Using renormalization Hopf algebra intuition on symmetric functions

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Why am I talking in this session?

and with a section called "Renormalization Hopf algebras"!?

In 2014-2015 I was on sabbatical.
I spent some of that time talking to Steph.

I wanted to understand why the renormalization Hopf algebra world seemed to have a different flavour than the rest of the combinatorial Hopf algebra world.

One paper came out of this `arXiv:1511.06337`.
It had errors (all due to me).
Now it is corrected and has appeared: E-JC **24**(3) (2017) #P3.10.

The Connes-Kreimer Hopf algebra

Take a polynomial algebra of rooted trees and use the following coproduct:

$$\cancel{\Delta(G) =} \qquad \Delta(t) = \sum_{\substack{S \subseteq V(t) \\ S \text{ antichain}}} \left(\prod_{s \in S} t_s \right) \otimes \left(t \setminus \prod_{s \in S} t_s \right)$$

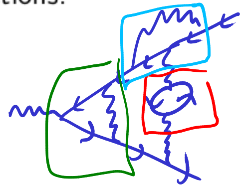
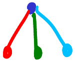
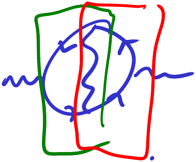
where t_s = subtree rooted at s

Eg:

$$\Delta(\Lambda) = 1 \otimes \Lambda + \bullet\bullet \otimes \bullet + \Lambda \otimes 1 + 2 \bullet \otimes \mathfrak{l}$$

Renormalization Hopf algebras of Feynman diagrams

Feynman diagrams are graphs which tell a story of particle interactions.

Eg:   

Take a polynomial algebra of them and use the following coproduct:

$$\Delta(G) = \sum_{\gamma} \gamma \otimes G/\gamma$$

product of divergent 1PI subgraphs

Important things from this perspective

- Add-a-root/graph insertion, Hochschild 1-cocycles.
- SubHopf algebras with one generator in each degree.
- Combinatorial specifications/Dyson-Schwinger equations.
- Feynman rules/character maps.
- Iterated coproduct, and pulling out particular terms.

Shape Hopf algebra

Use two layers.

Take the polynomial algebra generated by connected shapes, including skew.

Identify disconnected shapes with the monomial of components.

Make the coproduct match Schur functions:

$$\Delta(\lambda/\mu) = \sum_{\mu \leq \eta \leq \lambda} (\eta/\mu) \otimes (\lambda/\eta)$$

And nothing else.

. . . and symmetric functions

The shape Hopf algebra is the Hopf algebra of intervals in Young's lattice.
It is not cocommutative.

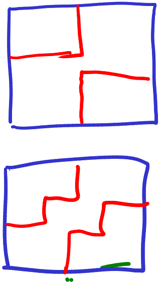
The symmetric function Hopf algebra is a quotient.
It is a cocommutative quotient.

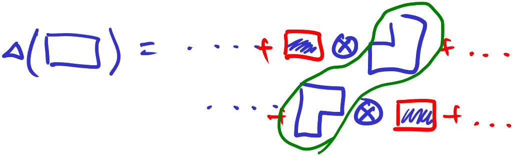
That's all we need to know.

A complicated proof of an easy fact

If α is a shape let α^* be the 180° rotation.
Their Schur functions are the same, written $\alpha \sim \alpha^*$

Proof.





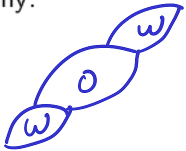
repeat

Cocommutativity was the key.

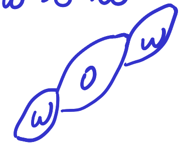
McNamara, van Willigenburg WOW shapes

Schematically:

$w \uparrow o \uparrow w$



$w \rightarrow o \rightarrow w$

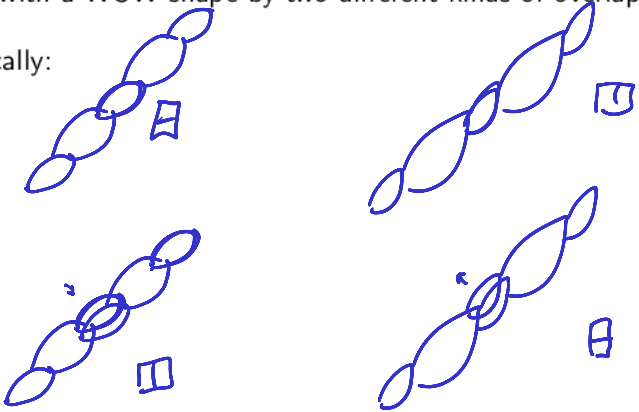


(These are only two of their four cases.)

Composition

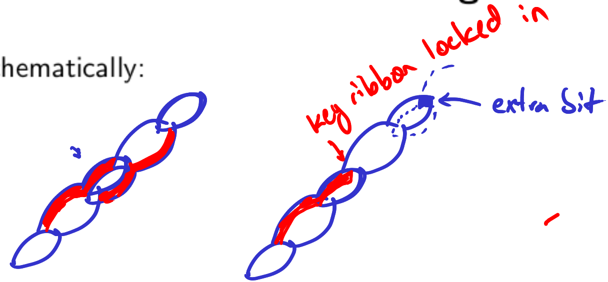
Compose with a WOW shape by two different kinds of overlap.

Schematically:



Key ribbons and almost rectangles

Schematically:



Note that the top and bottom key ribbons may differ.

Result

Theorem

Let β be a partition shape that is a rectangle with the lower right corner box removed. Let γ be a $W \to O \to W$ or $W \uparrow O \uparrow W$ shape with no loose end ribbons. Then

$$\beta \circ \gamma \sim \beta^* \circ \gamma.$$

What are loose end ribbons?

Prove the result by cocommutativity pulling out the key ribbon $\otimes$ stuff and stuff $\otimes$ key ribbon terms (with many details).

Conclusion

- This is much more specialized than what McNamara and van Willigenburg can prove about WOW-built skew Schur function identities.
- It can prove some cases that they can't.
- Generalizations?
- What about all those other renormalization Hopf algebra things?