# A combinatorial perspective on Dyson-Schwinger equations

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## An equation

$$\gamma_1(x) = P(x) - \gamma_1(x)(1 - sx\partial_x)\gamma_1(x)$$

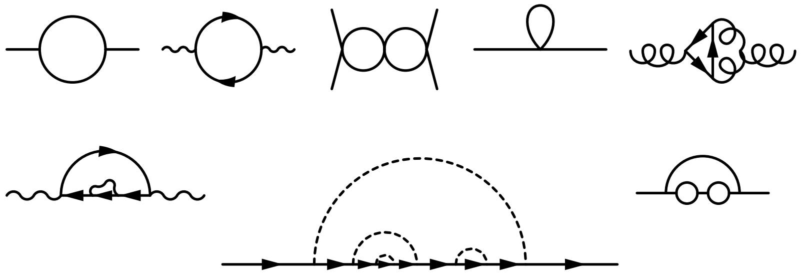
Important special cases

$$\gamma_1(x) = x - \gamma_1(x)(1 - 2x\partial_x)\gamma_1(x)$$
$$2\gamma_1(x) = \left(\frac{x}{3} + \frac{x^2}{4}\right) - \gamma_1(x)(1 - x\partial_x)\gamma_1(x)$$

$$\gamma_1^+(x) = P^+(x) - \gamma_1^+(x)^2 - (\gamma_1^+(x) - 2\gamma_1^-(x))x\partial_x\gamma_1^+(x)$$
$$\gamma_1^-(x) = P^-(x) - \gamma_1^-(x)^2 - (\gamma_1^+(x) - 2\gamma_1^-(x))x\partial_x\gamma_1^-(x)$$



## Feynman diagrams

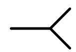


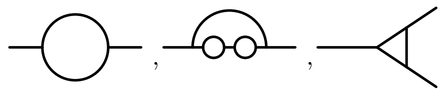
Said to be divergent if they have external leg structure matching an edge or vertex of the theory (for a renormalizable theory).

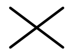
Said to be 1-particle irreducible (1PI) if they are 2-edge connected.

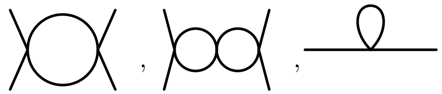


## Examples

In $\phi^3$ there is one edge type, , and one vertex type,  . Divergent 1PI Feynman graphs:

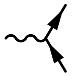


In $\phi^4$ there is one edge type, , and one vertex type,  . Divergent 1PI Feynman graphs:

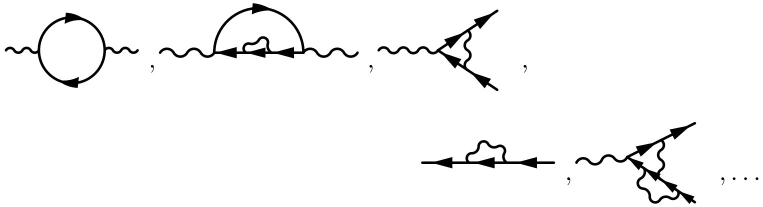


## More examples

In QED there are two edge types; a photon,  , and an electron,  , and one vertex type



Divergent 1PI Feynman graphs:





## The Hopf algebra

As a vector space — the $\mathbb{Q}$ span of Feynman graphs with $\mathbb{I}$ the empty graph.

As an algebra — with multiplication $m$ given by disjoint union.

As a coalgebra — with coproduct

$$\Delta(\Gamma) = \sum_{\substack{\gamma\subseteq\Gamma \\ \gamma \text{ product of divergent} \\ \text{1PI subgraphs}}} \gamma \otimes \Gamma/\gamma$$

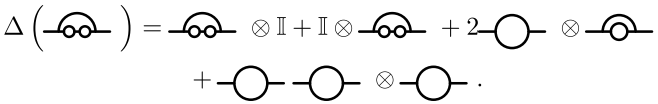
As a Hopf algebra — with antipode defined recursively by $S(\mathbb{I}) = \mathbb{I}$,

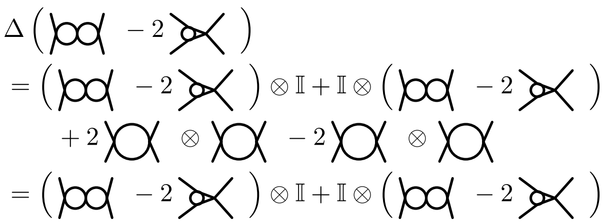
$$S(\Gamma) = -\Gamma - \sum_{\substack{\mathbb{I}\neq\gamma\subsetneq\Gamma \\ \gamma \text{ product of divergent} \\ \text{1PI subgraphs}}} S(\gamma)\Gamma/\gamma$$

on connected graphs, and extended as an anti-homomorphism.



## Examples

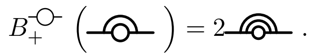






## $B_+$

Write $B_+^\gamma$ for insertion into the primitive graph $\gamma$. For example



The sum of the $B_+$s for a given loop number is a Hochschild 1-cocycle

$$\Delta\Sigma B_+ = (\mathrm{id} \otimes \Sigma B_+)\Delta + \Sigma B_+ \otimes \mathbb{I}.$$

# A Dyson-Schwinger equation

Consider
$$X(x) = \mathbb{I} - \sum_{k\geq 1} x^k B_+^k(X(x)Q(x)^k)$$
where $Q(x) = X(x)^{-s}$ with $s > 0$ an integer. Associate with each $B_+$ a Mellin transform
$$F^k(\rho_1, \ldots, \rho_n).$$

Write the combination ($X \mapsto G$, $B_+^k \mapsto F^k$, the $\rho_i$ mark the insertion places) as

$$G(x, L) = \sum \gamma_k(x) L^k \text{ with } \gamma_k(x) = \sum_{j\geq k} \gamma_{k,j} x^j.$$

Systems of equation are similar but messier.



# An example

Broadhurst and Kreimer; a bit of massless Yukawa theory.

$X(x) = \mathbb{I} - xB_+$  $\left(\frac{1}{X(x)}\right)$,

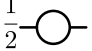
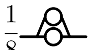
$X(x) = \mathbb{I} - x$  $-x^2$  $-x^3\Big($  $+$  $\Big) - \cdots$

$$F(\rho) = \frac{1}{q^2}\int d^4k \frac{k\cdot q}{(k^2)^{1+\rho}(k+q)^2} - \cdots \Big|_{q^2=\mu^2}.$$

Combine to get
$$G(x, L) = 1 - \frac{x}{q^2}\int d^4k \frac{k\cdot q}{k^2 G(x, \log k^2)(k+q)^2} - \cdots \Big|_{q^2=\mu^2}$$
where $L = \log(q^2/\mu^2)$.



# Four initial steps

1. From the renormalization group equation or the scattering type formula of Connes and Kreimer derive
$$\gamma_k(x) = \frac{1}{k}\gamma_1(x)(1 - sx\partial_x)\gamma_{k-1}(x),$$
2. Reduce to primitives with a single insertion place, that is Mellin transforms in a single variable $\rho$. For example with
$X = 1 - xB_+^{\frac{1}{2}}$  $\left(\frac{1}{X^2}\right)$
use
$q_1 = \frac{1}{2}$  $\quad q_2 = 0 \quad q_3 = \frac{1}{8}$  $-\frac{1}{8}$  $\quad \cdots$



3. Rewrite the (analytic) Dyson-Schwinger equation using the usual tricks
   - plug in $\sum \gamma_k L^k$
   - use $\partial_\rho^k x^{-\rho}|_{\rho=0} = (-1)^k \log^k(x)$
   - switch the order of $\int$ and $\partial$
$$\gamma \cdot L = \sum x^k (1 - \gamma\cdot\partial_{-\rho})^{1-sk}(e^{-L\rho} - 1)F^k(\rho)\Big|_{\rho=0}$$
where $\gamma \cdot U = \sum \gamma_k U^k$.
4. Reduce to geometric series Mellin transforms by noticing that for $\ell \geq 0$
$$x^k(1 - \gamma\cdot\partial_{-\rho})^{1-sk}\rho^\ell|_{\rho=0}$$
viewed as a series in $x$ has lowest term $x^{k+\ell}$.

## Finding $\gamma_1$

From step 3

$$\gamma \cdot L = \sum x^k (1 - \gamma \cdot \partial_{-\rho})^{1-sk} (e^{-L\rho} - 1) F^k(\rho) \Big|_{\rho=0}.$$

Take an $L$ derivative and set $L = 0$ to get

$$\gamma_1 = -\sum x^k (1 - \gamma \cdot \partial_{-\rho})^{1-sk} \rho F^k(\rho) \Big|_{\rho=0}.$$

Take two $L$ derivatives and use step 4 to get

$$2\gamma_2 = \sum_k x^k (1 - \gamma \cdot \partial_{-\rho})^{1-sk} r_k \frac{\rho}{1-\rho} \Big|_{\rho=0} = -\gamma_1 - \sum x^k r_k.$$

Write $P(x) = -\sum x^k r_k$ and use step 1

$$\gamma_1 = P(x) - \gamma_1 (1 - sx\partial_x)\gamma_1.$$



## As a recursive equation

View

$$\gamma_1(x) = P(x) - \gamma_1(x)(1 - sx\partial_x)\gamma_1(x)$$

as a recursive equation. At the level of coefficients

$$\gamma_{1,n} = p(n) + \sum_{j=1}^{n-1} (sj - 1)\gamma_{1,j}\gamma_{1,n-j}.$$

Assume $\gamma_{1,1} \neq 0$ and $f(x) = \sum \frac{p(n)}{n!} x^n$ has radius of convergence $\rho > 0$.
Let $a(n) = \frac{\gamma_{1,n}}{n!}$. The recursion becomes

$$a_n = \frac{p(n)}{n!} + \sum_{i=1}^{n-1} (si - 1) a_i a_{n-i} \binom{n}{i}^{-1}$$
$$= \frac{p(n)}{n!} + \left(\frac{sn}{2} - 1\right) \sum_{i=1}^{n-1} a_i a_{n-i} \binom{n}{i}^{-1}$$



## How bad is the growth of $\gamma_1$?

Idea:

$$a(n) \text{ is approximately } \frac{p(n)}{n!} + s a_1 a_{n-1}$$

giving a radius of $\min\left\{\rho, \frac{1}{sa_1}\right\}$ for $\sum a_n x^n$. For nonnegative series implement the idea by bounding on each side.

Easy direction:

$$a_n \geq \frac{p(n)}{n!} + s\frac{n-2}{n} a_1 a_{n-1}$$

Messy direction: for any $\epsilon > 0$ there is an $N > 0$ such that for $n > N$

$$a_n \leq \frac{p(n)}{n!} + s a_1 a_{n-1} + \epsilon \sum_{j=1}^{n-1} a_j a_{n-j}$$



## Why?

- Understanding the growth of $\gamma_1$ is understanding the growth of the whole theory.
- Expect a Lipatov bound $\gamma_{1,n} \leq c^n n!$.
- Does the first singularity of $\sum \frac{\gamma_{1,n}}{n!} x^n$ come from renormalon chains or from instantons?
- We've shown that a Lipatov bound for the primitives leads to a Lipatov bound on the whole theory.
- The radius is either the radius from the primitives or $\frac{1}{s\gamma_{1,1}}$, the first coefficient of the beta function.
- The moral is that the primitives control matters.

## As a differential equation

View

$$\gamma_1(x) = P(x) - \gamma_1(x)(1 - sx\partial_x)\gamma_1(x)$$

as a differential equation.

The $P(x) = x$ family is the last bastion of exact solutions,

$s = 1$: $\gamma_1(x) = x + xW\left(C\exp\left(-\frac{1+x}{x}\right)\right)$,

$s = 2$: $\exp\left(\frac{(1+\gamma_1(x))^2}{2x}\right)\sqrt{-x} + \operatorname{erf}\left(\frac{1+\gamma_1(x)}{\sqrt{-2x}}\right)\frac{\sqrt{\pi}}{\sqrt{2}} = C$

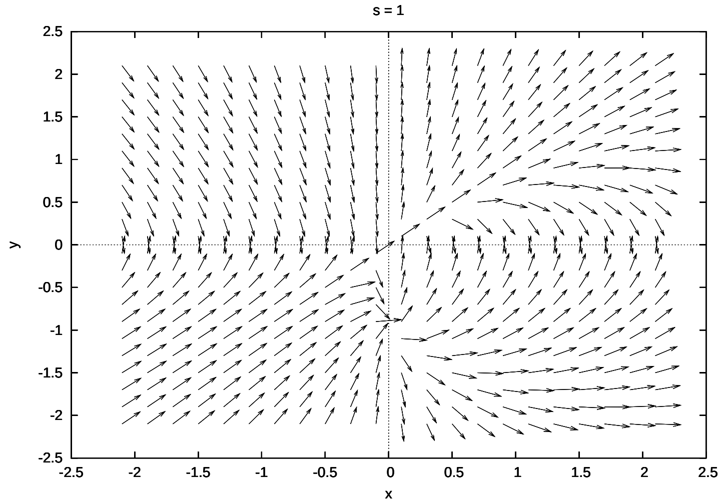
$s = 3/2$: $A\left(X\right) - x^{1/3}2^{1/3}A'\left(X\right) = C\left(B\left(X\right) - x^{1/3}2^{1/3}B'\left(X\right)\right)$ where $X = \frac{1+\gamma_1(x)}{2^{2/3}x^{2/3}}$

$s = 3$: $(\gamma_1(x)+1)A\left(X\right) - 2^{2/3}A'\left(X\right) = C\left((\gamma_1(x)+1)B\left(X\right) - 2^{2/3}B'\left(X\right)\right)$ where $X = \frac{(1+\gamma_1(x))^2+2x}{2^{4/3}x^{2/3}}$

where $A$ is the Airy Ai function, $B$ the Airy Bi function and $W$ the Lambert W function.

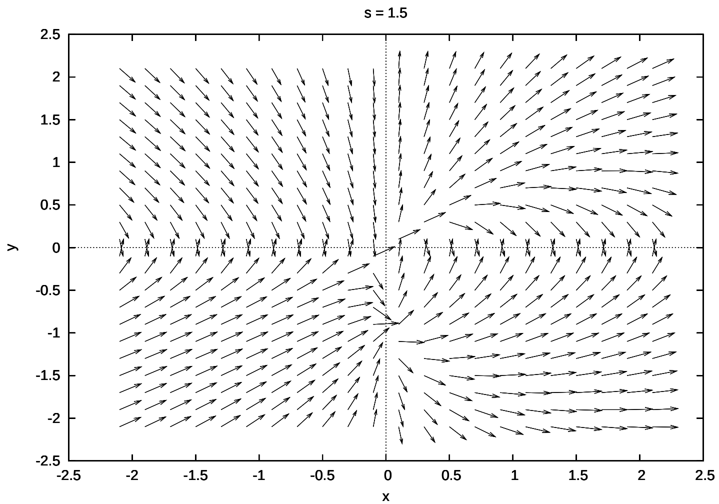


## $s = 1$



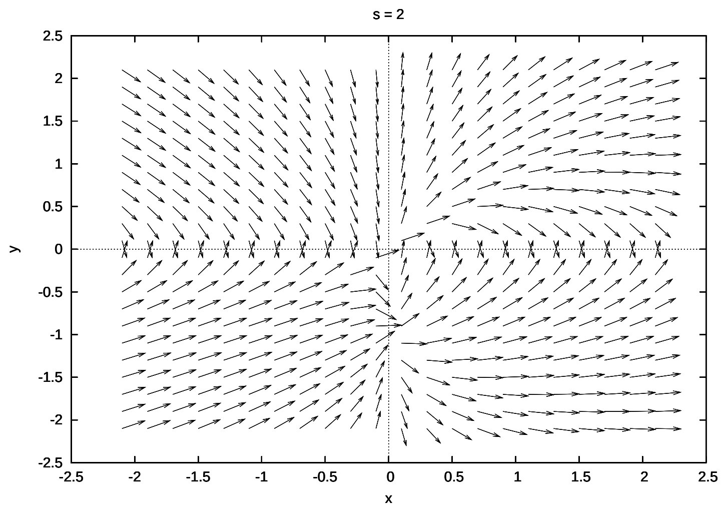



## $s = 3/2$

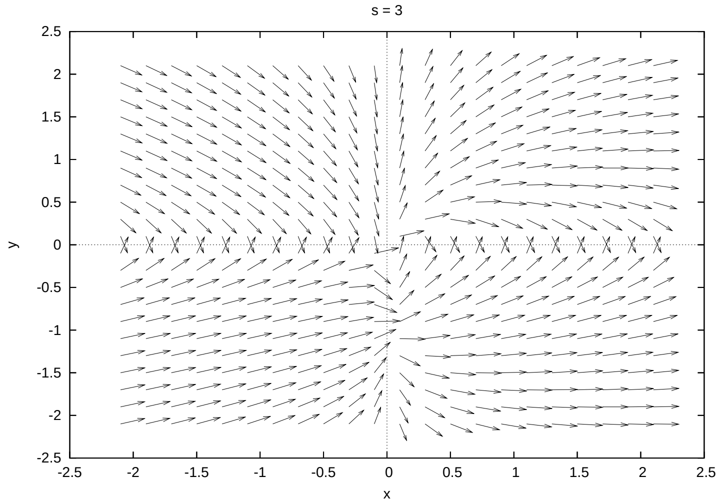





## $s = 2$



# $s = 3$






# QED as a single equation

By the Baker, Johnson, Willey analysis we can reduce to a single equation for the photon propagator.

$$2\gamma_1(x) = P(x) - \gamma_1(x)(1 - x\partial_x)\gamma_1(x)$$

$s = 1$ gives a term $B_+(\mathbb{I})$ independent of $X$ to take into account the fact that the photon propagator can not be inserted into the one loop graph.

To 2 loops

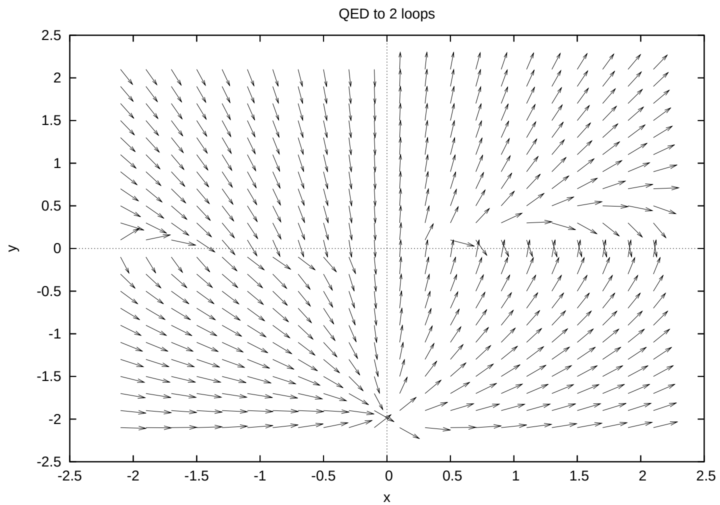
$$P(x) = \frac{x}{3} + \frac{x^2}{4}$$

To 4 loops we need to correct the primitives for our setup

$$P(x) = \frac{x}{3} + \frac{x^2}{4} + (-0.0312 + 0.06037)x^3 + (-0.6755 + 0.05074)x^4$$



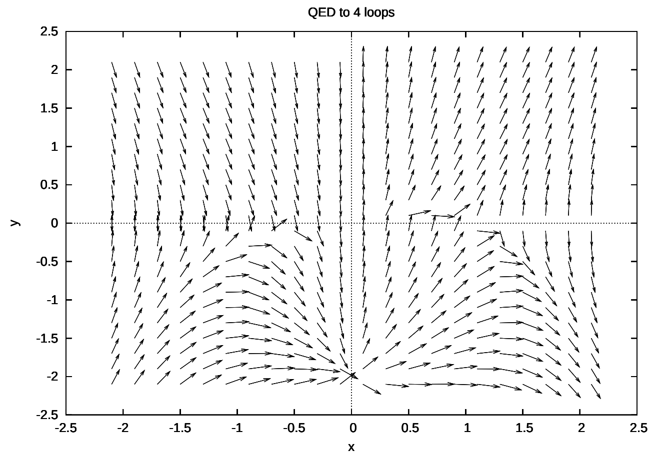
# QED to 2 loops



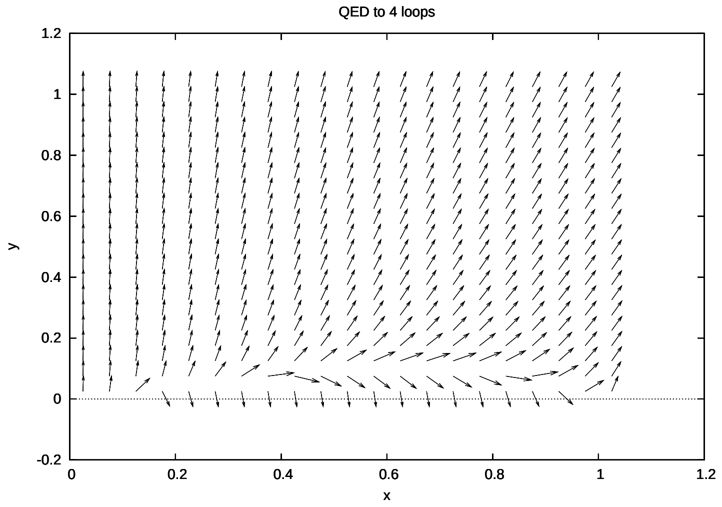



# QED to 4 loops

At 4 loops $P(0.992\ldots) = 0$ changing everything.



# Zoomed in






# We are here