Combinatorial approaches in quantum field theory

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Combinatorics providing insights in QFT

	Things you can do	Combinatorics to use
Mathematizing Renormalization	Connes-Kreimer Hopf alg. math framework for renormalization	Comb. Hopf alg how to decompose into sub objects graphs, trees
Evaluating individual Feynman graphs	High loop scalar integrals why particular #s show up	graph theory graph polynomials
Moving from scalar field theories to gauge theories	master integrals bootstrapping diff op	matroids corolla polynomial
Understanding Dyson-Schwinger equations		

Some refs

	References
Mathematizing Renormalization	arXiv:hep-th/0211136 arXiv:hep-th/0506190 arXiv:1202.3552
Evaluating individual Feynman graphs	arXiv:0804.1660 arXiv:0801.2856 arXiv:0910.5429 arXiv:1208.1890
Moving from scalar field theories to gauge theories	arXiv:1010.5804 arXiv:1208.6477 arXiv:1207.5460
Understanding Dyson-Schwinger equations	arXiv:hep-th/0605096 arXiv:0810.2249 arXiv:0805.0826 ← arXiv:1210.5457

Many of these have appeared in journals now. And there are many more.

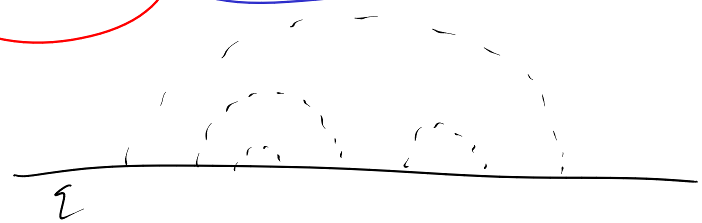
Simple nestings and chainings

Today there's only time to talk about one of these, so I will talk about Dyson-Schwinger equations.

An example in Yukawa theory (Broadhurst-Kreimer arXiv:hep-th/0012146)

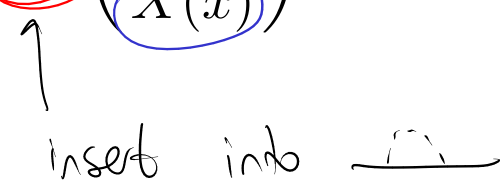
$$G(x,L) = 1 - \frac{x}{q^2}\int d^4k \frac{k \cdot q}{k^2 G(x, \log k^2/\mu^2)(k+q)^2} - \cdots \Big|_{q^2=\mu^2}$$

where $L = \log(q^2/\mu^2)$.



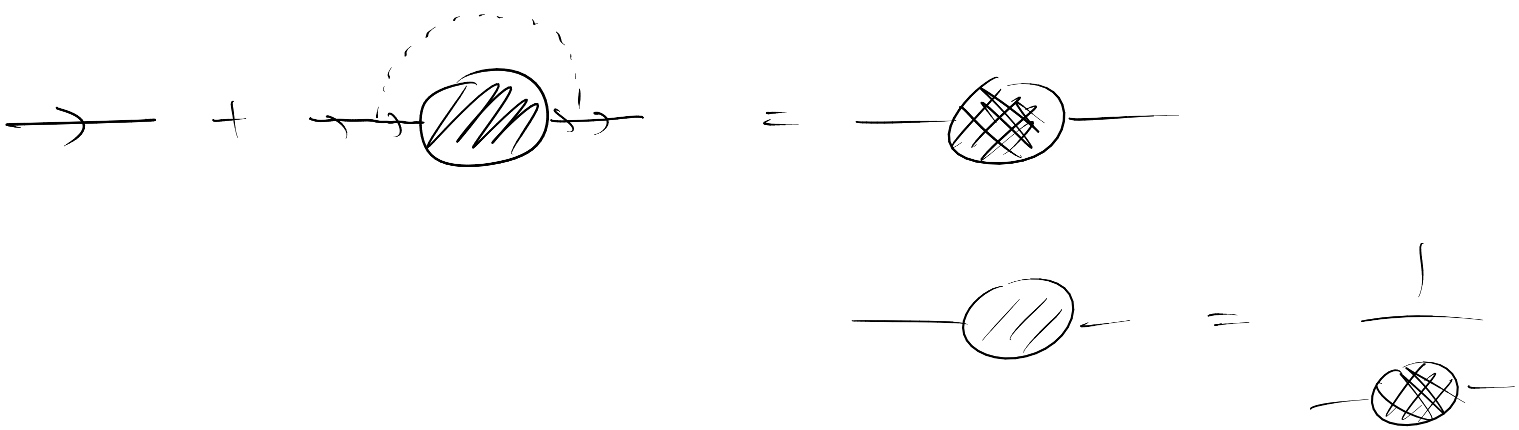
How to capture the combinatorics of the recursion?

$$X(x) = \mathbb{I} - xB_+\left(\frac{1}{X(x)}\right)$$



Combinatorial Dyson-Schwinger equations

We can capture other recursions in a similar language – this is equivalent to the diagrammatic viewpoint on Dyson-Schwinger equations.
~~Eg QED~~:



Putting the analysis back in

In the Yukawa example we had

$$G(x,L) = 1 - \frac{x}{q^2}\int d^4k \frac{k\cdot q}{k^2 G(x, \log k^2/\mu^2)(k+q)^2} - \cdots \Big|_{q^2=\mu^2}$$

- plug in $G(x,L) = 1 - \sum \gamma_k(x) L^k$
- use $\partial_\rho^k x^{-\rho}|_{\rho=0} = (-1)^k \log^k(x)$
- switch the order of $\int$ and ∂

to obtain

$$G(x,L) = 1 - xG(x, \partial_{-\rho})^{-1}(e^{-L\rho} - 1)F(\rho)\Big|_{\rho=0}$$

Where $F(\rho)$ is the integral for the primitive regularized by a parameter ρ which marks the insertion place.

Today's analytic Dyson-Schwinger equations

Beginning with a combinatorial Dyson-Schwinger equation

$$X = \mathbb{I} \pm \sum_{k\geq 1} x^k B_+^{\gamma_k}(XQ^k)$$

where $Q = X^{-s}$, *define* the analytic Dyson-Schwinger equation of to be

$$G(x, L) = 1 \pm \sum_{k\geq 1} x^k G(x, \partial_{-\rho})^{1-sk}(e^{-L\rho} - 1)F_k(\rho)|_{\rho=0}$$

where $F_k(\rho)$ is the Feynman integral for γ_k regularized by a parameter ρ which marks the insertion place.

More insertion places and systems get more complicated.

Rearranging Dyson-Schwinger equations

The Yukawa example is particularly nice and can in fact be solved.

This example works so well because the Dyson-Schwinger equation had

- One primitive graph
- which had a particularly nice integral (scaled just a geometric series)
- inserted into one place

The program of arXiv:0810.2249, Memoir. Am. Math. Soc. 211, no. 995, with an important improvement in arXiv:1302.0080, was to generalize this nice situation into a general reduction process for Dyson-Schwinger equations.

Some steps make combinatorial sense, others do not.

Finding the γ_k recurrence

Write

$$G(x, L) = 1 \pm \sum_{k \geq 1} \gamma_k(x) L^k$$

We can find a recurrence for γ_k in terms of lower γ_j – it is the renormalization group equation translated into this language:

$$\left(\frac{\partial}{\partial L} + \beta(x) \frac{\partial}{\partial x} \pm \gamma_1(x) \right) G(x, L) = 0$$

Extracting the coefficient of L^{k-1} gives a recurrence for γ_k

$$\gamma_k = \frac{1}{k} \gamma_1(x)(-\text{sign}(s) + |s| x \partial_x) \gamma_{k-1}(x)$$

for $k \geq 2$

Trading ρ for x

Notice that $\gamma_k(x)$ begins with an x^k term. So the lowest possible power of x in

$$x^k G(x, \partial_{-\rho})^{1-sk} \rho^\ell|_{\rho=0}$$

is

Consequently there is a unique sequence r_k such that

$$\sum_k x^k G(x, \partial_{-\rho})^{1-sk} (e^{-L\rho} - 1) F_k(\rho)\bigg|_{\rho=0}$$

$$= \sum_k x^k G(x, \partial_{-\rho})^{1-sk} (e^{-L\rho} - 1) \frac{r_k}{\rho(1-\rho)}\bigg|_{\rho=0}$$

The differential equation

Taking the coefficient of L and L^2 in

$$G(x,L) = 1 \pm \sum_k x^k G(x, \partial_{-\rho})^{1-sk}(e^{-L\rho} - 1)\frac{r_k}{\rho(1-\rho)}\bigg|_{\rho=0}$$

and then using the γ_k recurrence we get

$$\gamma_1(x) = -P(x) + \gamma_1(x)(\text{sign}(s) - |s|x\partial_x)\gamma_1(x)$$

where

$$P(x) = \sum_{k\geq 1} r_k x^k$$

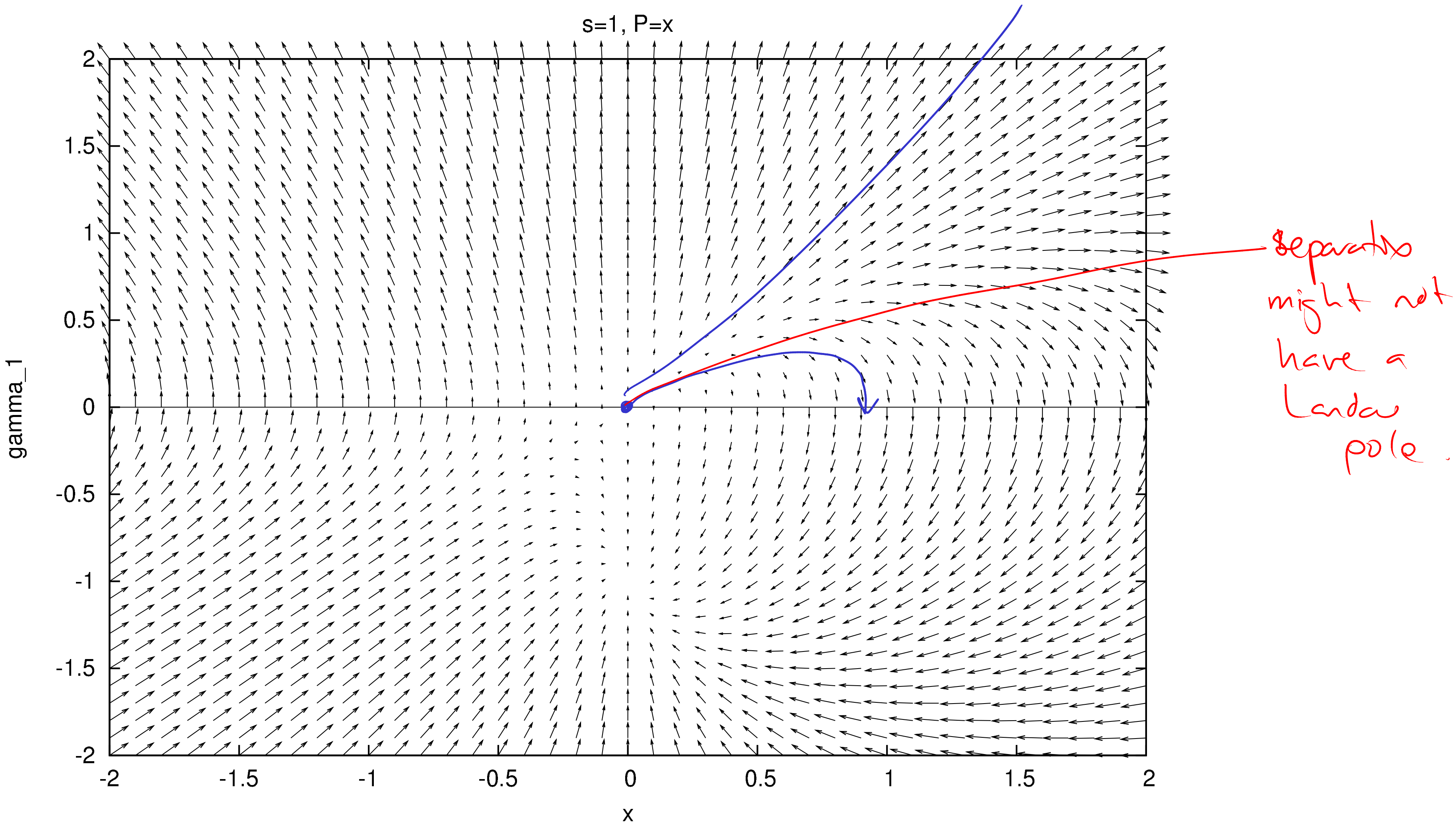
The differential equation in QED

Joint work with Guillaume van Baalen, Dirk Kreimer, and David Uminsky, arXiv:0805.0826.

In QED in the Baker, Johnson, Willey gauge, we only need to worry about the photon, so we are in the single equation case.

$s = 1$ because

Picture



There are two behaviours. The *separatrix* is the separating solution.

Results

If $P(x)$ is $\mathcal{C}^2$ and $P(x) > 0$ for $x \in (0, x_0)$ then either

- γ_1 crosses the x axis with a vertical tangent and returns to -1, or
- P and γ_1 have a common zero, or
- γ_1 is positive and exists for all x

In the last case if also $P(x) > 0$ for all $x > 0$ and $P(x)$ is increasing then either

- γ_1 is the separatrix and diverges in finite L (a Landau pole) iff

$$\int_{x_0}^{\infty} \frac{2dz}{z(\sqrt{1+4P(z)}-1)} < \infty$$

- γ_1 is larger than the separatrix and diverges in finite L regardless of P.

Other results

We also thought about other values of s including in arXiv:0906.1754 negative values of s which have quite a different flavour (spirals!) and form a model of massless QCD.

Looking at $s = 2$ we can give an explicit combinatorial solution as a sum over rooted connected chord diagrams

Marc Bellon and his collaborators have looked at the Wess-Zumino model, eg arXiv:1205.0022, and specific approximations to P.