G should be MZV.
- If $c_2^{(p)}(G) = 0$ then P_G should have less than maximal transcendental weight.
- If $c_2^{(p)}(G)$ is constant in some field extension then P_G should be a multiple polylogarithm evaluated at the roots.
- Some $c_2^{(p)}(G)$ are $\boxed{\text{proven}}$ to be coefficient sequences of modular forms.
- In this case P_G should be more exotic.

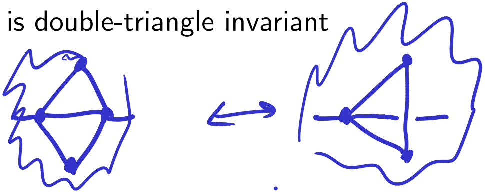
Known graph-related properties

proven

- If $\boxed{K}$ has a 3-separation then $c_2^{(p)}(G) = 0$.
- If K has an internal 4-edge-cut then $c_2^{(p)}(G) = 0$.
- If G has vertex width 3 then $c_2^{(p)}$ is a constant.
- c_2 is double-triangle invariant



more than one vertex on each side



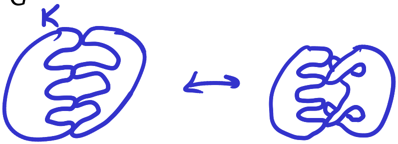
Known and conjectured symmetries

The period is proven to be invariant under

- Completion/decompletion

  ie given K 4-regular no matter which vertex we remove to get G get same P_G

- Planar duality for G
- Schnetz twist



The c_2 invariant should have these symmetries as well.

all calculated evidence supports this

Expanded Laplacian and more polynomials

We need some definitions to obtain our combinatorial rephrasings. Let

$$M_G = \begin{bmatrix} \Lambda & E^T \\ -E & 0 \end{bmatrix}$$

where $\Lambda = \mathrm{diag}(a_1, a_2, \ldots, a_n)$ and E is the signed incidence matrix with one row removed.

Then as another way to view the matrix tree theorem we have

$$\Psi_G = \det M_G$$

We also care about minors

$$\Psi^{I,J}_{G,K} = \det M_G(I,J)|_{a_e=0, e\in K}$$

Brown's denominator reduction revisited

Brown's integration algorithm is controlled by the denominators. These are polynomials with combinatorial meaning.

Let's revisit the sketch of the algorithm.

Spanning forest polynomials

from the minors

These polynomials can all be rewitten as sums over spanning forests. (All minors matrix tree theorem)

Eg if edges $1, 2, 3$ meet at a 3-valent vertex:

$\Psi_3^{1,2} =$ spanning forests  ← colouring says vertices of the same colour are in the same tree of the spanning forest

$\Psi^{13,23} =$

A result

Brown and Schnetz conjecture that for all p, 4-regular K, $v_1, v_2 \in V(K)$

$$c_2^{(p)}(K - v_1) = c_2^{(p)}(K - v_2)$$

I prove that if K has an odd number of vertices, $v_1, v_2 \in V(K)$, then

$$c_2^{(2)}(K - v_1) = c_2^{(2)}(K - v_2)$$

Two known results we need

Proposition (Brown and Schnetz)

$$c_2^{(p)}(G) = [\Psi^{1,2}_{G,3}\Psi^{13,23}_{G}]_p \mod p$$

Proposition (Corollary of Chevalley-Warning)

If f has total degree n in $x_1, x_2, \ldots, x_n$ then

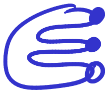
$$[f]_p = \textit{coefficient of } x_1^{p-1}\cdots x_n^{p-1} \textit{ in } f^{p-1} \mod p$$

Reduction to counting certain edge bipartitions

Apply these to our situation.

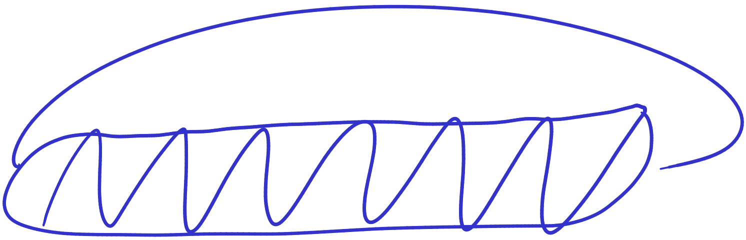
$$
\begin{aligned}
c_2^{(2)}(G) &= [\Psi_{G,3}^{1,2}\Psi_G^{13,23}]_2 \quad \text{mod } 2 \\
&= \text{coefficient of } x_1 \cdots x_n \text{ in } \Psi_{G,3}^{1,2}\Psi_G^{13,23} \quad \text{mod } 2
\end{aligned}
$$

= # of bipartitions of the edges
one giving a spanning tree of $G \setminus 123$
other giving a forest compatible with



mod

Proof sketch



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