Dyson-Schwinger equations and Renormalization Hopf algebras

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April 10, 2007
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Unfolding some recursive equations

Lets get our intuition going

$$X = \mathbb{I} + xB_+(X^2)$$

What does this count?

$$X = \mathbb{I} + xB_+(X^3)$$

What does this count?

$$X = \mathbb{I} - xB_+\left(\frac{1}{X}\right)$$

What does this count?

Answers

$$X = \mathbb{I} + xB_+(X^2)$$

counts computer science binary trees (separate slots for left and right children).

$$X = \mathbb{I} + xB_+(X^3)$$

counts ternary trees with separate slots for left, middle, and right children.

$$X = \mathbb{I} - xB_+\left(\frac{1}{X}\right)$$

counts plane rooted trees.

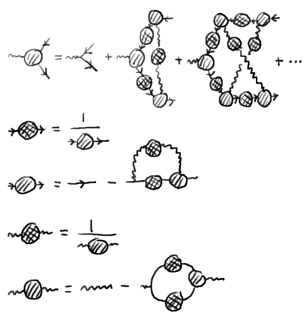
Dyson-Schwinger equations combinatorially

As the simple tree examples, or systems

$$X^r(x) = \mathbb{I} - \sum_{k\geq 1} x^k p_r(k) B_+^{k,r}(X^r(x)Q(x)^k)$$

where $Q(x) = \prod X^r(x)^{s_r}$ and r runs over the different external leg structures.

Example: QED



Dyson-Schwinger equations analytically

Example from Broadhurst and Kreimer [3].

$$X(x) = \mathbb{I} - xB_+\left(\frac{1}{X(x)}\right).$$

along with

$$F(\rho) = \frac{1}{q^2}\int d^4k \frac{k\cdot q}{(k^2)^{1+\rho}(k+q)^2} - \cdots \Big|_{q^2=\mu^2}$$

gives ($X \mapsto G$, $B_+ \mapsto F$)

$$G(x,L) = 1 - \frac{x}{q^2}\int d^4k \frac{k\cdot q}{k^2 G(x, \log k^2)(k+q)^2} - \cdots |_{q^2=\mu^2}$$

where $L = \log(q^2/\mu^2)$. The (analytic) Dyson-Schwinger equation for a bit of massless Yukawa theory.

Dyson-Schwinger equations physically

Equations of motion, analogous to the classical differential equations of motion.

By expanding in the coupling constant Dyson-Schwinger equations give perturbation theory.

But Dyson-Schwinger equations also contain non-perturbative information if we can extract it. Broadhurst and Kreimer [3] solved

$$G(x,L) = 1 - \frac{x}{q^2}\int d^4k \frac{k\cdot q}{k^2 G(x, \log k^2)(k+q)^2} - \cdots |_{q^2=\mu^2}$$

where $L = \log(q^2/\mu^2)$ parametrically with

$$G(x,L) = \frac{\sqrt{x}}{\exp(p^2)\mathrm{erfc}(p)} \qquad q^2 = \mu^2\left(\frac{\mathrm{erfc}p}{\mathrm{erfc}p_0}\right)^{1/2}$$

Other physical perspectives: `http://web.mit.edu/redingtn/www/netadv/Xdysonschw.html`

Dyson-Schwinger equations and B_+

The key is B_+.

All the Hopf algebras we're interested in are generated by one or more B_+ and so are the solutions of Dyson-Schwinger equations or quotients thereof.

B_+ is a 1-cocycle

$$\Delta B_+ = (\mathrm{id} \otimes B_+)\Delta + B_+ \otimes \mathbb{I}$$

A subpiece comes from the branches, or is the whole thing. Unique decomposition.

$(\mathcal{H}_{rt}, B_+)$ is universal for Hopf algebras with a 1-cocycle. Connes, Kreimer: [4].

B_+ and the universal law

The 1-cocycle property is the cohomological way to say unique decomposition.

Rooted trees are nice due to the unique decomposition of a tree into its root and the forest of its subtrees: B_+. For unlabelled trees, $\mathbf{T}(x) = \sum t(n)x^n$,

$$\mathbf{T}(x) = x\exp\left(\sum_{m\geq 1}\mathbf{T}(x^m)/m\right).$$

Which by Pólya's classical analysis gives the asymptotics

$$t(n) \sim C\rho^{-n}n^{-3/2}$$

Asymptotics of the form $C\rho^{-n}n^{-3/2}$ are ubiquitous for classes of rooted trees with recursive definitions, hence the term universal law.

Operators giving the universal law

How ubiquitous? Let $\mathcal{O}$ be the set of operators on power series built out of

1. $\mathbf{E}(x, \cdot)$ such that
   (a) $\mathbf{E}(x,y)$ has nonnegative coefficients and zero constant term,
   (b) $\mathbf{E}(a,b) < \infty \Rightarrow \exists \epsilon > 0, \mathbf{E}(a+\epsilon, b+\epsilon) < \infty$,
   (c) $\exists R > 0, [x^i y^j]\mathbf{E}(x,y) \leq R^{i+j}$.
2. MSet_M and Seq_M for all $M \subseteq \mathbb{Z}^{>0}$.
3. DCycle_M and Cycle_M for $\sum_{m\in M} 1/m = \infty$ or M finite.

using scalar multiplication from $\mathbb{R}^{\geq 0}$, addition, multiplication, and composition, and where if MSet_M, DCycle_M, or Cycle_M appear then scalars and coefficients of $\mathbf{E}$ must be integers.

Theorem 1. [Bell, Burris, – [1]] *Let $\Theta \in \mathcal{O}$ such that*

- *Θ is nonlinear*
- *$[x^n]\Theta(\mathbf{A}(x))$ depends only on $[x^i]\mathbf{A}(x)$ for $i < n$.*

Let $\mathbf{A}(x)$ be a power series

- *with nonnegative coefficients*
- *with zero constant term*
- *which diverges at its radius of convergence*
- *if MSet_M, DCycle_M, or Cycle_M appear in Θ then $\mathbf{A}(x)$ has integer coefficients.*

Then there is a unique $\mathbf{T}(x)$ satisfying

$$\mathbf{T}(x) = \mathbf{A}(x) + \Theta(\mathbf{T})(x).$$

The coefficients of $\mathbf{T}$ satisfy the universal law on their support.

B_+ and the first recursion

For an analytic Dyson-Schwinger equation write

$$G(x,L) = \sum \gamma_k(x) L^k \qquad \gamma_k = \sum_{j\geq k} \gamma_{k,j} x^j$$

The Hochschild closedness of B_+ is what permits us to rewrite the linearized coproduct which along with $S \star Y$ gives the recursion ([5])

$$\gamma_k(x) = \frac{1}{k}\gamma_1(x)(1 + rx\partial_x)\gamma_{k-1}(x)$$

B_+ and the second recursion

Again

$$G(x,L) = \sum \gamma_k(x) L^k \qquad \gamma_k = \sum_{j\geq k} \gamma_{k,j} x^j$$

The properties of B_+ don't care about connectedness which permits us to modify the primitives of the theory to

- reduce to one insertion place; univariate Mellin transforms.
- take away higher order behaviour of Mellin transforms; geometric series Mellin transforms.

which along with the other recursion gives ([6])

$$\gamma_{1,n} = p(n) + \sum_{j=1}^{n-1}(-rj-1)\gamma_{1,j}\gamma_{1,n-j}$$

B_+ and the growth of γ_1

$$\gamma_{1,n} = p(n) + \sum_{j=1}^{n-1} (-rj - 1)\gamma_{1,j}\gamma_{1,n-j}$$

is what we were able to analyze to show that the primitives determine the growth of the whole theory.

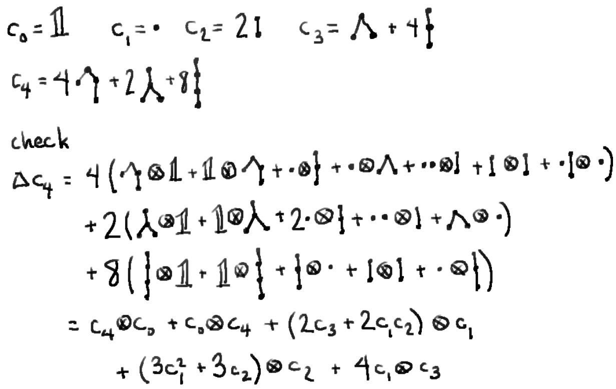
In particular Lipatov bounds $\gamma_{1,n} \leq c^n n!$ carry over.

B_+ and sub Hopf algebras

Today's punchline, solutions to Dyson-Schwinger equations are sub Hopf algebras. Bergbauer, Kreimer [2].

In the example

$$X = \mathbb{I} + xB_+(X^2)$$



The sub Hopf algebra result

Let $B_+^{d_n}$ be Hochschild 1-cocycles. Consider

$$X = \mathbb{I} + \sum x^n w_n B_+^{d_n}(X^{n+1})$$

write $X = \sum x^n c_n$. Then the Dyson-Schwinger equation has a unique solution

$$c_n = \sum w_m B_+^{d_m} \sum_{\substack{k_1+\cdots+k_m=n-m \\ k_i \geq 0}} c_{k_1} \cdots c_{k_{m+1}}$$

and the c_n generate a sub Hopf algebra

$$\Delta c_n = \sum_{k=0}^{n} P_k^n \otimes c_k$$

where the P_k^n are homogeneous polynomials of degree $n-k$ in the c_i, specifically

$$P_k^n = \sum_{\ell_1+\cdots+\ell_{k+1}} c_{\ell_1} \cdots c_{\ell_{k+1}}$$

The role of B_+ for the sub Hopf algebras

Bergbauer and Kreimer [2] give a very natural operadic proof and an elementary proof consisting of a triple induction.

The inductive proof has the advantage of showing explicitly the use of the Hochschild 1-cocycle property of B_+ and that no deep facts are needed.

References

[1] Jason Bell, Stanley Burris, and Karen Yeats, Counting Rooted Trees. Elec. J. Combin. **13** (2006), #R63. (Also arXiv:math.CO/0512432.)

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[4] A. Connes and D. Kreimer. Hopf algebras, renormalization and noncommutative geometry. Commum. Math. Phys. **199** (1998), 203-242. (Also arXiv:hep-th/9808042)

[5] Dirk Kreimer and Karen Yeats, An Étude in non-linear Dyson-Schwinger Equations. Nucl. Phys. B Proc. Suppl., **160**, (2006), 116-121. (Also arXiv:hep-th/0605096.)

[6] Dirk Kreimer and Karen Yeats, Recursion and Growth Estimates in Renormalizable Quantum Field Theory. arXiv:hep-th/0612179.