# Recursion and growth estimates in quantum field theory

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## Some recursive equations

Start in the middle

$$\gamma_k(x) = \frac{1}{k}\gamma_1(x)(1 + rx\partial_x)\gamma_{k-1}(x)$$

$$\gamma_{1,n} = p(n) + \sum_{j=1}^{n-1}(-rj-1)\gamma_{1,j}\gamma_{1,n-j}$$

How do we get these? How do we analyze them? What does it mean for quantum field theory?



## A Dyson-Schwinger equation

In the context of renormalization Hopf algebras consider

$$X(x) = \mathbb{I} - \sum_{k\geq 1} x^k p(k) B_+^k(X(x)Q(x)^k)$$

where $Q(x) = X(x)^r$ with $r < 0$ an integer. This carries the combinatorial information.

Consider the integral kernels for each $B_+$, namely the Mellin transforms

$$F^k(\rho_1, \ldots, \rho_s).$$

This adds the analytic information.

Write the combination ($X \mapsto G$, $B_+^k \mapsto F^k$) as $G(x, L) = \sum \gamma_k(x)L^k$ with $\gamma_k(x) = \sum_{j\geq k}\gamma_{k,j}x^j$.

Working with systems of equations only increases technical messiness.



## Example

From Broadhurst and Kreimer [1].

$$X(x) = \mathbb{I} - xB_+\left(\frac{1}{X(x)}\right).$$

So $Q(x) = 1/X(x)^2$ Combinatorially counts rooted trees.

$$F(\rho) = \frac{1}{q^2}\int d^4k \frac{k\cdot q}{(k^2)^{1+\rho}(k+q)^2} - \cdots \Big|_{q^2=\mu^2}$$

Combine to get

$$G(x, L) = 1 - \frac{x}{q^2}\int d^4k \frac{k\cdot q}{k^2 G(x, \log k^2)(k+q)^2} - \cdots|_{q^2=\mu^2}$$

where $L = \log(q^2/\mu^2)$. The (analytic) Dyson-Schwinger equation for a bit of massless Yukawa theory.

## The linearized coproduct

Define

$$\Delta_{\mathsf{lin}} = (P_{\mathsf{lin}} \otimes P_{\mathsf{lin}})\Delta$$

or in general

$$\Delta_{\mathsf{lin}}^{n-1} = \underbrace{(P_{\mathsf{lin}} \otimes \cdots \otimes P_{\mathsf{lin}})}_{n} \Delta^{n-1}$$

where $P_{\mathsf{lin}}$ projects onto the linear part of the Hopf algebra, that is, kills disjoint unions of graphs.

By the Hochschild closedness of $B_+$ we get

$$\Delta_{\mathsf{lin}} X = P_{\mathsf{lin}} X \otimes P_{\mathsf{lin}} X + P_{\mathsf{lin}} Q \otimes x\partial_x X$$

where $P_{\mathsf{lin}} Q = r P_{\mathsf{lin}} X$



## Extracting $\gamma_k(x)$ with $S \star Y$

Writing the (analytic) Dyson-Schwinger equation as

$$G(x, L) = \sum \gamma_k(x) L^k,$$

we know from Connes and Kreimer [2] that if

$$\sigma_1 = \partial_L \phi(S \star Y)|_{L=0}$$

and

$$\sigma_n = \frac{1}{n!} m^{n-1} \underbrace{(\sigma_1 \otimes \cdots \otimes \sigma_1)}_{n} \Delta^{n-1}$$

then

$$\gamma_k(x) = \sigma_k(X(x))$$

where $\phi$ is the renormalized Feynman rules, $m$ is multiplication, $S$ is the antipode, and $Y$ is the grading operator.



## Extracting $\gamma_k(x)$ with $\Delta_{\mathsf{lin}}$

But $\sigma_1$ only sees the linear part of the Hopf algebra so we can use $\Delta_{\mathsf{lin}}$ in place of $\Delta$. Giving

$$\gamma_k(x) = \frac{1}{k} \gamma_1(x)(1 + r x \partial_x) \gamma_{k-1}(x),$$

the first of the recursive equations we began with.

In the Yukawa example

$$\gamma_k(x) = \frac{1}{k} \gamma_1(x)(1 - 2x\partial_x) \gamma_{k-1}(x).$$



## The power of primitives

We need not restrict ourselves to connected primitives. We can choose a basis for the primitives which involves only one insertion place.

$p_1 =$ 

$p_2 = \frac{1}{4}$  $- \frac{1}{2}$ 

Mellin transforms become univariate: $F^k(\rho)$.

## More power of primitives

We can also expand $\rho(1-\rho)F^k(\rho)$ as a series making new primitives out of the higher order terms:

$$p_1 = p \qquad p_2 = B_+^p(B_+^p(\mathbb{I})) - \frac{1}{2}B_+^p(\mathbb{I})B_+^p(\mathbb{I})$$

so that

$$F^k(\rho) = \sum \rho^n F^{p_n}(\rho) = \sum \frac{r_n \rho^n}{\rho(1-\rho)}$$

Mellin transforms become geometric series:

$$F^k(\rho) = \frac{r_k}{\rho(1-\rho)}.$$



## Finding the messy $\gamma_1$ recursion

Rewrite the (analytic) Dyson-Schwinger equation

$$\gamma \cdot L = \sum p(k)x^k(1+\gamma\cdot\partial_{-\rho})^{-rk+1}(1-e^{-L\rho})F^k(\rho)\bigg|_{\rho=0}$$

where $\gamma \cdot U = \sum \gamma_k U^k$.

Take an $L$ derivative and set $L=0$ to get

$$\gamma_1 = \sum p(k)x^k(1+\gamma\cdot\partial_{-\rho})^{-rk+1}\rho F^k(\rho)\bigg|_{\rho=0}$$

This determines $\gamma_1$ recursively, but messily.



## Using the geometric series

We had

$$\gamma \cdot L = \sum p(k)x^k(1+\gamma\cdot\partial_{-\rho})^{-rk+1}(1-e^{-L\rho})F^k(\rho)\bigg|_{\rho=0}$$

$$\gamma_1 = \sum p(k)x^k(1+\gamma\cdot\partial_{-\rho})^{-rk+1}\rho F^k(\rho)\bigg|_{\rho=0}$$

Now use $\rho F^k(\rho) = r_k(1+\rho+\rho^2+\cdots)$. Take two $L$ derivatives of the DSE and set $L=0$ to get

$$2\gamma_2 = -\sum_k p(k)x^k(1+\gamma\cdot\partial_{-\rho})^{-rk+1}r_k(1+\rho+\rho^2+\cdots)\bigg|_{\rho}$$
$$= -\gamma_1 + \sum x^k p(k) r_k$$

Suck $r_k$ into the definition of $p(k)$ giving

$$\gamma_1 = \sum p(k)x^k - 2\gamma_2$$



## Finding the nice $\gamma_1$ recursion

We had

$$\gamma_1 = \sum p(k)x^k - 2\gamma_2$$

and the other recursion

$$\gamma_k(x) = \frac{1}{k}\gamma_1(x)(1+rx\partial_x)\gamma_{k-1}(x).$$

Together

$$\gamma_1 = \sum p(k)x^k - \gamma_1(1+rx\partial_x)\gamma_1,$$

or at the level of coefficients

$$\gamma_{1,n} = p(n) + \sum_{j=1}^{n-1}(-rj-1)\gamma_{1,j}\gamma_{1,n-j}.$$

the second of the recursive equations we began with.

## Growth of $\gamma_1$

How bad is the growth of $\gamma_1$?

Assume $\gamma_{1,1} \neq 0$ and $f(x) = \sum \frac{p(n)}{n!} x^n$ has nonzero radius of convergence $\rho$.

Let $a(n) = \frac{\gamma_{1,n}}{n!}$. The recursion becomes

$$\begin{aligned} a_n &= \frac{p(n)}{n!} + \sum_{i=1}^{n-1} (-ri-1) a_i a_{n-i} \binom{n}{i}^{-1} \\ &= \frac{p(n)}{n!} + \left(-\frac{rn}{2} - 1\right) \sum_{i=1}^{n-1} a_i a_{n-i} \binom{n}{i}^{-1} \end{aligned}$$

Idea:

$$a(n) \text{ is approximately } \frac{p(n)}{n!} - r a_1 a_{n-1}$$

giving a radius of $\min\left\{\rho, \frac{1}{-ra_1}\right\}$ for $\sum a_n x^n$.

Implement the idea by bounding on each side.



## Lower bound on $a_n$

Recall

$$a_n = \frac{p(n)}{n!} + \left(-\frac{rn}{2} - 1\right) \sum_{i=1}^{n-1} a_i a_{n-i} \binom{n}{i}^{-1}$$

so

$$a_n \geq \frac{p(n)}{n!} - r\frac{n-2}{n} a_1 a_{n-1}$$

Let $b_1 = a_1$, $\mathbf{B}(x) = \sum b_n x^n$ and

$$b_n = \frac{p(n)}{n!} - r\frac{n-2}{n} b_1 b_{n-1}$$

Then $\mathbf{B}''(x) = f''(x) - r b_1 x \mathbf{B}''(x)$ which can be solved for $\mathbf{B}''(x)$ to give radius $\min\left\{\rho, \frac{1}{-ra_1}\right\}$ for $\mathbf{B}(x)$.



## Upper bound on $a_n$

Recall

$$a_n = \frac{p(n)}{n!} + \left(-\frac{rn}{2} - 1\right) \sum_{i=1}^{n-1} a_i a_{n-i} \binom{n}{i}^{-1}$$

so for any $\epsilon > 0$ there is an $N > 0$ such that for $n > N$

$$a_n \leq \frac{p(n)}{n!} - r a_1 a_{n-1} + \epsilon \sum_{j=1}^{n-1} a_j a_{n-j}$$

Let $c_1 = a_1$, $\mathbf{C}(x) = \sum c_n x^n$,

$$c_n = \frac{p(n)}{n!} - r c_1 c_{n-1} + \epsilon \sum_{j=1}^{n-1} c_j c_{n-j}$$

if this is greater than $a_n$ and $c_n = a_n$ otherwise. Then

$$\mathbf{C}(x) = f(x) - r a_1 x \mathbf{C}(x) + \epsilon \mathbf{C}(x)^2 + P_\epsilon(x)$$

where $P_\epsilon$ is a polynomial to deal with initial terms.



## The radius of $\mathbf{C}(x)$

We have

$$\mathbf{C}(x) = f(x) - r a_1 x \mathbf{C}(x) + \epsilon \mathbf{C}(x)^2 + P_\epsilon(x)$$

The radius comes from the discriminant

$$(1 + r a_1 x)^2 - 4\epsilon(f(x) + P_\epsilon(x))$$

Clear poles

$$\frac{(1 + r a_1 x)^2}{f(x)} - 4\epsilon\left(1 + \frac{P_\epsilon(x)}{f(x)}\right)$$

Technical computation gives that $P_\epsilon(x)/f(x)$ is bounded as $\epsilon \to 0$ so conclude that the radius of $\mathbf{C}(x)$ is

$$\min\left\{\rho, \frac{1}{-ra_1}\right\}.$$

## Why?

Understanding the growth of $\gamma_1$ is understanding the growth of the whole theory.

Expect a Lipatov bound $\gamma_{1,n} \leq c^n n!$.

Does the first singularity of $\sum \frac{\gamma_{1,n}}{n!} x^n$ come from renormalon chains or from instantons?

We've shown that a Lipatov bound for the primitives leads to a Lipatov bound on the whole theory.

The radius is either the radius from the primitives or $\frac{1}{-r\gamma_{1,1}}$, the first coefficient of the beta function.

The moral is that the primitives control matters.



## References


[1] D.J. Broadhurst and D. Kreimer, Exact solutions of Dyson-Schwinger equations for iterated one-loop integrals and propagator-coupling duality. Nucl.Phys. B **600**, (2001), 403-422. (Also arXiv:hep-th/0012146).

[2] Alain Connes and Dirk Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem II, Commun.Math.Phys. **216** (2001) 215-241. (Also arXiv:hep-th/0003188)

[3] Dirk Kreimer and Karen Yeats, An Étude in non-linear Dyson-Schwinger Equations. Nucl. Phys. B Proc. Suppl., **160**, (2006), 116-121. (Also arXiv:hep-th/0605096.)

[4] Dirk Kreimer and Karen Yeats, Recursion and Growth Estimates in Renormalizable Quantum Field Theory. arXiv:hep-th/0612179.