Dyson-Schwinger equations III

Karen Yeats
Simon Fraser University

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les Houches

Analysing the differential equation

Joint work with Guillaume van Baalen, Dirk Kreimer, and David Uminsky.

Restrict to the single equation case with $s > 0$.

$$\gamma_1(x) = P(x) - \gamma_1(x)(1 - sx\partial_x)\gamma_1(x)$$

so

$$0 = \frac{d\gamma_1(x)}{dx} = \frac{\gamma_1(x) + \gamma_1(x)^2 - P(x)}{sx\gamma_1(x)}$$

to find nullcline

Exact solutions

Beyond $P(x) = x$ there's little hope for exact solutions. Even with $P(x) = x$, Maple can only do 4 of them.

$$\gamma_1(x) = x - \gamma_1(x)(1 - sx\partial_x)\gamma_1(x).$$

$s = 1$: $\gamma_1(x) = x + xW\left(C\exp\left(-\frac{1+x}{x}\right)\right)$,

$s = 2$: $\exp\left(\frac{(1+\gamma_1(x))^2}{2x}\right)\sqrt{-x} + \operatorname{erf}\left(\frac{1+\gamma_1(x)}{\sqrt{-2x}}\right)\frac{\sqrt{\pi}}{\sqrt{2}} = C$

$s = 3/2$: $A(X) - x^{1/3}2^{1/3}A'(X) = C\left(B(X) - x^{1/3}2^{1/3}B'(X)\right)$ where $X = \frac{1+\gamma_1(x)}{2^{2/3}x^{2/3}}$

$s = 3$: $(\gamma_1(x)+1)A(X) - 2^{2/3}A'(X) = C\left((\gamma_1(x) + 1)B(X) - 2^{2/3}B'(X)\right)$ where $X = \frac{(1+\gamma_1(x))^2+2x}{2^{4/3}x^{2/3}}$

where A is the Airy Ai function, B the Airy Bi function and W the Lambert W function.

Qualitative situation

Qualitatively, however, the basic shape doesn't change much with s.

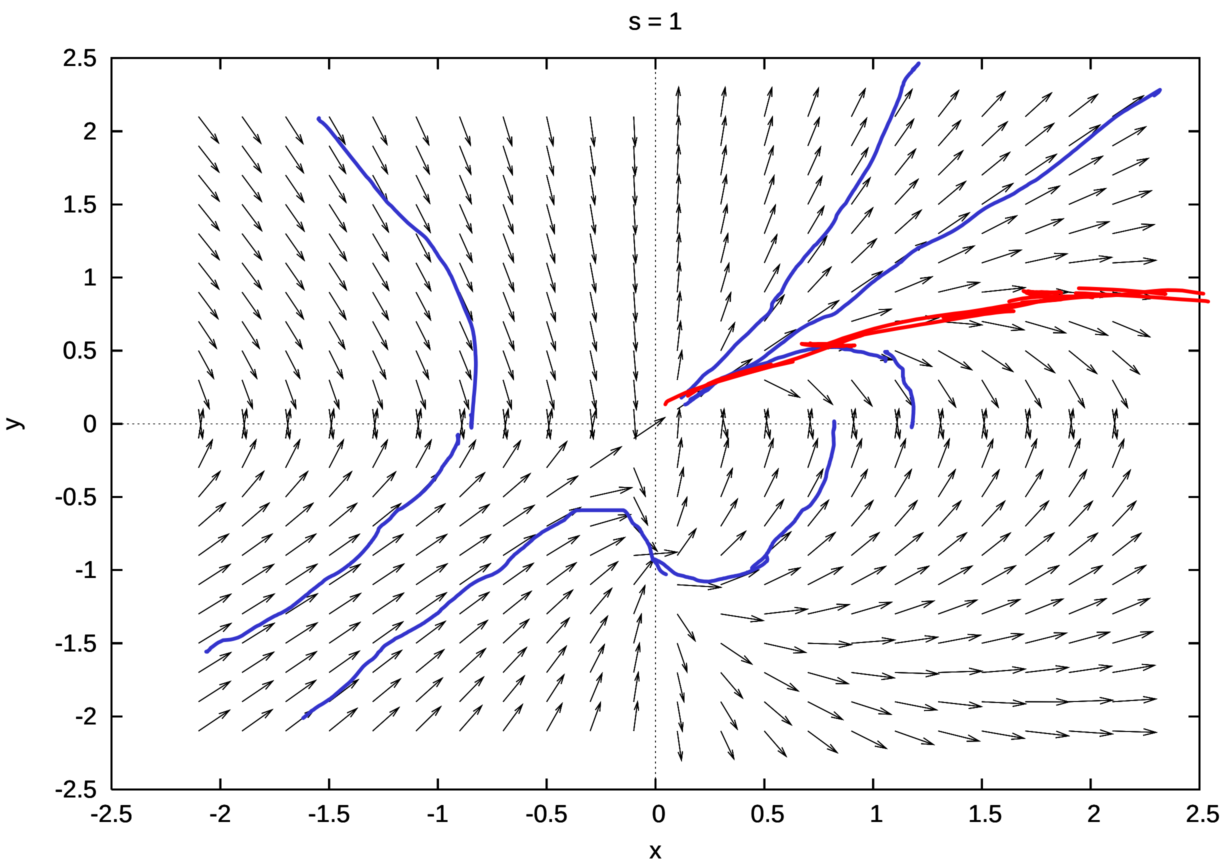
watch s animation here

$$X = \mathbb{1} \pm \sum_k x^k B_+^k (X Q^k)$$

$$P(x) = x, \; s = 1$$

$$Q = X^{-s}$$

What are the behaviours?



The running coupling

The β-function introduces a new differential equation

$L = \log q^2/\mu^2$

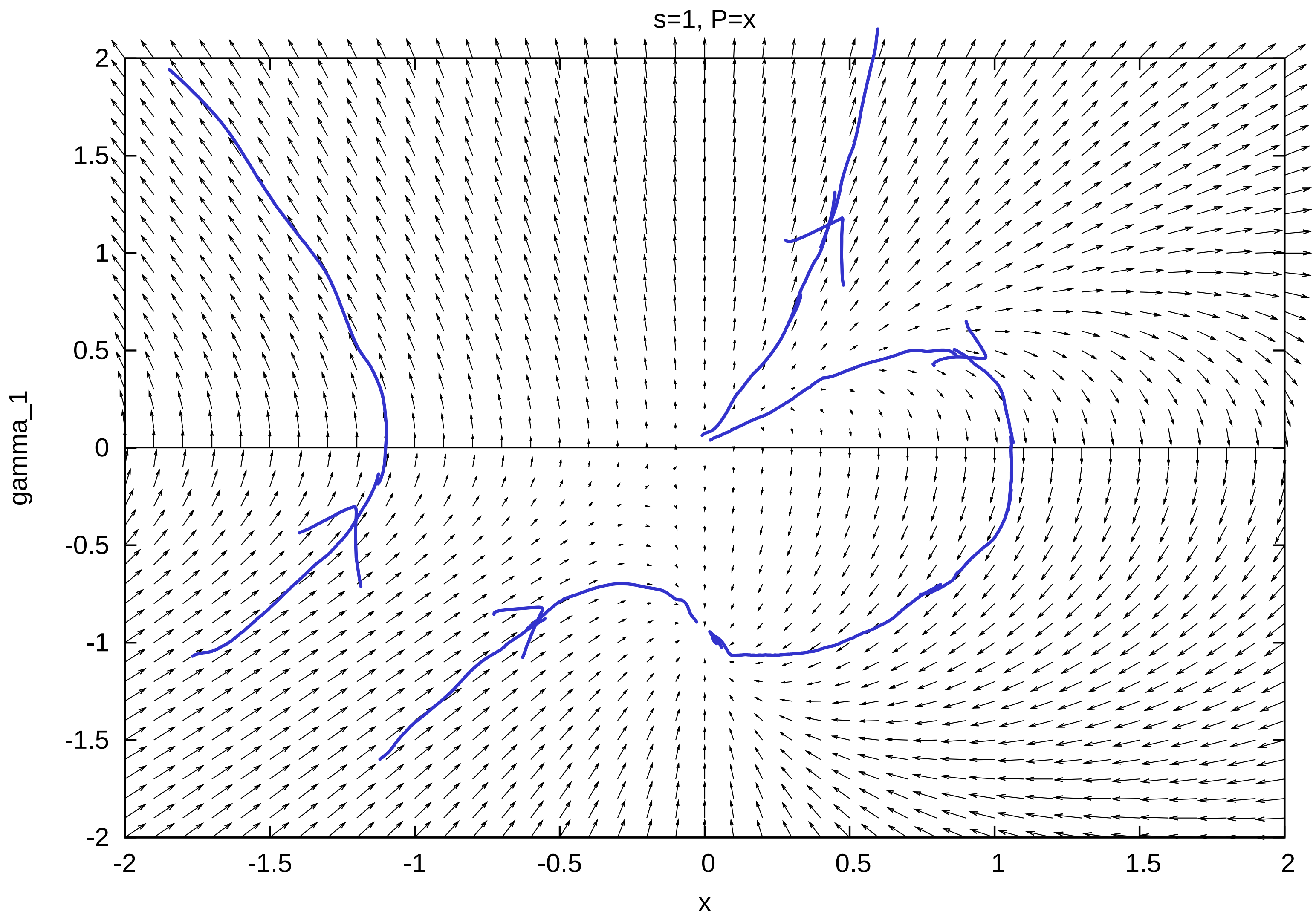
$$\frac{dx}{dL} = \beta(x(L)).$$

In the single equation case

$$\left\{ \begin{aligned} \frac{d\gamma_1}{dL} &= \gamma_1 + \gamma_1^2 - P \\ \frac{dx}{dL} &= sx\gamma_1 \end{aligned} \right.$$

The introduction of the running coupling removes the singularity at the origin.

Picture

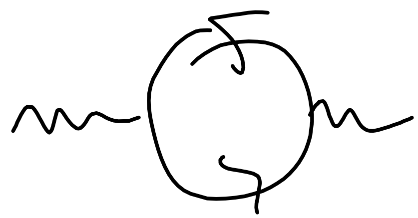


$P(x) > 0$

The picture near 0 is still very much the same for any $P(x) > 0$ with $P(0) = 0$.

QED lives in this world: by Johnson, Baker, Willey, the QED system can be reduced to a single equation for the photon.

$s = 1$ because The one loop photon graph



has no internal photon edge

$$X = \mathbb{1} - \sum x^k B_+^k (X Q^k)$$

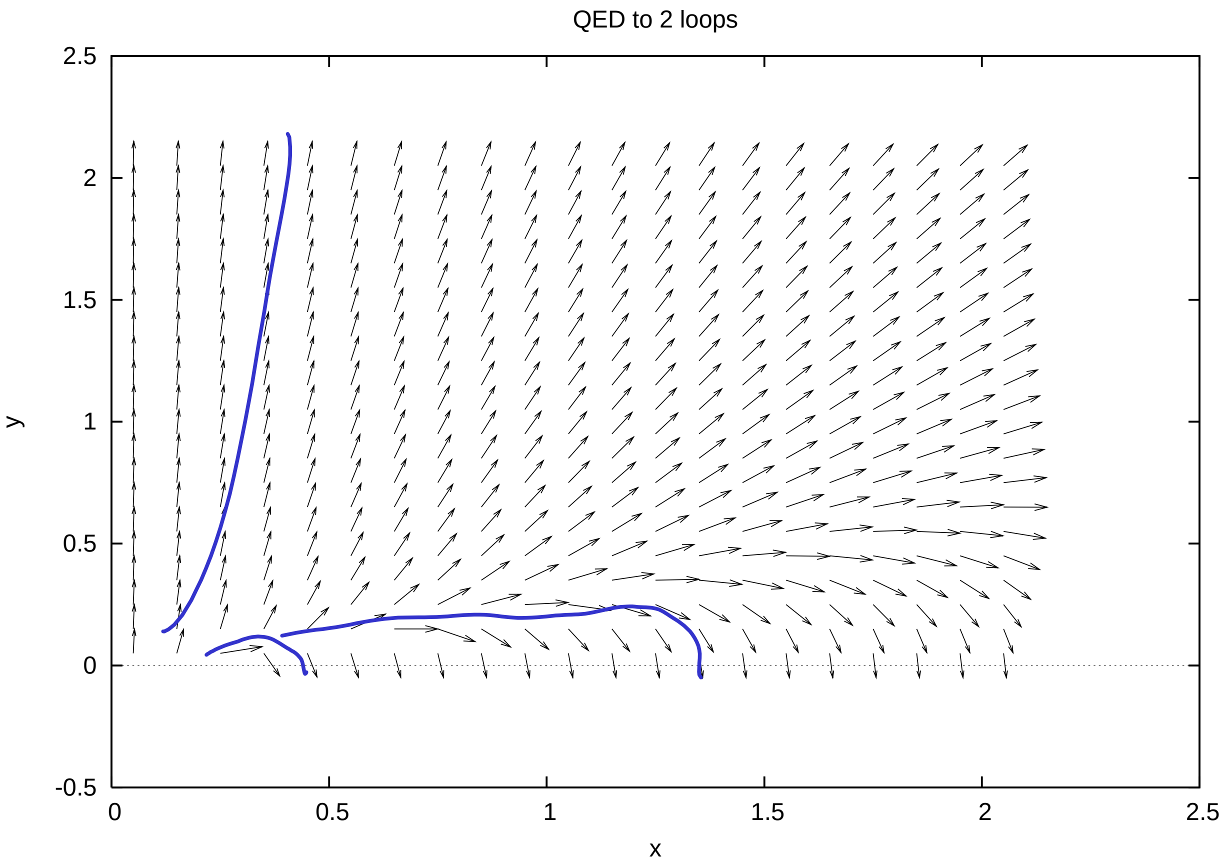
for $k=1$ $\qquad B_+^1 (X Q)$ need this to be X^0

$Q = X^{-1}$

so $s=1$

QED to 2 loops

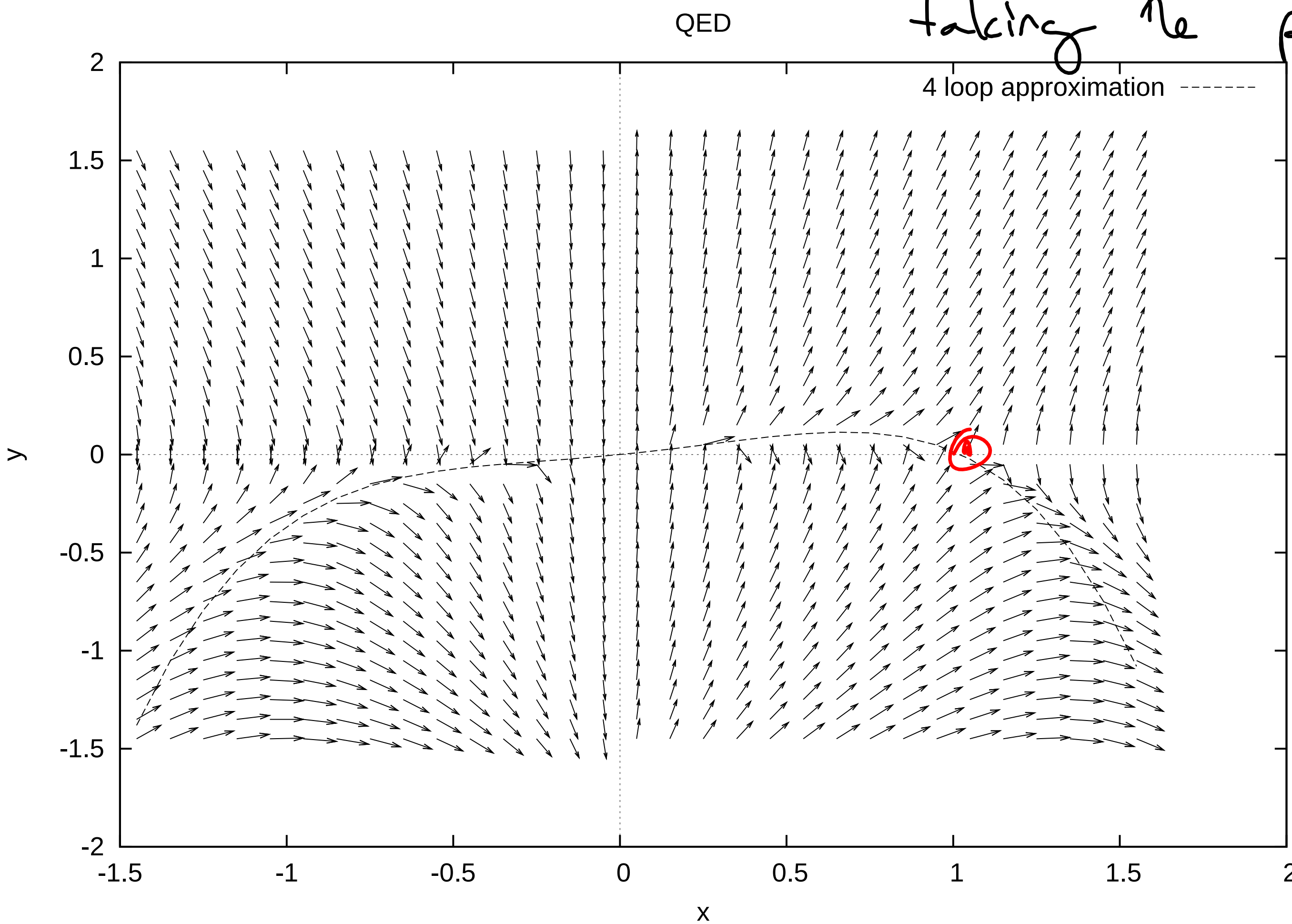
$$2\gamma_1(x) = \frac{x}{3} + \frac{x^2}{4} - \gamma_1(x)(1 - x\partial_x)\gamma_1(x)$$



QED to 4 loops

At 4 loops $P(0.992\ldots) = 0$.

This should be an artifact of taking the perturbative expansion beyond where its valid



Results – Global solutions

Let $s > 0$ and let P be $\mathcal{C}^2$ and positive for $x > 0$, then there exist global (in x) solutions if and only if

$$\int_{x_0}^{\infty} \frac{P(z)}{z^{1+2/s}} dz < \infty \tag{1}$$

for some $x_0 > 0$.

- Note for $P(x) = x$ $\quad \int_{x_0}^{\infty} \frac{1}{z^{2/s}} dz < \infty$. Global in x solutions iff $s < 2$
- For QED $s = 1$ $\quad P(x)$ can grow at most $o(x^2)$ for (1) to hold
- If $\lim_{x \to \infty} P(x) = c < \infty$ then for any s, (1) holds.

- If P satisfies (1) There is a unique separatix - all solutions above exist for all x all solutions below hit the x axis for some finite x

Results – Asymptotics

Let

$$\gamma_c(x) = \frac{\sqrt{1+4P(x)} - 1}{2}$$

null cline

Let $x_0, s > 0$. Assume that P is $\mathcal{C}^2$, positive for $x > 0$, increasing, and satisfies (1). Then every global solution with $\gamma_1(x_0) > \gamma_1^\star(x_0)$ satisfies

separating soln.

$$C_1 \; x^{\frac{1}{s}} \le \gamma_1(x) \le C_2 \; x^{\frac{1}{s}}$$

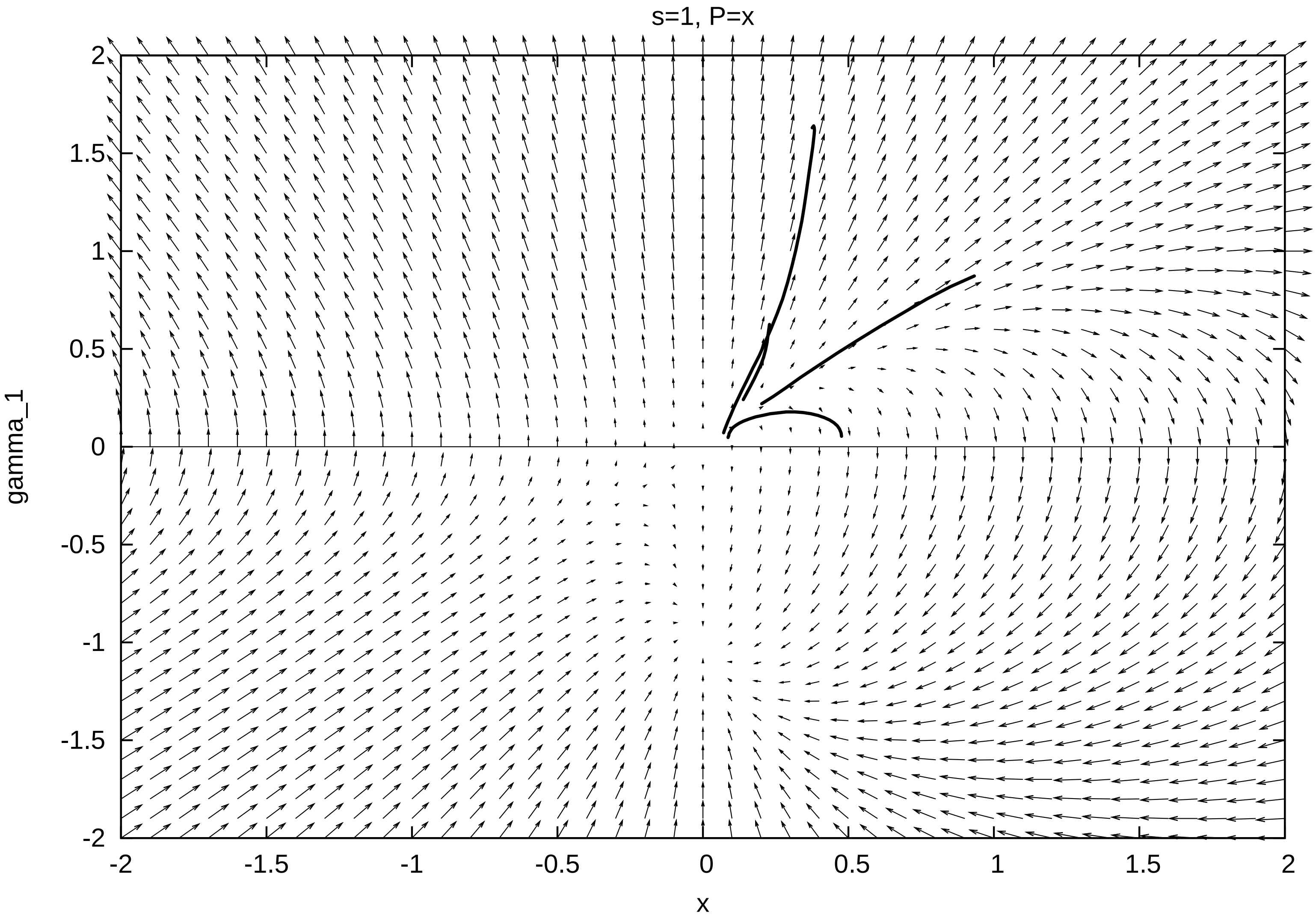
as $x \to \infty$ for some $0 < C_1 < C_2$, while the separatrix itself satisfies

$$\cancel{\gamma_c(x)} < \gamma_1^\star(x) \le \min \lim_{x\to\infty} \left\{ \gamma_c(x) \; , \; C \; x^{\frac{1}{s}} \right\}$$

for some $C > 0$.

In particular, if $\lim_{x\to\infty} P(x) < \infty$, the separatrix is the only global bounded solution.

Back to the L picture



Translation to L

- all solutions go to 0 as $L \to -\infty$

- The solutions which don't exist for all x
  all go to -1 as $L \to \infty$
  double valued as functions of x

- The solutions which exist for all x
  could go to ∞ in finite L or
  only as $L \to \infty$
  This is the question of Landau poles

Landau poles

Assume that P is a $\mathcal{C}^2$, positive, everywhere increasing function that satisfies (1). The separatrix $\gamma_1^\star$ is a Landau pole if and only if

$$\mathcal{L}(P) = \int_{x_0}^{\infty} \frac{\mathrm{d}z}{z\ \gamma_c(z)} = \int_{x_0}^{\infty} \frac{2\mathrm{d}z}{z(\sqrt{1+4P(z)}-1)} < \infty\ .$$

All other global solutions of are Landau poles, irrespective of the value of $\mathcal{L}(P)$.

Summary of $P(x) > 0$ for x near 0

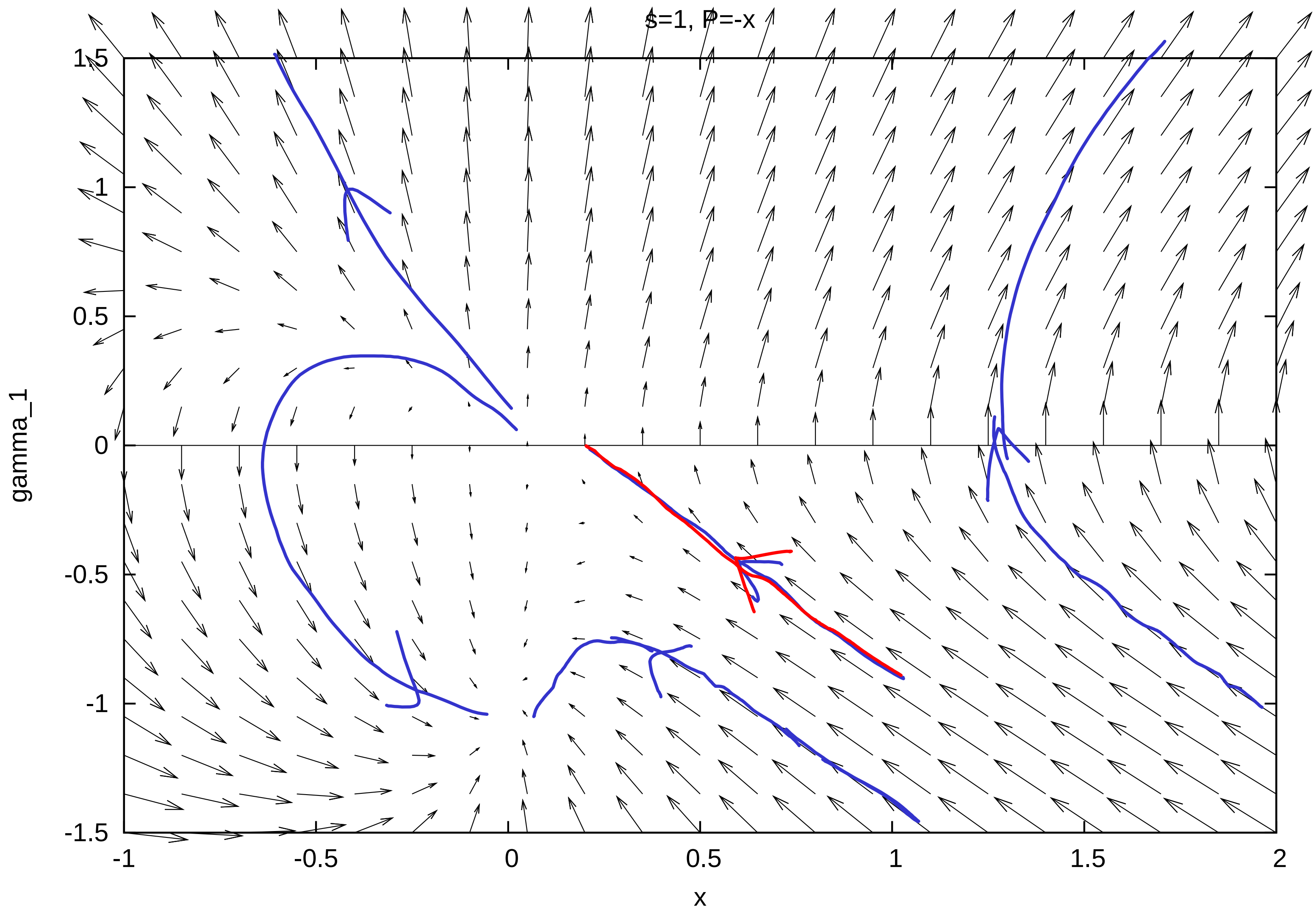
If $P(x)$ is $\mathcal{C}^2$ and $P(x) > 0$ for $x \in (0, x_0)$ then either

- γ_1 crosses the x axis with a vertical tangent and returns to -1, or
- P and γ_1 have a common zero, or
- γ_1 is a global positive solution

In the last case if also $P(x) > 0$ for all $x > 0$ and $P(x)$ is increasing then either

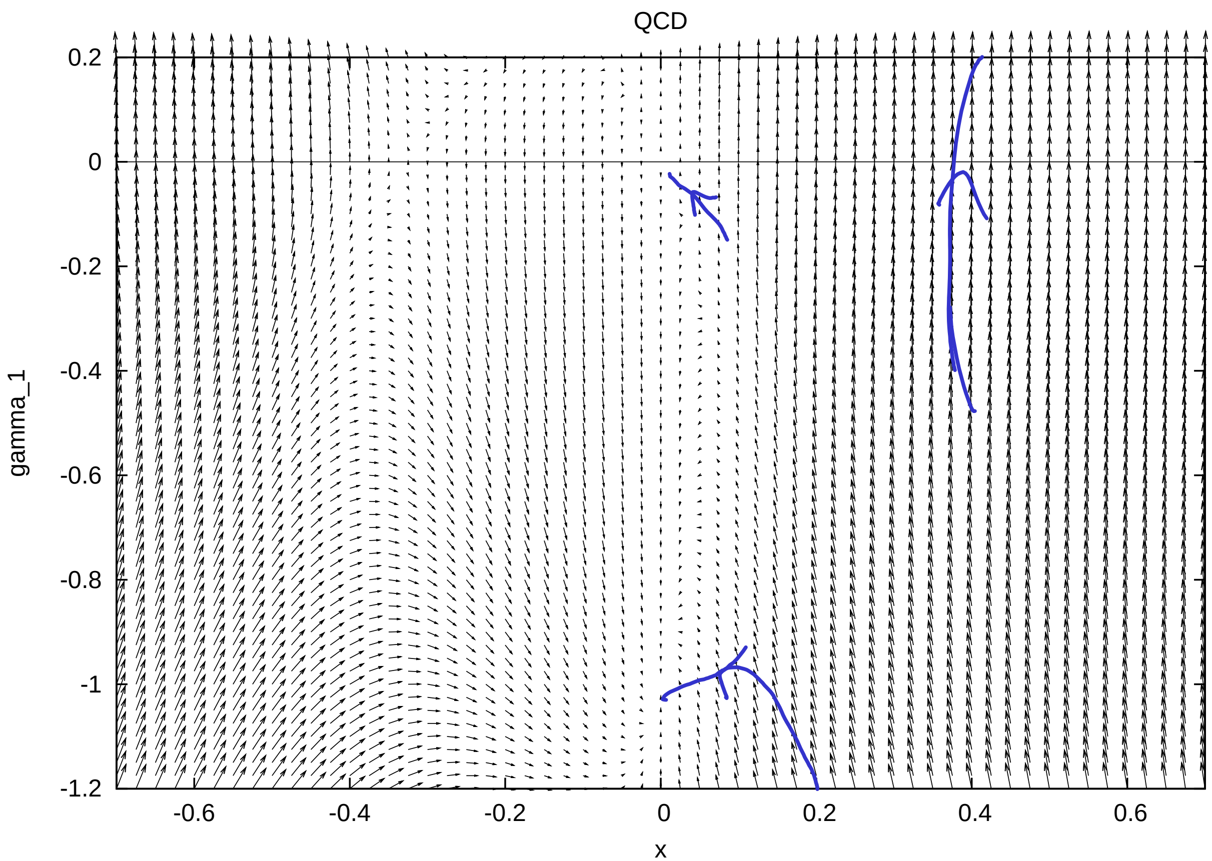
- γ_1 is the separatrix and may or may not diverge in finite L depending on P, or
- γ_1 is larger than that separatrix and necessarily diverges in finite L.

$P(x) < 0$ for x near 0

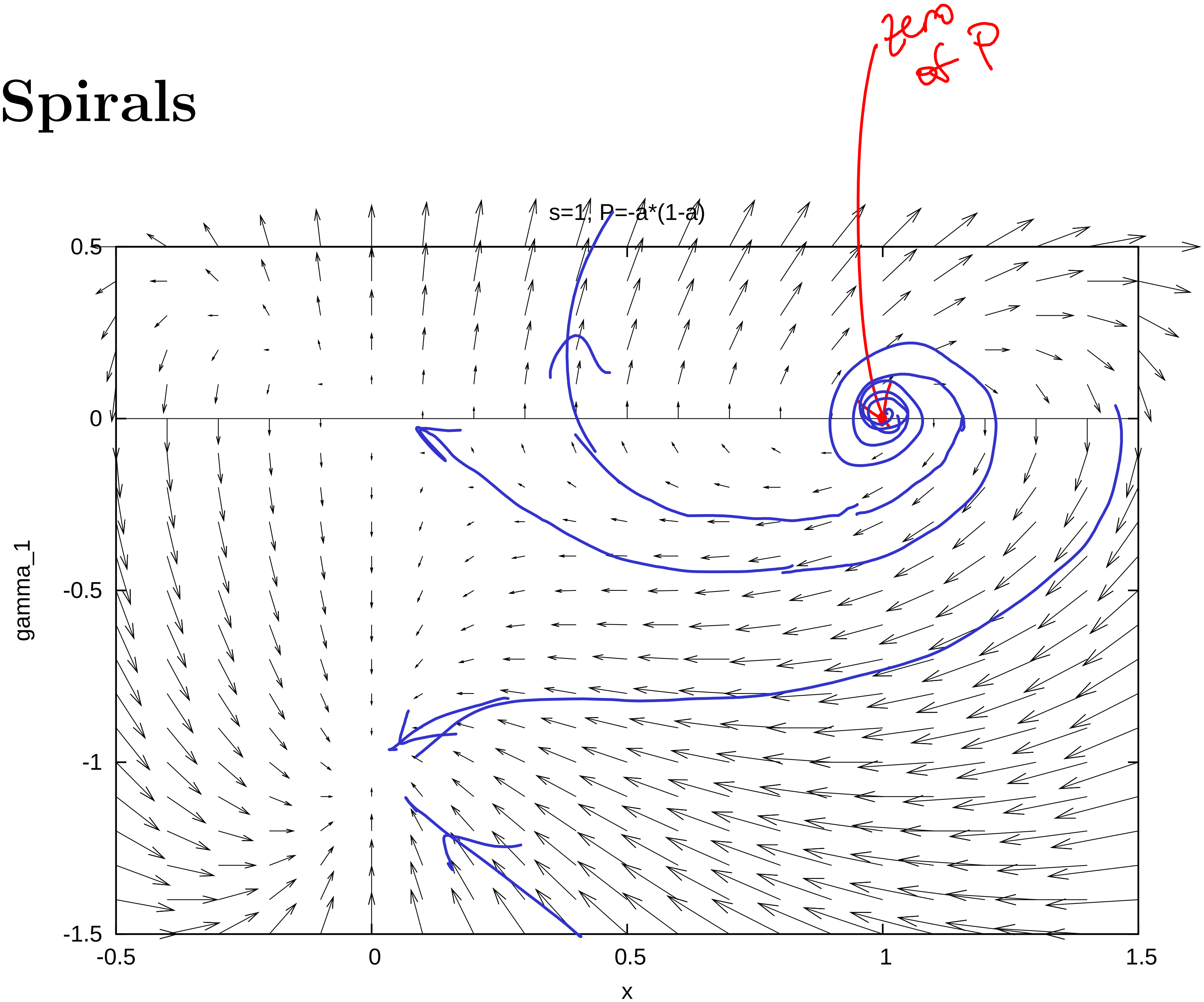


QCD

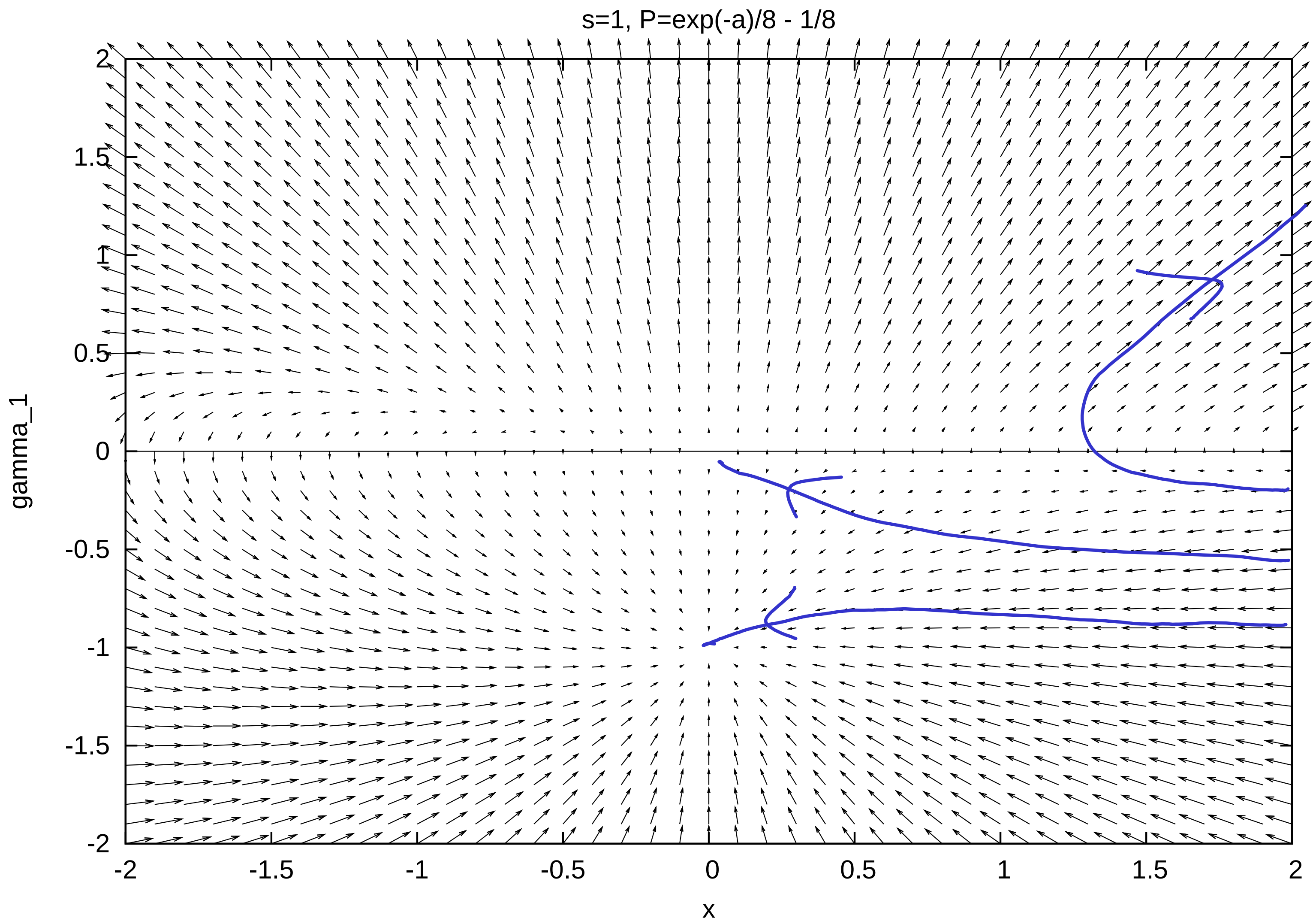
$P(x) < 0$ is the situation for massless QCD in background field gauge.



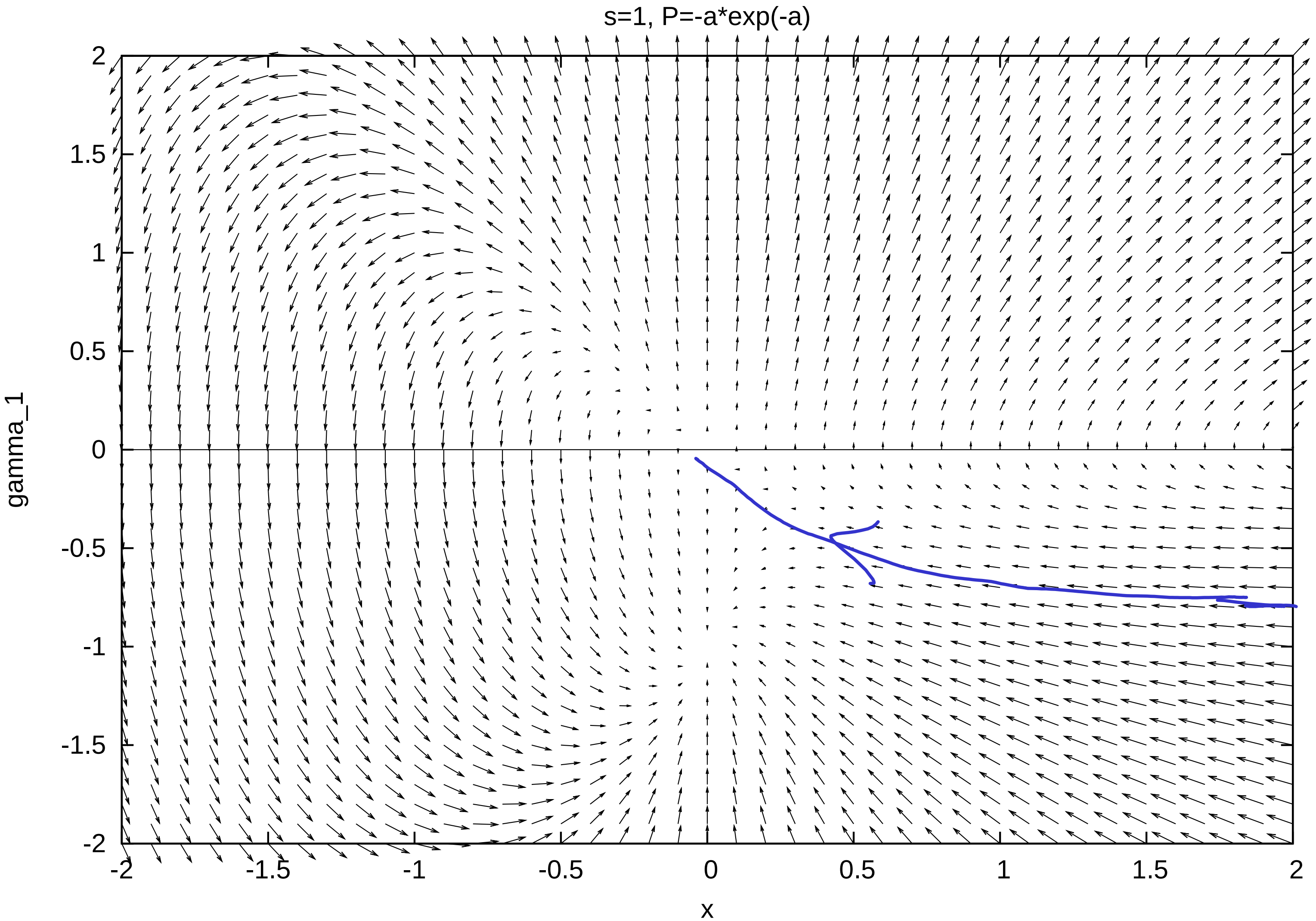
Spirals



$P > -1/4$



Delicacy



Conditions

Recall the QED condition (1)

$$\int_{x_0}^{\infty} \frac{P(z)}{z^{1+2/s}} dz < \infty$$

for some $x_0 > 0$.

The finiteness of the same quantity determines things here. Specifically with $s = 1$ and P negative

$$-\int_{x_0}^{\infty} \frac{P(z)}{z^3} dz < \infty \tag{2}$$

for some $x_0 > 0$.

Results

Assume P is $\mathcal{C}^2$, with $P(0) = 0$, $P'(0) < 0$, and $P(x) < 0$ for $x > 0$. Assume there is an x^* with $P(x^*) < -1/4$ and P concave on $[0, x^*]$.

- There is a unique solution which is 0 as $x \to 0$. Solutions below this approach -1 as $x \to 0$ and solutions above it cross the x-axis at some positive value.

- Assume $\gamma_1(x) > 0$ or $\gamma_1(x) < -1$.

  - If (2) holds then γ_1 is aymptotically linear as $x \to \infty$

  - Otherwise $\gamma_1 \sim \pm x \left(\frac{\gamma_1(x_0)^2}{x_0^2} + 2 \int_{x_0}^{x} \frac{-P(z)}{z^3} dz \right)^{\frac{1}{2}}$

- If further $\lim_{x\to\infty} P(x) = c > -1/4$ and $\lim_{x\to\infty} xP'(x) = 0$ then there is a unique solution with

$$\lim_{x\to\infty} \gamma_1(x) = -\frac{1 + \sqrt{1 + 4c}}{2}$$

Systems of equations in x

The question is what is the right question to ask

Need to visualize these.

Massless ϕ^4

vertex

$$\frac{d\gamma_1^+}{dL} = \gamma_1^+ - (\gamma_1^+)^2 - P^+(x)$$

propagator

$$\frac{d\gamma_1^-}{dL} = \gamma_1^- + (\gamma_1^-)^2 - P^-(x)$$

$$\frac{dx}{dL} = x(\gamma_1^+ + 2\gamma_1^-)$$

let's see some animations