Counting trees with applications to counting Feynman diagrams

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March 14, 2006
CIRM Renormalization and Galois theories

Unlabelled rooted trees

Let $t(n)$ be the number of unlabelled rooted trees with n vertices. Let $\mathbf{T}(x) = \sum_{n\geq 1} t(n)x^n$ be the corresponding generating function.

Decompose a rooted tree into its root and the forest of its subtrees; an arbitrary multiset of rooted trees.

$$\mathbf{T}(x) = x + x\mathsf{MSet}(\mathbf{T})(x).$$

That is

$$\mathbf{T}(x) = x \exp\left(\sum_{m\geq 1} \mathbf{T}(x^m)/m\right).$$

The radius of convergence, ρ, of $\mathbf{T}(x)$ is (the reciprocal of) Otter's tree constant. $\rho = 0.3383218568992076952\ldots$.

Pólya's analysis of rooted trees

Pólya converted the recursive equation

$$\mathbf{T}(x) = x \exp\left(\sum_{m\geq 1} \mathbf{T}(x^m)/m\right)$$

to a bivariate function

$$\mathbf{E}(x, y) = xe^y \exp\left(\sum_{m\geq 2} \mathbf{T}(x^m)/m\right).$$

The recursive equation is then $\mathbf{T}(x) = \mathbf{E}(x, \mathbf{T}(x))$.

Weierstrass preparation on $\mathbf{E}$ gives a square root singularity at ρ. Then the Cauchy integral theorem gives

$$t(n) \sim C\rho^{-n}n^{-3/2}$$

The universal law

Asymptotics of the form $C\rho^{-n}n^{-3/2}$ are ubiquitous for classes of rooted trees with recursive definitions, hence the term universal law.

- plane trees: $\mathbf{T}(x) = x + x\mathsf{Seq}(\mathbf{T})(x)$
- plane binary trees: $\mathbf{T}(x) = x + x\mathsf{Seq}_{\{2\}}(\mathbf{T})(x)$
- $(0, 1, 2, 3)$-trees: $\mathbf{T}(x) = x + x\mathsf{MSet}_{\{1,2,3\}}(\mathbf{T})(x)$
- trees with cyclically ordered subtrees at each vertex: $\mathbf{T} = x + x\mathsf{DCycle}(\mathbf{T})(x)$
- identity trees: $\mathbf{T}(x) = x + x\mathsf{Set}(\mathbf{T})(x)$
- labelled trees: $\mathbf{T}(x) = xe^{\mathbf{T}(x)}$

and anything defined by a huge swath of other recursive equations built out of (most) iterations of the basic building blocks above and others.

Recursive systems

The solutions of polynomial recursive systems also satisfy the universal law under reasonable conditions (independently: Drmota, Lalley, and Woods)

Suppose

$$y_1 = \Phi_1(x, y_1, \ldots, y_m)$$
$$\vdots$$
$$y_m = \Phi_m(x, y_1, \ldots, y_m)$$

with the Φ_i polynomials with real coefficients.

Note that geometric series can be converted to polynomials at the expense of a new variable: replace $1/(1-T)$ with a new variable F and add the equation

$$F = 1 + F \cdot T.$$

This is enough for systems coming from quantum field theory.

5 conditions on systems

1. The system is *nonlinear* if at least one of the Φ_i is nonlinear in $y_1, \ldots, y_m$.

2. The system is *nonnegative* if each Φ_i has nonnegative coefficients.

3. For $\overline{y} = (y_1, \ldots, y_m) \in \mathbb{R}[[x]]^m$ define the x-valuation by $\mathrm{val}(\overline{y}) = \min_i(\mathrm{val}(y_i))$ where $\mathrm{val}(\sum_{n=k}^{\infty} a_n x^n) = k$ with $a_k \neq 0$, and $\mathrm{val}(0) = \infty$. Define $d(\overline{y}, \overline{y}') = 2^{-\mathrm{val}(\overline{y} - \overline{y}')}$. Then the system is *proper* if

   $$d(\Phi(\overline{y}), \Phi(\overline{y}')) < K d(\overline{y}, \overline{y}') \qquad \text{for some } K < 1.$$

4. The system is *irreducible* if its dependency graph is strongly connected.

5. A power series $\mathbf{T}(x)$ is *aperiodic* if it cannot be written $\mathbf{T}(x) = x^a \mathbf{U}(x^d)$. The system is *aperiodic* if each component solution is aperiodic.

Theorem for systems

Theorem 1. *Suppose $\overline{y} = \Phi(\overline{y})$ is a polynomial system that is nonlinear, proper, nonnegative, and irreducible.*

Then all component solutions y_j have the same radius of convergence $\rho < \infty$ and have a square root singularity at ρ.

If furthermore the system is aperiodic then all y_j satisfy the universal law.

QED with 1 primitive per loop order

system drawn with diagrams goes here

$$T_1 = 1 + \sum_{k \geq 1} x^k \frac{T_1^{2k+1}}{(1 - (T_2 - 1))^{2k}(1 - (T_3 - 1))^k}$$

$$T_2 = 1 + x \frac{T_1}{(1 - (T_2 - 1))(1 - (T_3 - 1))}$$

$$T_3 = 1 + x \frac{T_1}{(1 - (T_2 - 1))^2}$$

A canonical subsystem; convergent, but captures renormalization.

Make the system nonnegative

$$T_1 = 1 + T_1 \sum_{k\geq 1} \left(x \frac{T_1^2}{(1-N_2)^2(1-N_3)} \right)^k$$
$$N_2 = x\frac{T_1}{(1-N_2)(1-N_3)}$$
$$N_3 = x\frac{T_1}{(1-N_2)^2}$$

Convert the geometric series

$$\Phi = \begin{cases} T_1 & = 1 + T_1 F \\ F & = xT_1^2F_2^2F_3 + xT_1^2F_2^2F_3F \\ N_2 & = xT_1F_2F_3 \\ F_2 & = 1 + F_2N_2 \\ N_3 & = xT_1F_2^2 \\ F_3 & = 1 + F_3N_3 \end{cases}$$

Φ is nonlinear, irreducible, and aperiodic. Φ^2 is proper.

QED universal law and radius

So the QED system satisfies the universal law; the number $t_1(n)$ of objects (particular sums of graphs) with n loops (per summand) satisfies

$$t_1(n) \sim C\rho^{-n}n^{-3/2}$$

What is the radius? Manipulate the system to get

$$-x+T_1+(6x-5)T_1^2+8T_1^3+(-12x-4)T_1^4+8xT_1^6 = 0$$

As a polynomial in T_1 this has discriminant

$$4096x^2(32x^2-8x+1)(-2+27x)^2$$

So the radius of the system is

$$\frac{2}{27}$$

This number belongs to QED; what is its physical meaning?

QED variants

Any polynomial number of primitives per loop order

$$T_1 = 1 + \sum_{k\geq 1} p(k)x^k \frac{T_1^{2k+1}}{(1-(T_2-1))^{2k}(1-(T_3-1))^k}$$

with T_2 and T_3 as before. The linear case is Cvitanović's gauge invariant sectors. The radii gently decrease: only down to 0.046 by the polynomial k^{28}.

Use gauge invariance first (Johnson, Baker, Willey) to reduce to

$$T = \sum_{k\geq 1} \left(\frac{x}{1-T} \right)^k = \frac{x}{1-T-x}$$

This gives large Schröder numbers A006318. The radius is $3-2\sqrt{2} = 0.17157287525380990247\ldots$ which is considerably larger than $2/27 = 0.\overline{074}$ showing how powerful gauge invariance is.

Other theories

We can play the same game for other theories, ϕ^3, ϕ^4, mixed ϕ^3 ϕ^4, ...

The universal law continues to hold for reasonable, convergent series of primitives. The radii don't end up being particularly nice.

Bonus slides:

Operators giving the universal law

Let $\mathcal{O}$ be the set of operators on power series built out of

1. $\mathbf{E}(x, \cdot)$ such that
   (a) $\mathbf{E}(x, y)$ has nonnegative coefficients and zero constant term,
   (b) $\mathbf{E}(a, b) < \infty \Rightarrow \exists \epsilon > 0, \mathbf{E}(a + \epsilon, b + \epsilon) < \infty$,
   (c) $\exists R > 0, [x^i y^j]\mathbf{E}(x, y) \leq R^{i+j}$.
2. MSet_M and Seq_M for all $M \subseteq \mathbb{Z}^{>0}$.
3. DCycle_M and Cycle_M for $\sum_{m \in M} 1/m = \infty$ or M finite.

using scalar multiplication from $\mathbb{R}^{\geq 0}$, addition, multiplication, and composition, and where if MSet_M, DCycle_M, or Cycle_M appear then scalars and coefficients of $\mathbf{E}$ must be integers.

Theorem 2. *Let $\Theta \in \mathcal{O}$ such that*

- *Θ is nonlinear*
- *$[x^n]\Theta(\mathbf{A}(x))$ depends only on $[x^i]\mathbf{A}(x)$ for $i < n$.*

Let $\mathbf{A}(x)$ be a power series

- *with nonnegative coefficients*
- *with zero constant term*
- *which diverges at its radius of convergence*
- *if MSet_M, DCycle_M, or Cycle_M appear in Θ then $\mathbf{A}(x)$ has integer coefficients.*

Then there is a unique $\mathbf{T}(x)$ satisfying

$$\mathbf{T}(x) = \mathbf{A}(x) + \Theta(\mathbf{T})(x).$$

The coefficients of $\mathbf{T}$ satisfy the universal law on their support.

References

[1] Jason Bell, Stanley Burris, and Karen Yeats, *Counting Rooted Trees: The Universal Law $t(n) \sim C \cdot \rho^{-n} \cdot n^{-3/2}$*. arxiv:math.CO/0512432

[2] Philippe Flajolet and Robert Sedgewick, *Analytic Combinatorics.* http://algo.inria.fr/flajolet/Publications/books.html

[3] *Dirk Kreimer and Karen Yeats upcoming*